Optimal strategies associated with optionality features in financial contracts

presented by

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Examples of free boundary value problems in finance

- Early exercise right in American options
- Resetting terms in a derivative contract
- Reload right in employee stock options
- Early conversion into shares by bondholders and recall by bond issuer in a convertible bond
- Guaranteed floor protection in equity linked annuities

Formulated as optimal stopping problems or obstacle problems.
In his Nobel Award winning paper, Merton (1973) wrote:

“... because options are specialized and relatively unimportant financial securities, the amount of time and space devoted to the development of a pricing theory might be questioned”

Ross (1987) wrote in New Palgrave Dictionary of Economics

“This does not mean, however, that there are no important gaps in the (option pricing) theory. Perhaps of most important, beyond numerical results ... very little is known about most American options which expire in finite time ... Despite such gaps, when judged by its ability to explain the empirical data, option pricing theory is the most successful theory not only in finance, but in all of economics.
Valuation of contingent claims

- A contingent claim is a random variable $X$ that represents the time $T$ payoff from a “seller” to a “buyer”.

- A contingent claim is said to be marketable or attainable if there exists a self-financing trading such that

$$V_T(\omega) = X(\omega) \text{ for all } \omega \in \Omega.$$ 

Risk neutral valuation principle

The time $t$ value of a marketable contingent claim $X$ is equal to $V_t$, the time $t$ value of the portfolio which replicates $X$.

$$V_t^* = V_t/B_t = E_Q[X/B_T|F_t], \quad t = 0, 1, \ldots, T$$

for all risk neutral probability measure $Q$. 
Martingales

The process $Z$ is said to be a *martingale* if

$$E[Z_{t+s}|F_t] = Z_t \quad \text{for all } s \text{ and } t \geq 0.$$  

- A martingale is the mathematical formalization of the concept of a fair game.

- If discounted prices of derivative securities can be modelled as martingales, this implies that no market participant can consistently make (or lose) money by trading in derivatives.
Martingales pricing theory

A risk neutral probability measure (martingale measure) is a probability measure $Q$ such that

1. $Q(\omega) > 0$ for all $\omega \in \Omega$; and

2. the discounted price process $S^*_n$ is a martingale under $Q, n = 1, \cdots, N$.

$$E_Q[S^*_n(t + s)|F_t] = S^*_n(t), \quad t \text{ and } s \geq 0.$$ 

Theorems

1. There are no arbitrage opportunities if and only if there exists a martingale measure $Q$.

2. If $Q$ is a martingale measure and $\mathcal{H}$ is a self financing trading strategy, then $V^*$, the discounted value process corresponding to $\mathcal{H}$, is a martingale under $Q$. 
Completeness theorem

If the market is complete, that is, all contingent claims can be replicated, then equivalent martingale measures are unique.

Fundamental theorem of asset pricing

In an arbitrage free complete market, there exists a unique equivalent martingale measure.

Risk neutral pricing formula

In an arbitrage free complete market, arbitrage prices of contingent claims are their discounted expected values under the risk neutral (equivalent martingale) measure.
Riskless hedging principle

Consider the writer of a derivative whose underlying asset has the following price process

\[
\frac{dS}{S} = \mu \ dt + \sigma \ dZ. 
\]

Let \(V(S,t)\) denote the price of the derivative. Black-Scholes (1973) derived the governing equation for \(V(S,t)\) by following the dynamic hedging principle. Form the portfolio \(\pi\) that contains \(\Delta\) units of the underlying asset and shorts one unit of the derivative

\[
\pi = -V + \Delta S. 
\]

By Ito’s lemma:

\[
d\pi = - \left[ \frac{\partial V}{\partial t} \ dt + \frac{\partial V}{\partial S} \ dS + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \ dt \right] + \Delta \ dS. 
\]
Suppose we choose $\Delta = \frac{\partial V}{\partial S}$ so that the stochastic components are cancelled.

$$d\pi = - \left( \frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \right) dt. $$

The riskfree portfolio should earn the riskfree interest rate, otherwise there is arbitrage opportunity. We then have

$$d\pi = r\pi \ dt \quad \Rightarrow \quad - \left( \frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \right) dt = r \left( -V + \frac{\partial V}{\partial S} S \right) dt,$$

and obtain the Black-Scholes equation

$$\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0.$$  

Note that the expected rate of return $\mu$ does not appear in the equation.

**Risk neutral valuation**

$$V(S, t) = e^{-r(T-t)} E^*[V_0(S_T)].$$

The option price is the discounted expectation of the terminal payoff under the risk neutral measure.
Market price of risk

Suppose we write formally the stochastic process of $V$ as

$$\frac{dV}{V} = \rho_V \ dt + \sigma_V \ dZ.$$

We obtain

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} - \rho_V V = 0 \quad \text{and} \quad \sigma_V = \frac{\sigma S \frac{\partial V}{\partial S}}{V}. $$

Since the option and the stock are hedgeable, they share the same market price of risk

$$\frac{\rho_V - r}{\sigma_V} = \frac{\mu - r}{\sigma}$$

from which we can deduce

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial V^2} + r S \frac{\partial V}{\partial S} - r V = 0.$$
American put option

Early exercise right \[ \Rightarrow \quad P(S, t) \geq \max(X - S, 0). \]

For European put price function, \( P_E(0, t) = X e^{-r(T-t)} < X \) so that \( P_E \) is below the intrinsic value \( X - S \) when \( S \) is sufficiently low. There exists a critical asset price \( S^*(t) \) at which it is optimal to exercise the put option.
Smooth pasting condition

At the critical asset price $S^*(t)$

$$P(S^*(t), t) = X - S^*(t)$$

$$\frac{\partial P}{\partial S}(S^*(t), t) = -1$$
Proof of the smooth pasting condition

\[ P(S, t) = \max_{b(t)} f(S, \tau; b(t)); \text{ write } F(S, b) = f(S, \tau; b(t)). \]

The total derivative of \( F \) with respect to \( b \) along the boundary \( S = b \) is

\[ \frac{dF}{db} = \frac{\partial F}{\partial S}(S, b)\big|_{S=b} + \frac{\partial F}{\partial b}(S, b)\big|_{S=b}. \]

Let \( b^* \) be the value of \( b \) that maximizes \( F \). When \( b = b^* \), we have

\[ \frac{\partial F}{\partial b}(S, b^*) = 0 \quad \text{(first order condition)}. \]

For a put, we write \( h(b) = F(b, b) = X - b \) and

\[ \frac{dh}{db}\big|_{b=b^*} = \frac{d}{db}(X - b)\big|_{b=b^*} = -1 \]

so that

\[ \frac{\partial F}{\partial S}(b^*, b^*) = -1 \quad \Leftrightarrow \quad \frac{\partial P}{\partial S}(S^*(t), t) = -1. \]
Continuation region and stopping region

In the stopping region where the holder should exercise the put optimally

\[ P(S, t) = X - S. \]

Substituting the put price function into the Black-Scholes equation

\[ \left( \frac{\partial}{\partial t} + \mathcal{L} \right) (X - S) = -rX < 0. \]

This is equivalent to

\[ d\pi < r\pi \, dt \]

indicating that the rate of return from the portfolio is less than \( r \).
Linear complementarity formulation

In the stopping region: \(-\left(\frac{\partial V}{\partial t} + \mathcal{L}V\right) > 0\)

In the continuation region: \(\frac{\partial V}{\partial t} + \mathcal{L}V = 0\).

\[
\begin{cases}
-\left(\frac{\partial V}{\partial t} + \mathcal{L}V\right) \geq 0 \quad \text{and} \quad V \geq V_0(S) \\
\min \left(-\left(\frac{\partial V}{\partial t} + \mathcal{L}V\right), V - V_0(S)\right) = 0.
\end{cases}
\]
Obstacle problem

The string is held fixed at $A$ and $B$, and it must pass smoothly over the obstacle.

$-u''(u - f) = 0$, $f(x)$ is the obstacle function; $-u'' \geq 0$ and $u \geq f$.

- string must be above or on the obstacle
- string must have negative or zero curvature
- string and its slope must be continuous at contact points

Free boundary value problem: We do not know a priori the contact points of the string on the obstacle.
Optimal stopping time

\[ P(S, t) = \sup_{t \leq \xi \leq T} E^*\{e^{-r(\xi-t)}(X - S_\xi)\} \]

where

\[ \xi^* = \inf_\xi\{t \leq \xi \leq T : P(S_\xi, \xi) = X - S_\xi\}, \]

that is, \( \xi^* \) is the first time that the price of the derivative drops down to its exercise payoff.
The price of an American put satisfies the following linear complementarity problem:

\[
\frac{\partial P}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 P}{\partial S^2} + rS \frac{\partial P}{\partial S} - rP \leq 0
\]  
\tag{i}

\[P \geq X - S \]  
\tag{ii}

\[
\left[ \frac{\partial P}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 P}{\partial S^2} + rS \frac{\partial P}{\partial S} - rP \right] [P - (X - S)] = 0
\]  
\tag{iii}

to be solved in \( \{(S, t) : S > 0, 0 < t < T\} \) and \( P(S, T) = (X - S)^+ \).

**Sketch of the proof:** For any stopping time \( \xi, \ t < \xi < T \), by Ito's calculus,

\[
e^{-r\xi} P(S_{\xi}, \xi) = e^{-rt} P(S_t, t) + \int_t^\xi e^{-ru} \left( \frac{\partial}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2}{\partial S^2} + rS \frac{\partial}{\partial S} - r \right) P(S_u, u) \, du
\]

\[
+ \int_t^\tau e^{-ru} \sigma S_u \frac{\partial P}{\partial S}(S_u, u) \, dW_u
\]

\[
E^* \{ e^{-r(\xi-t)} P(S_{\xi}, \xi) \} \leq P(S_t, t) \text{ by (i) & Doob's optional stopping theorem and by (ii), we have } E^* \{ e^{-r(\xi-t)} (X - S) \} \leq P(S_t, t).
\]

If the optimal stopping time \( \xi^* \) is taken, then we have equality.
Delayed exercise compensation

The holder of the American put should be compensated by a continuous cash flow when the put should be exercised optimally. The discounted expectation for the above continuous cash flow is

$$e^{-r\xi} \int_0^{S^*(\tau - \xi)} (rX - qS_\xi) \psi(S_\xi; S) \, dS_\xi.$$

This is because the holder would earn interest of amount $rXd\xi$ from the cash received and would lose dividends of amount of $qS_\xi \, d\xi$ from the short position of the asset.
American put price function

\[ P(S, t) = e^{-r\tau} \int_0^X (X - S_T) \psi(S_T; S) \ dS_T \]

\[ + \int_0^\tau e^{-r\xi} \int_0^{S^*(\tau-\xi)} (rX - qS_\xi) \psi(S_\xi; S) \ dS_\xi d\xi \]

where \( \tau = T - t \) and \( \xi \) is the time lapsed after the current time \( t \).

\[ P(S, t) = X e^{-r\tau} N(-d_2) - S e^{-q\tau} N(-d_1) \]

\[ + \int_0^\tau rX e^{-r\xi} N(-d_{\xi,2}) - qSe^{-1\xi} N(-d_{\xi,1}) \ d\xi \]

where

\[ d_1 = \frac{\ln \frac{S}{X} + \left( r - q + \frac{\sigma^2}{2} \right) \tau}{\sigma \sqrt{\tau}}, \quad d_2 = d_1 - \sigma \sqrt{\tau} \]

\[ d_{\xi,1} = \frac{\ln \frac{S}{S^*(\tau-\xi)} + \left( r - q + \frac{\sigma^2}{2} \right) \xi}{\sigma \sqrt{\xi}}, \quad d_{\xi,2} = d_{\xi,1} - \sigma \sqrt{\xi}. \]
Proof of the integral representation of the early exercise premium

Let
\[ G(S, t; \xi, T) = \frac{e^{-r(T-t)}}{\xi \sigma \sqrt{2\pi(T-t)}} \exp \left( -\frac{1}{2\sigma^2(T-t)} \left[ \ln \frac{S}{\xi} + \left( r - q - \frac{\sigma^2}{2} \right)(T-t) \right]^2 \right). \]

Write \( V(\xi, u) = G(S, t; \xi, u) \), which is considered as a function of \( \xi \) and \( u \); then \( V(\xi, u) \) satisfies the adjoint equation:

\[ \mathcal{L}^* V = -\frac{\partial V}{\partial u} + \frac{\sigma^2}{2\xi^2} (\xi^2 V) - (r - q) \frac{\partial}{\partial \xi} (\xi V) - r V = 0 \]

\[ V(\xi, t) = \delta(\xi - S). \]

We have

\[ -\mathcal{L} P(S, t) = \begin{cases} 0 & (S, t) \in \text{continuation region} \\ rX - qS & (S, t) \in \text{stopping region} \end{cases}. \]
Consider

\[\int_{t+\epsilon}^{T} du \int_{0}^{S^{*}(u)} (rX - q\xi)G^{*}(\xi, u; S, t) \, d\xi\]

\[= - \int_{t+\epsilon}^{T} du \int_{0}^{\infty} G^{*}(\xi, u; S, t) \mathcal{L}P \, d\xi\]

\[= - \int_{t+\epsilon}^{T} du \int_{0}^{\infty} [G^{*}(\xi, u; S, t) \mathcal{L}P - P \mathcal{L}^{*}G^{*}(\xi, u; S, t)] \, d\xi\]

\[= - \int_{t+\epsilon}^{T} du \int_{0}^{\infty} \left\{ \frac{\partial}{\partial u}(G^{*}P) + \frac{\sigma^{2}}{2} \frac{\partial}{\partial \xi} \left( \xi^{2}G^{*}\frac{\partial P}{\partial \xi} \right) \right.\
\left. - \frac{\sigma^{2}}{2} \frac{\partial}{\partial \xi} \left[ P \frac{\partial}{\partial \xi}(\xi^{2}G^{*}) \right] \right\} \, d\xi.\]

Note that when \(\xi \to 0\) and \(\xi \to \infty\), we have

\(\xi^{2}G^{*}\frac{\partial P}{\partial \xi} \to 0, \quad P \frac{\partial}{\partial \xi}(\xi^{2}G^{*}) \to 0\) and \(\xi PG^{*} \to 0\).
Hence, we obtain

$$\int_0^\infty G^*(\xi, t + \epsilon; S, t) P(\xi, t + \epsilon) \, d\xi = \int_0^\infty G^*(\xi, T; S, t) P(\xi, T) \, d\xi$$

$$+ \int_T^{t+\epsilon} du \int_0^{S^*(u)} (rX - q\xi)G^*(\xi, u; S, t) \, d\xi.$$

Lastly, we let $\epsilon \to 0$ so that

$$P(S, t) = \int_0^\infty G(S, t; \xi, T)(X - \xi) \, d\xi$$

$$+ \int_T^T du \int_0^{S^*(u)} (rX - q\xi)G(S, t; \xi, u) \, d\xi.$$
American fixed strike lookback call

Let \( C_{fix}(S, M, \tau; K) \) denote the price function of an American fixed strike lookback call with payoff \((M - K)^+\), where \( M = \max_{T_0 \leq u \leq t} S_u \) is the current realized maximum of the asset price.

Linear complementarity formulation

\[
\frac{\partial C_{fix}}{\partial \tau} - \mathcal{L}C_{fix} \geq 0, \quad C_{fix} \geq (M - K),
\]

\[
\left( \frac{\partial C_{fix}}{\partial \tau} - \mathcal{L}C_{fix} \right) [C_{fix} - (M - K)] = 0, \quad 0 < S < M, \tau > 0,
\]

with auxiliary conditions:

\[
\frac{\partial C_{fix}}{\partial M} \big|_{S=M} = 0,
\]

\[
C_{fix}(S, M, 0) = (M - K)^+.
\]
Stopping region

\[ S(K) = \{(S, M, \tau) \in \{0 < S \leq M\} \times (0, \infty) : C_{fix}(S, M, \tau) = (M - K)^+\}. \]

Proposition

The stopping region \( S(K) \) and the price function \( C_{fix}(S, M, \tau; K) \) of the American fixed strike lookback call observe the following properties:

(i) \( C_{fix}(S, M, \tau; K_2) - C_{fix}(S, M, \tau; K_1) \leq K_1 - K_2 \) \hspace{1cm} \text{if} \hspace{1cm} K_1 > K_2,

(ii) \( S(K_1) \subset S(K_2) \) \hspace{1cm} \text{if} \hspace{1cm} K_1 > K_2,

(iii) Suppose \((S, M, \tau) \in S(K)\) and \(0 < \lambda_1 \leq 1, \lambda_2 \geq 1, 0 < \lambda_3 \leq 1\), we have

\[ (\lambda_1 S, \lambda_2 M, \lambda_3 \tau) \in S(K). \]
The exercise boundaries (solid curves) of the American fixed strike lookback call option with varying maturity $\tau$ are plotted in the $S-M$ plane.
• At a given $\tau$, the stopping region is lying to the left and above of the corresponding exercise boundary.

• The dotted lines are asymptotic lines of the exercise boundaries, corresponding to the exercise boundaries of the zero-strike counterparts.

• The stopping region of the Russian option lies to the left of the dotted line: $M = S\xi^*(\infty; 0)$. 
Proof of Proposition

(i) Define the function $V(S, M, \tau; K) = C_{fix}(S, M, \tau; K) + K$. The linear complementarity formulation for $V(S, M, \tau; K)$ is given by

$$\frac{\partial V}{\partial \tau} - LV \geq rK, \quad V \geq \max(M, K)$$

$$\left[\frac{\partial V}{\partial \tau} - LV - \max(M, K)\right][V - \max(M, K)] = 0,$$

with auxiliary conditions:

$$\frac{\partial V}{\partial M}\big|_{S=M} = 0 \quad \text{and} \quad V(S, M, 0; K) = \max(M, K).$$

By virtue of the comparison principle, we have

$$V(S, M, \tau; K_1) \geq V(S, M, \tau; K_2) \quad \text{if} \quad K_1 > K_2,$$

and hence the result.
(ii) From (i), for $K_1 > K_2$, we have

\[(C.1) \quad C_{fix}(S, M, \tau; K_1) - (M - K_1) \geq C_{fix}(S, M, \tau; K_2) - (M - K_2).\]

Suppose $(S, M, \tau) \in S^C(K_2)$, where $S^C(K_2)$ denotes the continuation region. In the continuation region, the option value is strictly greater than the exercise payoff so that

\[C_{fix}(S, M, \tau; K_2) > M - K_2.\]

Combining with Inequality (C.1), we can deduce

\[C_{fix}(S, M, \tau; K_1) > M - K_1,\]

so that $(S, M, \tau) \in S^C(K_1)$.

Hence, we establish $S^C(K_2) \subset S^C(K_1)$; and so $S(K_1) \subset S(K_2)$. 

(iii) Since $C_{fix}(S, M, \tau)$ is monotonically increasing with respect to both $S$ and $
abla$, and the exercise payoff is independent of $S$ and $\tau$, we deduce that if $(S, M, \tau) \in S(K)$, then

$$(\lambda_1 S, M, \lambda_3 \tau) \in S(K) \quad \text{for all} \quad 0 < \lambda_1 \leq 1 \text{ and } 0 < \lambda_3 \leq 1.$$ 

- $(S, M, \tau) \in S(K)$ would imply $(S, \lambda_2 M, \tau) \in S(K)$, for all $\lambda_2 \geq 1$.

- Suppose $(S, M, \tau) \in S(K)$, then $(S/\lambda_2, M, \tau) \in S(K)$ for $\lambda_2 \geq 1$. 

By virtue of the linear homogeneity property of the price function and result in (i), we obtain

\[
C_{fix}(S, \lambda_2 M, \tau; K) = \lambda_2 C_{fix} \left( \frac{S}{\lambda_2}, M, \tau; \frac{K}{\lambda_2} \right) \\
\leq \lambda_2 \left[ C_{fix} \left( \frac{S}{\lambda_2}, M, \tau; K \right) + \left( 1 - \frac{1}{\lambda_2} \right) K \right] \\
= \lambda_2 \left[ M - K + \left( 1 - \frac{1}{\lambda_2} \right) K \right] = \lambda_2 M - K.
\]

\(C_{fix}(S, \lambda_2 M, \tau; K)\) cannot fall below the exercise payoff \(\lambda_2 M - K\).

Combining the results

\[
C_{fix}(S, \lambda_2 M, \tau; K) = \lambda_2 M - K,
\]

that is, \((S, \lambda_2 M, \tau) \in S(K)\).
**Proposition**

Suppose $M^*(S, \tau; K)$ denotes the exercise boundary of the American fixed strike lookback call in the $S-M$ plane, then $M^*(S, \tau; K)$ observes

(i) $\lim_{\tau \to 0^+} M^*(S, \tau; K) = K$ for all $S$,

(ii) $M^*(S, \tau; K)$ is a monotonically increasing with respect to $S$ and $\tau$,

(iii) $\lim_{S \to 0^+} M^*(S, \tau; K) = K$ for all $\tau$, 
(iv) $M^*(S, \tau; 0)$ is a linear function of $S$. Furthermore, $\frac{M^*(S, \tau; 0)}{S}$ is a monotonically increasing function of $\tau$ and

$$\lim_{S \to \infty} \frac{M^*(S, \tau; K)}{S} = \frac{M^*(S, \tau; 0)}{S} \quad \text{for} \quad K > 0.$$ 

(v) For $K > 0$, by virtue of the linear homogeneity property of $M^*(S, \tau; K)$, we obtain

$$\lim_{S \to \infty} \frac{M^*(S, \tau; K)}{S} = \lim_{S \to \infty} \frac{M^*(\frac{S}{K}, \tau; 1)}{\frac{S}{K}} = \lim_{K \to 0} \frac{M^*(\frac{S}{K}, \tau; 1)}{\frac{S}{K}}$$

$$= \lim_{K \to 0} \frac{M^*(S, \tau; K)}{S} = \frac{M^*(S, \tau; 0)}{S}.$$
American fixed strike lookback put

payoff = \((K - m)^+\), where \(m\) is the current realized minimum of asset price. Linear complementarity formulation that governs its price function \(P_{fix}(S, m, \tau)\) is given by

\[
\begin{align*}
\frac{\partial P_{fix}}{\partial \tau} - \mathcal{L}P_{fix} & \geq 0, \quad P_{fix} \geq (K - m), \\
\left(\frac{\partial P_{fix}}{\partial \tau} - \mathcal{L}P_{fix}\right)[P_{fix} - (K - m)] & = 0, \quad 0 < m < S, \tau > 0,
\end{align*}
\]

with auxiliary conditions:

\[
\begin{align*}
\frac{\partial P_{fix}}{\partial m}\big|_{S=m} & = 0 \\
P_{fix}(S, m, 0) & = (K - m)^+.
\end{align*}
\]
Proposition

The exercise boundary $m^*(S, \tau; K)$ of the American fixed strike lookback put satisfies the following properties:

(i) $\lim_{\tau \to 0^+} m^*(S, \tau; K) = K$ for all $S$,

(ii) $m^*(S, \tau; K)$ is monotonically increasing with respect to $S$,

(iii) $\lim_{S \to \infty} m^*(0, \tau; K) = K$ for all $\tau$,

(iv) $\lim_{S \to 0} \frac{m^*(S, \tau; K)}{S} = 1$ for all $\tau$. 
The exercise boundaries of the American fixed strike lookback put option with varying maturity $\tau$ are plotted in the $S-m$ plane.

All exercise boundaries tend to the oblique asymptotic line: $m = S$ as $S \to 0^+$, and the horizontal asymptotic line: $m = K$ as $S \to \infty$. 
Proof of Proposition

(iv) For $\alpha \geq 1$, we observe that

$$(K - m)^+ \leq (K - \alpha S)^+ + \alpha S - m$$

so that

$$(E.1) \quad P_{fix}(S, m, \tau; K) \leq \alpha P\left(S, \tau; \frac{K}{\alpha}\right) + C_{f\ell}(S, m, \tau; \alpha),$$

where $P\left(S, \tau; \frac{K}{\alpha}\right)$ denotes the price function of the American vanilla put option with strike price $\frac{K}{\alpha}$.

Let $S_P^*(\tau; \frac{K}{\alpha})$ be the critical asset price of the American vanilla put with payoff $\left(\frac{K}{\alpha} - S\right)^+$. 
Consider the point $(\hat{S}, \hat{m})$ in the $S$-$m$ plane which lies inside the region

$$R_\alpha = \left\{(S, m) : \ m \leq S_\eta^*(\tau; \alpha) \quad \text{and} \quad S \leq S_p^* \left( \tau; \frac{K}{\alpha} \right) \right\},$$

$(\hat{S}, \hat{m})$ lies in the corresponding stopping region of both the American floating strike call and American vanilla put. We then have

$$P(\hat{S}, \tau; \frac{K}{\alpha}) = \frac{K}{\alpha} - \hat{S} \quad \text{and} \quad C_f(\hat{S}, \hat{m}, \tau; \alpha) = \alpha \hat{S} - \hat{m}.$$

Now, we argue that $(\hat{S}, \hat{m})$ also lies in the stopping region of the American fixed strike put.

It suffices to show that

$$P_{fix}(\hat{S}, \hat{m}, \tau; K) = K - \hat{m}.$$
Combining the results in Eqs. (E.1) and (E.2), we obtain $P_{fix}(\hat{S}, \hat{m}, \tau; K) \leq K - \hat{m}$. Since the option value of the American fixed strike put cannot fall below its exercise payoff, the result in Eq. (E.3) is then established.

We take the limit $\alpha \to \infty$ and observe that

$$\lim_{\alpha \to \infty} \eta^*(\tau; \alpha) = 1 \quad \text{and} \quad \lim_{\alpha \to \infty} S^*_p \left( \tau; \frac{K}{\alpha} \right) = 0$$

for all $\tau$. As $\alpha \to \infty$, $R_\alpha$ shrinks to an infinitesimally small triangular wedge with the oblique side: $S = m$.

Hence, we can deduce that as $S \to 0^+$, the exercise boundaries $m^*(S, \tau; K)$, for all $\tau$, tend to the oblique asymptotic line: $S = m$. 
American options


American path dependent options


Reset features


Reload feature of executive stock options


Interaction of callable and conversion rights


Equity-linked annuities
