Real Options in Strategic Investment Games

between Two Asymmetric Firms

presented by

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This is a joint work with Jean Kong.
Four basic characteristics of strategic investment games

1. Irreversibility of investment (at least partially).
2. Possibility of delay of the investment.
3. Uncertainty about the profit stream generated by the investment.
4. Presence of competitors who can make a similar investment.

- In a deterministic world, the possibility of postponement has no value.
- Real game option model.
Statement of the problem

• Examine the strategic investment games between two firms that compete for optimal entry in a project that generates uncertain revenue flow – under asymmetry on both the sunk cost of investment and revenue flows.

• Identify the set of key parameters that characterize various forms of equilibriums: pre-emptive, dominant and simultaneous equilibriums.
Pre-emptive equilibrium: Both firms have an incentive to become the leader. This occurs when there is no clear advantage of one firm over the other as the leader.

Dominant equilibrium: One of the two firms may be seen to be always better off by serving as the follower, its competitor can wait until the optimal leader threshold is reached.

Simultaneous equilibrium: When the pre-emptive thresholds of both firms happen to coincide, the two firms enter simultaneously, even though at sub-optimal conditions of entry.
Symbols used in the two-player real options model

Two firms, $i \in \{1, 2\}$. Competitor of Firm $i$ is labelled as Firm $i'$.

Monopoly and duopoly states, $j \in \{m, d\}$

Revenue flow for Firm $i$ at state $j$

$$\pi_{ij}^i = D_{ij}\theta_t$$

where $D_{ij}$ is a constant multiplier.

The underlying stochastic aggregate economic factor $\theta_t$ is governed by the Geometric Brownian process

$$d\theta_t = \mu \theta_t \, dt + \sigma \theta_t \, dZ_t$$

$\mu$ is the drift rate, which is taken to be less than the riskless interest rate $r$. 
Asymmetry in sunk costs

Let $K_i$ be the sunk cost of Firm $i$, and define the revenue adjusted cost $\tilde{K}_{ij}$ by

$$\tilde{K}_{ij} = K_i/D_{ij}.$$ 

Negative externalities: $0 < D_{id} < D_{im}, \ i = 1, 2$, representing lower revenue flow when operated under the duopoly state.

- Negative externalities may induce keen competition between the two firms.
- The two firms may be worse off by rushing to pre-empt.

Positive externalities: $0 < D_{im} < D_{id}, \ i = 1, 2$. 

Typical of dynamic games, the problem is solved backward

- Solve for the *follower value function* and the corresponding investment threshold. Let $\theta_{i_f}^*$ be the threshold for optimal entry as a follower for Firm $i$ and $t_{i_f}^*$ denote the optimal time of investment (stopping time).

Follower’s value function of Firm $i = F_i(\theta)$

$$
= \begin{cases} 
\max_{t_{i_f}^*} E_t \left[ \int_{t_{i_f}^*}^{\infty} e^{-r(u-t)} D_{id} \theta_u du - e^{-r(t_{i_f}^*-t)} K_i \right], & \theta < \theta_{i_f}^* \\
E_t \left[ \int_t^{\infty} e^{-r(u-t)} D_{id} \theta_u du - K_i \right], & \theta \geq \theta_{i_f}^* 
\end{cases}
$$

$$
= \begin{cases} 
\frac{K_i}{\beta - 1} \left( \frac{\theta}{\theta_{i_f}^*} \right)^\beta, & \theta < \theta_{i_f}^* \\
\frac{D_{id}}{r-\mu} \theta - K_i, & \theta \geq \theta_{i_f}^*
\end{cases}
$$

where

$$
\theta_{i_f}^* = \frac{\beta}{\beta - 1} (r-\mu) \tilde{K}_{id}, \quad \beta = \frac{1}{2} \left[ 1 - \frac{2\mu}{\sigma^2} + \sqrt{\left(1 - \frac{2\mu}{\sigma^2}\right)^2 + \frac{8r}{\sigma^2}} \right], \quad \beta > 1.
$$
Pre-emptive leader value function

Leader value function at the moment $t_{ip}^*$ at which Firm $i$ takes the pre-emptive action

$$L_{i(p)}(\theta) = E_{t_{ip}^*} \left[ \int_{t_{ip}^*}^{t_{i'}^f} e^{-r(u-t_{ip}^*)} D_{im}\theta_u \, du + \int_{t_{i'}^f}^{\infty} e^{-r(u-t_{i'}^f)} D_{id}\theta_u \, du - K_i \right]$$

$$= \begin{cases} 
\frac{D_{im} - D_{id}}{r-\mu} \theta_{i'}^* \left( \frac{\theta}{\theta_{i'}^*} \right)^\beta + \frac{D_{im}}{r-\mu} \theta - K_i, & \theta < \theta_{i'}^* \\
\frac{D_{id}}{r-\mu} - K_i, & \theta \geq \theta_{i'}^*. 
\end{cases}$$

When $\theta < \theta_{i'}^*$, the first term $-\frac{D_{im} - D_{id}}{r-\mu} \theta_{i'}^* \left( \frac{\theta}{\theta_{i'}^*} \right)^\beta$ represents the "negative option value" associated with the drop in revenue flow from $D_{im}\theta_t$ to $D_{id}\theta_t$ (under negative externalities assumption) upon the entry of the follower.
Dominant leader value function

Suppose Firm $i$ is the dominant leader, it can wait until $\theta_t$ reaches $\theta_{i\ell}^*$ for optimal entry. When $\theta > \theta_{i\ell}^*$, the leader value function becomes $L_{i}^{(p)}(\theta)$. The dominant leader value function is

$$L_{i}^{(d)}(\theta) = \frac{K_i}{\beta - 1} \left( \frac{\theta}{\theta_{i\ell}^*} \right)^\beta - \frac{D_{im} - D_{id}}{r - \mu} \theta_{i'f}^* \left( \frac{\theta}{\theta_{i'f}^*} \right)^\beta, \quad \theta < \theta_{i\ell}^*,$$

where

$$\theta_{i\ell}^* = \frac{\beta}{\beta - 1} (r - \mu) \tilde{K}_{im}.$$

Note that

$$\theta_{i\ell}^* < \theta_{i'f}^* \iff \tilde{K}_{im} < \tilde{K}_{i'd}.$$


**Pre-emptive thresholds**

Firm $i$ would choose to pre-empt when the pre-emptive leader value function $L_i^{(p)}(\theta)$ first surpasses the follower value function $F_i(\theta)$.

\[
\text{pre-emptive threshold } = \theta_{ip}^* = \inf\{\theta : L_i^{(p)}(\theta) > F_i(\theta)\}.
\]

Under negative externalities, we have

\[
\theta_{ip}^* < \theta_{il}^* < \theta_{if}^*.
\]

Measure of pre-emptive incentive of Firm $i$:

\[
\phi_i(\theta) = \frac{L_i^{(p)}(\theta) - F_i(\theta)}{K_i}, \quad i = 1, 2.
\]

Since pre-emption must occur before either firm’s follower entry threshold is reached, we are only interested in $\phi_i(\theta)$ over the interval $[0, \min(\theta_{if}^*, \theta_{if}'^*)]$. Define

\[
\theta_{ip}^* = \inf\{\theta \in [0, \min(\theta_{if}^*, \theta_{if}'^*)] : \phi_i(\theta) > 0\}.
\]
Strategic equilibriums

1. Under positive externalities, both firms do not have the incentive to pre-emptive so that $\theta_{ip}^*$ and $\theta_{i'p}^*$ do not exist.

2. Under negative externalities, it may occur that
   a. only one of the pre-emptive thresholds exists;
   b. both pre-emptive thresholds exist and one is strictly smaller than the other;
   c. the two pre-emptive thresholds are equal.
Positive externalities

Assume $\theta_{i_f}^* \leq \theta_{i_l}^*$ (without loss of generality), and observe that

$$\theta_{i_f}^* < \theta_{i_l}^* \quad \text{and} \quad \theta_{i_f}^* < \theta_{i_l}^*,$$

there are 3 possible orderings:

(i) $\theta_{i_f}^* \leq \theta_{i_l}^* < \theta_{i_l}^* \leq \theta_{i_l}^*$

(ii) $\theta_{i_f}^* \leq \theta_{i_l}^* \leq \theta_{i_l}^* \leq \theta_{i_l}^*$

(iii) $\theta_{i_f}^* < \theta_{i_l}^* < \theta_{i_f}^* \leq \theta_{i_l}^*$

- **Simultaneous equilibrium** occurs when $\theta_{i_l}^* \geq \theta_{i_f}^*$ since Firm $i$ would wait until $\theta_{i_f}^*$ to invest.

- **Sequential equilibrium** occurs when $\theta_{i_l}^* \leq \theta_{i_f}^*$ since Firm $i$ would enter as leader at $\theta = \theta_{i_l}^*$. 
Negative externalities

(i) $\theta_{ip}^* < \theta_{il}^* < \theta_{if}^*$; (ii) $\bar{R}_i = \frac{\theta_{if}^*}{\theta_{il}^*} = \frac{\tilde{K}_{id}}{\tilde{K}_{im}} > 1$.

$\bar{R}_i$ measures Firm $i$’s first mover incentive, where lower $\bar{R}_i$ means higher incentive.

The extent that Firm $i$’ influences the strategic entry decision of Firm $i$ would depend on the two parameters $R_{if}$ and $R_{il}$.

(i) follower threshold ratio $R_{if} = \frac{\theta_{if}^*}{\theta_{i'f}^*} = \frac{\tilde{K}_{id}}{\tilde{K}_{i'd}}$; lower $R_{if}$, higher comparative advantage as follower for Firm $i$;

(ii) leader threshold ratio $R_{il} = \frac{\theta_{il}^*}{\theta_{i'l}^*} = \frac{\tilde{K}_{im}}{\tilde{K}_{i'm}}$; lower $R_{il}$, higher comparative advantage as leader for Firm $i$. 
Proposition

Define \( q(x) = \left( \frac{1}{\beta} - 1 \right)^{1/\beta - 1} \). Under negative externalities, the pre-emptive threshold \( \theta_{ip}^* \) of Firm \( i \) exists if and only if \( 0 < R_{if} < q(\overline{R}_i) \), where \( q(\overline{R}_i) > 1 \).

Corollaries

1. The pre-emptive thresholds of both firms exist if and only if

\[
\frac{1}{q(\overline{R}_{i'})} < R_{if} < q(\overline{R}_i),
\]

where \( \overline{R}_{i'} = \theta_{i'f}^*/\theta_{i'l}^* \). When both \( \theta_{ip}^* \) and \( \theta_{i'p}^* \) exist, the two regions in the \( \overline{R}_{if} - \overline{R}_{il} \) plane

\[
\{(\overline{R}_{if}, \overline{R}_{il}) : \theta_{ip}^* > \theta_{ip}^*\} \quad \text{and} \quad \{(\overline{R}_{if}, \overline{R}_{il}) : \theta_{ip}^* < \theta_{i'p}^*\}
\]

are separated by the curve:

\[
\{(\overline{R}_{if}, \overline{R}_{il}) : \theta_{ip}^* = \theta_{i'p}^*\}.
\]
Behaviors of $\phi_i(\theta)$ within the domain $[0, \min(\theta_{ip}^*, \theta_{i'f}^*))$ under varying values of $R_{if}$. When $0 < R_{if} < q(\overline{R}_i)$, $\phi_i(\theta)$ has at least one root so that the pre-emptive threshold $\theta_{ip}^*$ exists.
2. When $R_{if} \not\in \left(\frac{1}{q(R_{i'})}, q(R_i)\right)$, only one pre-emptive threshold exists. Specifically, we have (i) for $0 < R_{if} \leq \frac{1}{q(R_{i'})}$, $\theta^*_i$ exists but not $\theta^*_{i'p}$; and (ii) for $R_{if} \geq q(R_i)$, $\theta^*_{i'p}$ exists but not $\theta^*_i$.

3. In the $R_{if}$-$R_{il}$ plane, the region where both pre-emptive thresholds exist is bounded by the following inequalities:

$$R_{if} < q(R_i), R_{il} > \frac{R_{if}}{R_i} \quad \text{and} \quad R_{il} > \frac{R_{if}}{R_i} q^{-1} \left(\frac{1}{R_{if}}\right).$$
Characterization of the relative magnitude of the pre-emptive thresholds, $\theta^*_i$ and $\theta'^*_i$, in the $R_{if}-R_{i\ell}$ plane. For a fixed value of the parameter $\overline{R}_i$, the region where both pre-emptive thresholds exist is bounded by $R_{if} < q(\overline{R}_i), R_{i\ell} > q^{-1}\left(\frac{1}{R_{if}}\right) R_{if}/\overline{R}_i$ and $R_{i\ell} > R_{if}/\overline{R}_i$. Under negative externalities, the feasible region is given by $R_{i\ell} > R_{if}/\overline{R}_i$. 
Strategies under negative externalities

1. **Sequential equilibrium**

Firm $i$ dominates, that is, $\theta^*_i p$ exists but not $\theta^*_{i' p}$.

- Firm $i'$ would never take pre-emptive action.
- Since $\theta^*_{i l} < \theta^*_{i' f}$, hence Firm $i$ chooses to invest at $\theta = \theta^*_{i l}$ as optimal leader entry.
- Firm $i'$ invests optimally at $\theta = \theta^*_{i' f}$.

2. **Keen competition**

The pre-emptive thresholds $\theta^*_{i p}$ and $\theta^*_{i' p}$ both exist.

Assume $\theta^*_{i p} < \theta^*_{i' p}$ (without loss of generality)

Firm $i$ does not have to take pre-emptive action to invest at $\theta^*_{i p}$. 
(i) When $\theta_{i'l} \leq \theta_{i'p}^*$

[which is equivalent to $\beta R_i^\beta (R_{i'l} - 1) \leq \beta (R_i R_{i'l} - R_{if}) + R_{if}^\beta$]

Firm $i$ will invest at $\theta_{i'l}^*$ as optimal leader entry since the potential pre-emptive action of Firm $i'$ has no impact.

(ii) When $\theta_{i'l} > \theta_{i'p}^*$

Firm $i$ enters at $\theta_{i'p}^*$ as pre-emptive entry. When $\theta_t$ lies between $\theta_{ip}$ and $\theta_{i'p}^*$, Firm $i$ will choose to delay entry till as close to $\theta_{i'p}^*$ as possible.

3. Simultaneous pre-emptive entry

Under the special case where $\theta_{ip}^* = \theta_{i'p}^*$, both firm will choose simultaneous pre-emptive entry at $\theta_{ip}^*$. 
Influence of market uncertainty as measured by volatility of $\theta_t$

- An increase in volatility may hasten or delay pre-emptive entry.

With increasing volatility $\sigma$, $\theta^*_ip$ increases while $\theta^*_ip$ decreases. The parameter values used in the calculations: $R_{if} = \frac{69}{100}$, $R_{il} = \frac{57}{50}$, $R_i = \frac{69}{57}$ (corresponding to $\theta^*_i\ell = 50$, $\theta^*_i\ell = 57$, $\theta^*_i = 69$, $\theta^*_i\ell f = 100$), $r = 4\%$ and $\mu = 1.6\%$.
Relative magnitude of the pre-emptive thresholds

Recall

\[ \phi_i(\theta) = \frac{L_i^p(\theta) - F_i(\theta)}{K_i} = -a_i \theta^\beta + b_i \theta - 1, \quad \theta < \min(\theta_{i_f}^*, \theta_{i'_f}^*), \ i = 1, 2, \]

let

\[ q_i = \frac{1}{\beta - 1} \left[ \beta \left( \frac{1}{\theta_{i_f}^*} - \frac{1}{\theta_{i_f}^*} \right) \left( \frac{1}{\theta_{i'_f}^*} \right)^{\beta-1} + \left( \frac{1}{\theta_{i_f}^*} \right)^{\beta} \right] \]

\[ b_i = \frac{\beta}{\beta - 1} \frac{1}{\theta_{i_f}^*}, \]

define

\[ \theta_{i_P}^* = \inf \{ \theta \in [0, \min(\theta_{i_f}^*, \theta_{i'_f}^*)] : \phi_i(\theta) > 0 \} . \]
Proposition

Negative externalities and existence of both pre-emptive thresholds \( \theta^*_i \) and \( \theta^*_{i'p} \) are assumed. Under the scenario of \( R_{if} < 1 \) and \( R_{i\ell} > 1 \), the pre-emptive thresholds observe

1. \( \phi_i(\hat{\theta}) > 0 \implies \theta^*_i > \theta^*_{i'p} \)

2. \( \phi_i(\hat{\theta}) = 0 \)
   
   (a) \( \frac{d\phi_i}{d\theta}(\hat{\theta}) > 0 \implies \theta^*_i = \theta^*_{i'p} \)
   
   (b) \( \frac{d\phi_i}{d\theta}(\hat{\theta}) < 0 \implies \theta^*_i > \theta^*_{i'p} \)

3. \( \phi_i(\hat{\theta}) < 0 \)
   
   (a) \( \frac{d\phi_i}{d\theta}(\hat{\theta}) > 0 \implies \theta^*_i < \theta^*_{i'p} \)
   
   (b) \( \frac{d\phi_i}{d\theta}(\hat{\theta}) < 0 \implies \theta^*_i > \theta^*_{i'p} \).
New observations/results

• Simultaneous entry is triggered by the coincidence of the two pre-emptive thresholds, $\theta_{ip}^* = \theta_{i'p}^*$, which represents a wider set of scenarios than the simple case of identical pre-emptive threshold in symmetric firms.

• Cost asymmetry alone always yields leadership by the low-cost firm.

• Under negative externalities and existence of both $\theta_{ip}^*$ and $\theta_{i'p}^*$, simple algebraic inequalities are established to determine whether (i) $\theta_{ip}^* < \theta_{i'p}^*$, (ii) $\theta_{i\ell}^* \leq \theta_{i'p}^*$. 
The assumption of “asymmetry on cost only” would limit their consideration of strategic equilibriums to $R_{i\ell} < 1$ and $R_{if} < 1$.

Earlier papers postulate that simultaneous entry occurs when $K \in (1, K^{**})$, where $K$ is the ratio of the high cost to the low cost and $K^{**} = \max \left\{ \frac{1}{q(R_i)}, 1 \right\}$.

- Under negative externalities, we always have $q(R_i) > 1$.

- This leads to $K^{**} = 1$ so the low-cost firm always leads. Also, there will be no simultaneous entry.