Advanced Numerical Methods
Solution to Homework Two

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1. We consider stock prices of $0, $4, $8, $12, $16, $20, $24, $28, $32, $36 and $40. With \( r = 0.10, \Delta t = 0.0833, \Delta S = 4, \sigma = 0.30, X = 21, T = 0.3333 \), we obtain the table of option values shown below. The American put option price is $1.56.

<table>
<thead>
<tr>
<th>Stock Price ($)</th>
<th>Time To Maturity (Months)</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>0.00 0.00 0.00 0.00 0.00</td>
</tr>
<tr>
<td>36</td>
<td>0.00 0.00 0.00 0.00 0.00</td>
</tr>
<tr>
<td>32</td>
<td>0.01 0.00 0.00 0.00 0.00</td>
</tr>
<tr>
<td>28</td>
<td>0.07 0.04 0.02 0.00 0.00</td>
</tr>
<tr>
<td>24</td>
<td>0.38 0.30 0.21 0.11 0.00</td>
</tr>
<tr>
<td>20</td>
<td>1.56 1.44 1.31 1.17 1.00</td>
</tr>
<tr>
<td>16</td>
<td>5.00 5.00 5.00 5.00 5.00</td>
</tr>
<tr>
<td>12</td>
<td>9.00 9.00 9.00 9.00 9.00</td>
</tr>
<tr>
<td>8</td>
<td>13.00 13.00 13.00 13.00 13.00</td>
</tr>
<tr>
<td>4</td>
<td>17.00 17.00 17.00 17.00 17.00</td>
</tr>
<tr>
<td>0</td>
<td>21.00 21.00 21.00 21.00 21.00</td>
</tr>
</tbody>
</table>

The explicit scheme takes the form

\[
(P_{j+1}^{n+1})_{cont} = P_j^n + \Delta t \left[ \frac{\sigma^2 S_j^2 P_{j+1}^n - 2P_j^n + P_{j-1}^n}{\Delta S^2} + r S_j \frac{P_{j+1}^n - P_{j-1}^n}{2\Delta S} - rP_j^n \right], \quad S_j = j\Delta S,
\]

coupled with the application of the dynamic programming procedure at each node

\[
P_{j+1}^{n+1} = \max \left( (P_{j+1}^{n+1})_{cont}, X - S_j \right).
\]

The bold figures in the Table represent the exercise payoff values. For a given time to expiry, provided that the stock price level is sufficiently low, the American put option should be optimally exercised. When \( S \) is sufficiently large, the American put has negligible value.

2. Expand \( f(x_0 - 2\Delta x) \) and \( f(x_0 - \Delta x) \) at \( x_0 \) into Taylor series, where

\[
\begin{align*}
f(x_0 - 2\Delta x) &= f(x_0) - 2\Delta xf'(x_0) + 4\Delta x^2 f''(x_0) + O(\Delta x^3) \\
f(x_0 - \Delta x) &= f(x_0) - \Delta xf'(x_0) + \Delta x^2 f''(x_0) + O(\Delta x^3).
\end{align*}
\]

We determine \( \alpha_{-2}, \alpha_{-1} \) and \( \alpha_0 \) such that

\[
\alpha_{-2} f(x_0 - 2\Delta x) + \alpha_{-1} f(x_0 - \Delta x) + \alpha_0 f(x_0) = f'(x_0) + O(\Delta x^2).
\]

Collecting like terms, we obtain

\[
(\alpha_{-2} + \alpha_{-1} + \alpha_0) f(x_0) + (-2\alpha_{-2} - \alpha_{-1}) \Delta x f'(x_0) + (4\alpha_{-2} + \alpha_{-1}) \Delta x^2 f''(x_0) = f'(x_0) + O(\Delta x^2).
\]

\[
(\alpha_{-2} + \alpha_{-1} + \alpha_0) = 1, \\
(-2\alpha_{-2} - \alpha_{-1}) = 0, \\
(4\alpha_{-2} + \alpha_{-1}) = 0.
\]
The corresponding linear system of equations for \( \alpha_{-2}, \alpha_{-1} \) and \( \alpha_0 \) is
\[
\begin{pmatrix}
1 & 1 & 1 \\
-2 & -1 & 0 \\
4 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
\alpha_{-2} \\
\alpha_{-1} \\
\alpha_0
\end{pmatrix}
= \begin{pmatrix}
0 \\
1/\Delta x \\
0
\end{pmatrix}.
\]

The solution of the system gives \( \alpha_{-2} = \frac{1}{2\Delta x}, \alpha_{-1} = -\frac{2}{\Delta x} \) and \( \alpha_0 = \frac{3}{2\Delta x} \). The corresponding finite difference formula is the one-sided backward difference formula for the first order derivative, namely
\[
f'(x_0) \approx f(x_0 - 2\Delta x) - 4f(x_0 - \Delta x) + 3f(x_0).
\]

By changing \( \Delta x \) to \(-\Delta x\), we deduce that the one-sided forward difference formula for the first order derivative is given by
\[
f'(x_0) \approx -f(x_0 + 2\Delta x) + 4f(x_0 + \Delta x) - 3f(x_0).
\]

3. Truncation error of the Crank-Nicolson scheme

\[
\begin{align*}
&= \frac{V(j \Delta x, (n + 1)\tau) - V(j \Delta x, n\tau)}{\Delta \tau} \\
&+ \frac{\sigma^2}{4} \left[ \frac{V((j + 1) \Delta x, (n + 1)\Delta \tau) - 2V(j \Delta x, (n + 1)\Delta \tau) + V((j - 1) \Delta x, (n + 1)\Delta \tau)}{\Delta x^2} \\
&+ \frac{V((j + 1) \Delta x, n\Delta \tau) - 2V(j \Delta x, n\Delta \tau) + V((j - 1) \Delta x, n\Delta \tau))}{\Delta x^2} \right] \\
&- \frac{1}{2} \left( r - \frac{\sigma^2}{2} \right) \left[ \frac{V((j + 1) \Delta x, (n + 1)\Delta \tau) - V((j - 1) \Delta x, (n + 1)\Delta \tau)}{2\Delta x} \\
&+ \frac{V((j + 1) \Delta x, n\Delta \tau) - V((j - 1) \Delta x, n\Delta \tau))}{2\Delta x} \right] \\
&- \frac{r}{2} [V(j \Delta x, (n + 1)\Delta \tau) + V(j \Delta x, n\Delta \tau)].
\end{align*}
\]

Expanding each term using the Taylor series expansion at the intermediate time level \( (j \Delta x, (n + \frac{1}{2}) \Delta \tau) \), we obtain

\[
\begin{align*}
&= \frac{V(j \Delta x, (n + 1)\tau) - V(j \Delta x, n\tau)}{\Delta \tau} \\
&\quad \left\{ \left[ V + \frac{\partial V}{\partial \tau} \frac{\Delta \tau}{2} + \frac{1}{2} \frac{\partial^2 V}{\partial \tau^2} \left( \frac{\Delta \tau}{2} \right)^2 + \frac{1}{6} \frac{\partial^3 V}{\partial \tau^3} \left( \frac{\Delta \tau}{2} \right)^3 + \cdots \right] \\
&\quad - \left[ V - \frac{\partial V}{\partial \tau} \frac{\Delta \tau}{2} + \frac{1}{2} \frac{\partial^2 V}{\partial \tau^2} \left( \frac{\Delta \tau}{2} \right)^2 - \frac{1}{6} \frac{\partial^3 V}{\partial \tau^3} \left( \frac{\Delta \tau}{2} \right)^3 + \cdots \right] \right\} / \Delta \tau \\
&= \frac{\partial V}{\partial \tau} + \frac{1}{24} \frac{\partial^3 V}{\partial \tau^3} (\Delta \tau)^2 + O(\Delta \tau^4).
\end{align*}
\]

Here, we adopt the convention that any derivative with no argument specified would
implicitly implies that the derivative is evaluated at \((j\Delta x, \left( n + \frac{1}{2} \right) \Delta \tau)\). Note that

\[
[V((j + 1)\Delta x, n\Delta \tau) - 2V(j\Delta x, n\Delta \tau) + V((j - 1)\Delta x, n\Delta \tau)]/(\Delta x)^2
\]

\[
= \frac{\partial^2 V}{\partial x^2}(j\Delta x, n\Delta \tau) + \frac{\Delta x^2 \partial^4 V}{12 \partial x^4}(j\Delta x, n\Delta \tau) + O(\Delta x^4)
\]

\[
= \frac{\partial^2 V}{\partial x^2} - \frac{\Delta \tau}{2} \frac{\partial^3 V}{\partial x^2 \partial \tau} + \frac{1}{2} \left( \frac{\Delta \tau}{2} \right)^2 \frac{\partial^4 V}{\partial x^2 \partial \tau^2} + \cdots
\]

\[
+ \frac{\Delta x^2}{12} \left[ \frac{\partial^4 V}{\partial x^4} - \frac{\Delta \tau}{2} \frac{\partial^5 V}{\partial x^4 \partial \tau} + \frac{1}{2} \left( \frac{\Delta \tau}{2} \right)^2 \frac{\partial^6 V}{\partial x^4 \partial \tau^2} + \cdots \right] + O(\Delta x^4),
\]

and

\[
[V((j + 1)\Delta x, (n + 1)\Delta \tau) - 2V(j\Delta x, (n + 1)\Delta \tau) + V((j - 1)\Delta x, (n + 1)\Delta \tau)]/(\Delta x)^2
\]

\[
= \frac{\partial^2 V}{\partial x^2} + \frac{\Delta \tau}{2} \frac{\partial^3 V}{\partial x^2 \partial \tau} + \frac{1}{2} \left( \frac{\Delta \tau}{2} \right)^2 \frac{\partial^4 V}{\partial x^2 \partial \tau^2} + \cdots
\]

\[
+ \frac{\Delta x^2}{12} \left[ \frac{\partial^2 V}{\partial x^4} + \frac{\Delta \tau}{2} \frac{\partial^3 V}{\partial x^2 \partial \tau} + \frac{1}{2} \left( \frac{\Delta \tau}{2} \right)^2 \frac{\partial^4 V}{\partial x^2 \partial \tau^2} + \cdots \right] + O(\Delta x^4).
\]

Combining the results, we have

\[
\frac{\sigma^2}{4} \left[ \frac{V((j + 1)\Delta x, (n + 1)\Delta \tau) - 2V(j\Delta x, (n + 1)\Delta \tau) + V((j - 1)\Delta x, (n + 1)\Delta \tau)}{\Delta x^2} \right.
\]

\[
\left. + \frac{V((j + 1)\Delta x, n\Delta \tau) - 2V(j\Delta x, n\Delta \tau) + V((j - 1)\Delta x, n\Delta \tau)}{\Delta x^2} \right]
\]

\[
= \frac{\sigma^2}{2} \left[ \frac{\partial^2 V}{\partial x^2} + O(\Delta \tau^2) + O(\Delta x^2) \right].
\]

Similarly,

\[
\frac{1}{2} \left( r - \frac{\sigma^2}{2} \right) \left[ \frac{V((j + 1)\Delta x, (n + 1)\Delta \tau) - V((j - 1)\Delta x, (n + 1)\Delta \tau)}{2\Delta x} \right.
\]

\[
\left. + \frac{V((j + 1)\Delta x, n\Delta \tau) - V((j - 1)\Delta x, n\Delta \tau)}{2\Delta x} \right]
\]

\[
= \frac{1}{2} \left( r - \frac{\sigma^2}{2} \right) \left[ \frac{\partial V}{\partial x} + O(\Delta \tau^2) + O(\Delta x^2) \right],
\]

and

\[
r \left[ V(j\Delta x, (n + 1)\Delta \tau) + V(j\Delta x, n\Delta \tau) \right]
\]

\[
= rV + O(\Delta \tau^2).
\]

By noting that \( V\left( j\Delta x, \left( n + \frac{1}{2} \right) \Delta \tau \right) \) satisfies the Black-Scholes equation, we obtain truncation error of the Crank-Nicholson scheme = \( O(\Delta \tau^2) + O(\Delta x^2) \).

4. For the given pricing formulation of the floating strike lookback put, the binomial parameters are determined by equating the mean and variance:

\[
(1 - \alpha)\Delta x - \alpha \Delta x = \left( q - r - \frac{\sigma^2}{2} \right) \Delta t
\]

\[
(1 - \alpha)\Delta x^2 + \alpha \Delta x^2 = \sigma^2 \Delta t
\]
so that $\Delta x = \sigma \sqrt{\Delta t}$ and $(1 - 2\alpha)\Delta x = \left( q - r - \frac{\sigma^2}{2} \right) \Delta t = \left( q - r - \frac{\sigma^2}{2} \right) \frac{\Delta x^2}{\sigma^2}$ giving

$$\alpha = \frac{1}{2} + \frac{\Delta x}{2} \left( \frac{r - q}{\sigma^2} + \frac{1}{2} \right).$$

Here, $\alpha$ denotes the probability of down move in the binomial tree.

The term $-qV$ is similar to the discount term $-rV$ in the usual Black-Scholes equation. This gives rise to the discount factor $\frac{1}{1 + q\Delta t}$ in the binomial pricing formula. In summary, the binomial scheme takes the form

$$V^n_j = \frac{1}{1 + q\Delta t} \left[ \alpha V^{n+1}_{j-1} + (1 - \alpha)V^{n+1}_j \right], \quad j \geq 0.$$  

When $j = 0$, the binomial formula involves the grid point at $j = -1$, which is outside the computational domain. Suppose we approximate the Neumann boundary condition: $\frac{\partial V}{\partial x}(0, t) = 0$ using the one-sided finite difference formula, that is, $V^{-1}_{n+1} = V^0_{n+1}$. The numerical boundary value is given by

$$V^0_n = \frac{1}{1 + q\Delta t} \left[ \alpha V^{n+1}_0 + (1 - \alpha)V^{n+1}_1 \right].$$

The truncation error is defined to be

$$\left[ \alpha V^{n+1}_0 + (1 - \alpha)V^{n+1}_1 - (1 + q\Delta t)V^n_0 \right]/\Delta t.$$  

We expand each term by performing the Taylor expansion at $(0, (n + 1)\Delta t)$ [adopting the notation that the price function $V$ and its derivatives are evaluated at the grid $(0, (n + 1)\Delta t)$].

$$V^{n+1}_1 = V + \frac{\partial V}{\partial x} \Delta x + \frac{\partial^2 V}{\partial x^2} \frac{\Delta x^2}{2} + O(\Delta x^3)$$

$$V^n_0 = V - \frac{\partial V}{\partial t} \Delta t + \frac{\partial^2 V}{\partial t^2} \frac{\Delta t^2}{2} + O(\Delta t^3).$$

Now, we consider

$$\alpha V^{n+1}_0 + (1 - \alpha)V^{n+1}_1 - (1 + q\Delta t)V^n_0$$

$$= \alpha V + (1 - \alpha) \left( V + \frac{\partial V}{\partial x} \Delta x + \frac{\partial^2 V}{\partial x^2} \frac{\Delta x^2}{2} + \cdots \right)$$

$$- (1 + q\Delta t) \left( V - \frac{\partial V}{\partial t} \Delta t + \frac{\partial^2 V}{\partial t^2} \frac{\Delta t^2}{2} + \cdots \right)$$

$$= -qV\Delta t + \frac{\partial V}{\partial t} \Delta t + (1 - \alpha) \frac{\partial V}{\partial x} \Delta x + (1 - \alpha) \frac{\partial^2 V}{\partial x^2} \frac{\Delta x^2}{2} + O(\Delta t^2).$$
Recall that

\[(1 - \alpha)\Delta x = \alpha\Delta x + \left(q - r - \frac{\sigma^2}{2}\right)\Delta t\]

\[(1 - \alpha)\Delta x^2 = \sigma^2\Delta t - \alpha\Delta x^2,\]

the last expression can be expressed as

\[-qV\Delta t + \frac{\partial V}{\partial t}\Delta t + \left(q - r - \frac{\sigma^2}{2}\right)\frac{\partial V}{\partial x}\Delta t + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial x^2}\Delta t\]

\[+ \alpha \frac{\partial V}{\partial x}\Delta x - \frac{\alpha}{2} \frac{\partial^2 V}{\partial x^2}\Delta x^2 + O(\Delta t^2).\]

The sum of the first four term is zero since \(V\) satisfies the governing equation. Now, the truncation error

\[= \left[\alpha \frac{\partial V}{\partial x}\sigma \sqrt{\Delta t} - \alpha \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial x^2}\Delta t\right] / \Delta t + O(\Delta t)\]

which is seen to be inconsistent.

5. We write the FTCS scheme in the form of the standard 2-level-4-point scheme:

\[V_j^{n+1} = \left(\frac{\sigma^2}{2} S_j^2 \frac{\Delta \tau}{\Delta S^2} + r S_j \frac{\Delta \tau}{2 \Delta S}\right) V_{j+1}^n + \left(1 - \sigma^2 S_j^2 \frac{\Delta \tau}{\Delta S^2} - r \Delta \tau\right) V_j^n + \left(\frac{\sigma^2}{2} S_j^2 \frac{\Delta \tau}{\Delta S^2} - r S_j \frac{\Delta \tau}{2 \Delta S}\right) V_j^{n-1}.\]

In order to avoid spurious oscillations, it suffices to have all the coefficients to be positive. That is,

\[\frac{\sigma^2}{2} S_j^2 \frac{\Delta \tau}{\Delta S^2} + r S_j \frac{\Delta \tau}{2 \Delta S} > 0\]

\[1 - \sigma^2 S_j^2 \frac{\Delta \tau}{\Delta S^2} - r \Delta \tau > 0\]

\[\frac{\sigma^2}{2} S_j^2 \frac{\Delta \tau}{\Delta S^2} - r S_j \frac{\Delta \tau}{2 \Delta S} > 0.\]

The first inequality is always satified. The last two inequalities are satisfied provided that

\[\Delta S < \frac{\sigma^2 S_j}{r} \quad \text{and} \quad \frac{1}{\Delta \tau} > \frac{\sigma^2 S_j^2}{\Delta S^2} + r.\]

6. Let the first barrier be an up-stream barrier \(B_H\) while the second barrier be the down-stream barrier \(B_L\). The sequential barrier option is equivalent to a one-sided up-and-out barrier with a rebate paid upon knock-out. The rebate is a one-sided down-and-out barrier option. We assume a call payoff of the sequential barrier option. Let \(B_j^n\) and \(R_j^n\) denote the numerical option value of the sequential barrier option and the rebate barrier option at the \((j, n)^{th}\) node, respectively.

**Design of the computational domain**

For the sequential barrier option, we set the right boundary to coincide with the upstream barrier \(B_H\). The left boundary must lie sufficiently far to the left end.
For the down-and-out barrier option (treated as the rebate upon breaching the up-barrier in the sequential barrier option), we set the left boundary to coincide with the downstream barrier $B_L$. The right boundary must lie sufficiently far to the right end.

**Boundary conditions**

(i) At $j = 0$, the sequential barrier option is deep out-of-the-money so that the option value is close to zero. We set

$$B^n_0 = 0, \quad \text{for all } n.$$  

(ii) At $j = N$, which corresponds to the up-barrier $B_H$, the sequential barrier option is apparently “knocked” out, receiving the down-and-out barrier option as the rebate. Therefore, we have

$$B^n_N = R^n_N, \quad \text{for all } n.$$  

(iii) At $j = M$, the (rebate) down-and-out barrier option is deep in-the-money so that it is almost like a forward contract. Accordingly, the second order derivative of the price function with respect to the stock price is close to zero. That is,

$$R^n_M = \frac{5R^n_{M-1} - 4R^n_{M-2} + R^n_{M-3}}{2}, \quad \text{for all } n.$$  

(iv) At $j = L$, the rebate barrier option is knocked out at $x = B_L$ with zero value. That is

$$R^n_L = 0, \quad \text{for all } n.$$  

6
Both price functions of the sequential barrier option and the rebate barrier option satisfy the Black-Scholes equation. We start with computation of the rebate barrier option, then continue with the sequential barrier option. The explicit finite difference schemes in both option calculations take identical form:

\[
R_{j+1}^n = \left[ \frac{\mu + c}{2} R_j^n + (1 - \mu) R_{j+1}^n + \frac{\mu - c}{2} R_{j-1}^n \right] e^{-r \Delta \tau},
\]

\[j = L + 1, \ldots, M - 1, \quad n = 0, 1, 2, \ldots,
\]

\[
B_{j+1}^n = \left[ \frac{\mu + c}{2} B_j^n + (1 - \mu) B_{j+1}^n + \frac{\mu - c}{2} B_{j-1}^n \right] e^{-r \Delta \tau},
\]

\[j = 1, \ldots, N - 1, \quad n = 0, 1, 2, \ldots.
\]

7. Write

\[
\begin{pmatrix}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_3
\end{pmatrix} =
\begin{pmatrix}
\alpha_{11} & 0 & 0 \\
\alpha_{21} & \alpha_{22} & 0 \\
\alpha_{31} & \alpha_{32} & \alpha_{33}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
\]

where

\[
M = \begin{pmatrix}
\alpha_{11} & 0 & 0 \\
\alpha_{21} & \alpha_{22} & 0 \\
\alpha_{31} & \alpha_{32} & \alpha_{33}
\end{pmatrix}
\]

The entries in \( M \) are determined by the relation:

\[MM^T = \Sigma.\]

This yields

\[
\begin{pmatrix}
\alpha_{11}^2 & \alpha_{11}\alpha_{21} & \alpha_{31}\alpha_{11} \\
\alpha_{11}\alpha_{21} & \alpha_{21}^2 + \alpha_{22}^2 & \alpha_{21}\alpha_{31} + \alpha_{22}\alpha_{32} \\
\alpha_{31}\alpha_{11} & \alpha_{21}\alpha_{31} + \alpha_{22}\alpha_{32} & \alpha_{31}^2 + \alpha_{32}^2 + \alpha_{33}^2
\end{pmatrix}
= \begin{pmatrix}
1 & 0.6 & 0.5 \\
0.6 & 1 & 0.7 \\
0.5 & 0.7 & 1
\end{pmatrix},
\]

and accordingly,

\[
\begin{aligned}
\alpha_{11}^2 &= 1, & \alpha_{11}\alpha_{21} &= 0.6, & \alpha_{31}\alpha_{11} &= 0.5 \\
\alpha_{21}^2 + \alpha_{22}^2 &= 1, & \alpha_{21}\alpha_{31} + \alpha_{22}\alpha_{32} &= 0.7, & \alpha_{31}^2 + \alpha_{32}^2 + \alpha_{33}^2 &= 1.
\end{aligned}
\]

We take \( \alpha_{11} = 1 \) (we choose the positive root without loss of generality), then

\[
\alpha_{21} = 0.6, \quad \alpha_{22} = \sqrt{1 - 0.6^2} = 0.8 \quad \text{(choosing the positive root)}.
\]

Also, \( \alpha_{31} = 0.5 \), then

\[
(0.6)(0.5) + 0.8\alpha_{32} = 0.7 \quad \text{so that}
\]

\[
\alpha_{32} = \frac{0.7 - 0.3}{0.8} = 0.5.
\]

Lastly, \( \alpha_{33} = \sqrt{1 - \alpha_{31}^2 - \alpha_{32}^2} = 1/\sqrt{2} \) (choosing the positive root). Hence

\[
\begin{pmatrix}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_3
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 \\
0.6 & 0.8 & 0 \\
0.5 & 0.5 & 1/\sqrt{2}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}.
\]
8. Since \( \tau_{AV} = \frac{c + \tilde{c}}{2} \) so that
\[
\text{var}(\tau_{AV}) = \text{var} \left( \frac{c + \tilde{c}}{2} \right) = \frac{1}{4} \text{var}(c) + \frac{1}{4} \text{var}(\tilde{c}) + \frac{1}{2} \text{cov}(c, \tilde{c}).
\]

As \( \tilde{c} \) is generated using \(-\epsilon^{(i)}\) while \( c \) is generated using \( \epsilon^{(i)} \), we expect to have
\[
\text{var}(c) = \text{var}(\tilde{c})
\]
so that
\[
\text{var}(\tau_{AV}) = \frac{1}{2} \text{var}(c) + \frac{1}{2} \text{cov}(c, \tilde{c}). \tag{A}
\]

Now, we apply the following criterion of determining the trade-off between computational work units and variances: \( \sigma_1^2/\sigma_2^2 < W_2/W_1 \). Since the amount of computational work to compute \( \tau_{AV} \) is about twice that of \( c \), the control variate is preferred in terms of computational efficiency provided that
\[
\text{var}(\tau_{AV}) < \frac{\text{var}(c)}{2}. \tag{B}
\]

Based on Eq. (A), Ineq (B) is equivalent to
\[
\text{cov}(c, \tilde{c}) < 0.
\]

Since we have chosen \(-\epsilon^{(i)}\) for computing \( \tilde{c}_i \), the chances are high that \( c_i \) and \( \tilde{c}_i \) are negatively correlated. Hence, the antithetic variates method improves computational efficiency.

9. In this case, \( \Delta t = 0.5, \lambda = 0.03, \sigma = 0.25, r = 0.06 \) and \( q = 0 \) so that \( u = 1.1360, d = 0.8803, R = 1.0305, p_u = 0.6386, p_d = 0.3465, \) and the probability on default branches is \( e^{\lambda \Delta t} = e^{0.015} = 0.0149 \). Note that \( p_u + p_d \neq 1 \) (due to the possibility of default). Rather, we have \( p_u + p_d = 1 - 0.0149 = 0.9851 \). The ratio of up move probability and down move probability remain the same as in the non-defaultable case, that is,
\[
\frac{p_u}{p_d} = \frac{(R - d)/(u - d)}{(u - R)/(u - d)} = \frac{R - d}{u - R} = \frac{1.0305 - 0.8803}{1.1360 - 1.0305}.
\]

This leads to the tree shown in the Figure. The bond is called at nodes \( B \) and \( D \) and this forces exercise. Without the call, the value at node \( D \) would be 129.55, the value at node \( B \) would be 115.94, and the value at node \( A \) would be 105.18. The value of the call option to the bond issuer is therefore 105.18 – 103.72 = 1.46.
10. In this case, $\Delta t = 1, \lambda = 0.02/0.7 = 0.2857, \sigma = 0.25, r = 0.05, q = 0, u = 1.2023, d = 0.8318, a = 1.0513, p_u = 0.6557, p_d = 0.3161$, and the probability of a default is 0.0282. The calculations are shown in the Figure. The values at the nodes include the value of the coupon paid just before the node is reached. The value of the convertible is 105.21. The value if there is no conversion is calculated by working out the present value of the coupons and principal at 7%. It is 94.12. The value of the conversion option is therefore 11.09. Calling at node $D$ makes no difference because the bond will be converted at that node anyway. Calling at node $B$ (just before the coupon payment) does make a difference. It reduces the value of the convertible at node $B$ to $115. The value of the bond at node $A$ is reduced by 2.34. This is a reduction in the value of the conversion option. A dividend payment would affect the way the tree is constructed.