Solutions to Quiz (Version 1)

Problem 1. (15 points) Compute the order of the given element in a group (just write down your answer, no details needed).

(a). $9 \in \mathbb{Z}_{12}$. (b). $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i \in \mathbb{C}^*$. (c). $\begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \in GL(2, \mathbb{R})$.

Answer: (a). 4. (b). 8. (c). 4.

Problem 2. (15 points) Determine whether the given set of invertible 2×2 matrices is a subgroup of $GL(2, \mathbb{R})$ (just answer "Yes" or "No", no reasons needed)

(1). The set of 2×2 matrices with positive determinant.

(2). The set of 2×2 diagonal matrices with positive numbers on the diagonal.

(3). The set of 2×2 matrices with negative determinant.

Answer: (1). Yes. (2). Yes. (c). No.

Problem 3. (25 points) (no details needed). Let $\sigma \in S_8$ be of the form

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 7 & 5 & 1 & 8 & 4 & a & b \end{pmatrix}.$$

Suppose σ is an **odd** permutation,

(1). Find a and b. (2). Decompose σ as a product of disjoint cycles.

(3). Compute the order of σ. (4). Comupte σ^{2020}.

(5). Decompose σ^{-1} as a product of disjoint cycles.

Answer: (1). $a = 2, b = 3$. (2). $\sigma = (164)(358)(27)$. (3). 6. (4). $2020 = 6 \cdot 336 + 4$, so

$$\sigma^{2020} = \sigma^4 = (164)^4(358)^4(27)^4 = (164)(358).$$

(5). $\sigma^{-1} = (146)(385)(27)$.

Problem 4. (20 points) Let G be a group, H be the subset given by $H = \{a \in G \,|\, a^2 = e\}$.
(1). Suppose G is **abelian**, prove that H is a subgroup.
(2). Give an example of a group G such that $H = \{a \in G \,|\, a^2 = e\}$ is not a subgroup of G.

Proof of (1). If $a, b \in H$, then $a^2 = b^2 = e$. $(ab)^2 = a^2b^2 = e$, in the first $=$, we used $ab = ba$ as G is abelian. so $ab \in H$. This proves that H is closed. Since $e^2 = e$, so $e \in H$. If $a \in H$, $a^2 = e$, $(a^2)^{-1} = e^{-1}$, $(a^{-1})^2 = e$, so $a^{-1} \in H$. This proves $a \in H$ implies that $a^{-1} \in H$. So H is a subgroup of G.
(2). Our example of G must be non-abelian. We take $G = S_3$, then $H = \{e, (12), (23), (13)\}$, which is NOT a subgroup of G.

Problem 5. (15 points). Let G be a group, suppose $a \in G$ has order 100. Prove that $a^n = e$ (where n is an integer) implies that n is a multiple of 100.

Proof. By the division algorithm, $n = 100q + r$, where $0 \leq r < 100$. $e = a^n = a^{100q+r} = (a^{100})^q a^r = a^r$. Since 100 is the order of a, for any $1 \leq k < 100$, $a^k \neq e$. So $a^r = 0$ and $0 \leq r < 100$ imply that $r = 0$. This proves $n = 100q$ is a multiple of 100.

Problem 6. (10 points). Let a and b be elements of a group G. (1) Prove that for every positive integer m, $(ab)^m = a(ba)^m a^{-1}$. (2). Prove that ab and ba have the equal order.

Proof. (1). $(ab)^m = abab \cdots ab$ (m copies of ab) $= a(ba)^{m-1}b = a(ba)^{m-1}baa^{-1} = a(ba)^m a^{-1}$.
(2) $(ab)^m = e$ iff $a(ba)^m a^{-1} = e$ iff $a^{-1}a(ba)^m a^{-1}a = a^{-1}ea$ iff $(ba)^m = e$. This proves $(ab)^m = e$ iff $(ba)^m = e$. So ab and ba have the equal order.