# Solution to Homework 1 and 2

**Problem 1, Homework 1.** page 46, problems 11. 12. 13. 14. 15. 16. 17. 18. (just answer "yes" or "no", no reasons needed.)

11. Yes
12. No, because some elements has no inverse, e.g. matrices with a zero in (1,1) entry.
13. Yes 14. Yes
15. No, because some elements has no inverse, e.g. matrices with a zero in (1,1) entry.
16. Yes 17. Yes 18. Yes

**Problem 2, Homework 1.** Determine whether the given set $S$ with the given operation is a group (just answer "yes" or "no", no reasons needed.)
(1). $S$ is the set of positive real numbers, the binary operation in the multiplication.
Yes, check the three conditions.

(2). $S$ is the set of complex numbers $x + iy$ with imaginary part $y \geq 0$, the binary operation is the addition.
No. Check $i$ has no additive inverse in $S$.

(3). $S = \{even, odd\}$, the binary operation on $S$ is given as

$$even + even = even, even + odd = odd, odd + even = odd, odd + odd = even$$

Yes. Think of *even* as 0 and *odd* as 1, it is just like $\mathbb{Z}_2$. You can also check the defining conditions of a group.

(4). $\mathbb{R}$ is the set of real numbers, the binary operation is $*$

$$a * b = a + b - 2$$

Yes. Note that the identity is 2 for this operation.

(5). $S$ is the set of all positive real numbers, the binary operation $*$ is the usual division, that is, $a * b = \frac{a}{b}$.
No. $*$ is not associative. ( $a * (b * c) = \frac{ac}{b}$, while $(a * b) * c = \frac{ab}{c}$)

Problem 1, Homework 2 Compute the order of elements (just give the answers, no details needed):
(1). $-1, -i, \frac{1}{2} + \frac{\sqrt{3}}{2}i, 3$ in $G = \mathbb{C}^*$,
(2). $5, 6, 8$ in $\mathbb{Z}_{12}$.
(3). $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ in $GL(2, \mathbb{R})$.

(1). order of $-1$ is 2, order of $-i$ is 4. Notice that $\frac{1}{2} + \frac{\sqrt{3}}{2}i = e^{\frac{2\pi i}{6}}$, so its order is 6. The order of 3 is infinity.

(2). order of $5, 6, 8$ are $12, 2, 3$ respectively.

(3). The first matrix has order 4, the second has order 6. Note that the second matrix has eigenvalues $\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, which are both order 6. So after diagonlizing it you will immediately see the order of the matrix.

Problem 2, Homework 2 Page 55, problems 1,3,4,5,6,7,8,9 (just answer "yes" or "no", no reasons needed.)

1. Yes 3. Yes 4. Yes 5.Yes
6. No. the additive identity 0 is not in the set.

7. 2.($\mathbb{Q}^+$) and 6.(The set $\{\pi^n | n \in \mathbb{Z}\}$)
8. No. Product of two such matrices has determinant 4.
9. Yes

**Problem 3, Homework 2.** Let $G$ be a group, suppose that $a^2 = e$ for all $a \in G$, prove that $G$ is an abelian group.

*Proof.* Let $G$ be a group, and suppose that $a^2 = e$ for all $a \in G$.
Let $a, b \in G$. Then $a^2 = e = b^2$. By definition of inverse, $a = a^{-1}, b = b^{-1}$.
Then $aba^{-1}b^{-1} = abab = (ab)^2 = e$, by assumption.
Therefore multiplying first $b$ then $a$ to the right, $ab = ba$.
Since $a, b$ are arbitrary, $G$ is abelian.

□

**Problem 4, Homework 2.** Let $G$ be a **finite** group and $S$ be a non-empty subset of $G$. Suppose that $S$ is closed under the binary operation of $G$, prove that $S$ is a subgroup of $G$.

*Proof.* We use the property that $S$ is a subgroup of $G$ iff for all $x, y \in S$, $xy^{-1} \in S$.
Suppose that $S$ is closed under the binary operation $*$ in G.
Let $x, y \in S$. Consider the cyclic subgroup $\langle y \rangle = \{y^n | n \in \mathbb{Z}$
Since $G$ is finite, the cyclic subgroup is also finite. We split into two cases.
If $y = e$, then $y^{-1} = e$ and hence $xy^{-1} = xe = x \in S$.
If $y \neq e$, then there exist the least positive integer $m$ such that $y^m = e$, $m > 1$
By definition of inverse, $y^{-1} = y^{m-1} = y * \cdots * y \in S$, by closure of the binary operation.
Hence in this case, $xy^{-1} \in S$ also, by closure of the binary operation.
Hence $S$ is a subgroup of $G$.

□