Solution to Homework 3

Problem 1. Let $\sigma \in S_8$ be the element

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 4 & 3 & 2 & 1 & 5 & 8 & 7 \end{pmatrix}.$$

(1) Compute σ^2.

$$\sigma^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 2 & 3 & 4 & 6 & 1 & 7 & 8 \end{pmatrix}.$$

(2) Decompose σ as a product of disjoint cycles.

$$\sigma = (1\ 6\ 5)(2\ 4)(7\ 8)$$

(3)Compute the order of σ.

We see that the disjoint cycle permute each other, and (1 6 5) has order 3, while the 2-cycles has order 2. So the order of $\sigma = \text{lcm}(2, 3) = 6$.

(4). Compute σ^{-1}.

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 3 & 2 & 6 & 1 & 8 & 7 \end{pmatrix}.$$

or

$$\sigma^{-1} = (1\ 5\ 6)(2\ 4)(7\ 8)$$

Problem 2. Let $\sigma \in S_8$ be of the form

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 6 & 3 & 2 & a & b & 1 & 7 \end{pmatrix}.$$

Suppose σ is an **odd** permutation,

(1). Find a and b.

Since σ is a bijection, a and b could be 4 or 5 only. If $a = 5$ and $b = 4$, then

$\sigma = (1\ 8\ 7)(2\ 6\ 4) = (1\ 7)(1\ 8)(2\ 4)(2\ 6)$, which is an even permutation. If $a = 4$ and $b = 5$, then $\sigma = (1\ 8\ 7)(2\ 6\ 5\ 4) = (1\ 7)(1\ 8)(2\ 4)(2\ 5)(2\ 6)$, which is an odd permutation. So $a = 4$ and $b = 5$ is the correct answer.

(2). Decompose σ as a product of disjoint cycles.

$$\sigma = (1\ 8\ 7)(2\ 6\ 5\ 4)$$

(3). Compute the order of σ.
Note that $(1\ 8\ 7)$ has order 3 and $(2\ 6\ 5\ 4)$ has order 4. Hence order of $\sigma = \text{lcm}(3, 4) = 12$

(4). Decompose σ^{-1} as a product of disjoint cycles.
σ^{-1} is just taking the inverse of each cycle in the product of disjoint cycle of σ. Now $(1\ 8\ 7)^{-1} = (1\ 7\ 8)$ and $(2\ 6\ 5\ 4)^{-1} = (2\ 4\ 5\ 6)$. Hence

$$\sigma^{-1} = (1\ 7\ 8)(2\ 4\ 5\ 6)$$

(5). Comupte σ^{2019}.
We note that σ has order 12, and $2019 = 12 \times 168 + 3$. Hence

$$\sigma^{2019} = (\sigma^{12})^{168}\sigma^3 = \sigma^3 = ((1\ 8\ 7)(2\ 6\ 5\ 4))^3 = (1\ 8\ 7)^3(2\ 6\ 5\ 4)^3 = (2\ 4\ 5\ 6)$$

(Note the step $((1\ 8\ 7)(2\ 6\ 5\ 4))^3 = (1\ 8\ 7)^3(2\ 6\ 5\ 4)^3$ is possible since disjoint cycles commute each other!)

Problem 3. Give an example of a subgroup in S_4 that has order 6.
we can naturally consider S_3 as a subset of S_4, by considering an element in S_4 just permuting the $\{1, 2, 3\}$ in the set $\{1, 2, 3, 4\}$, i.e. leaving 4 fixed. Then this subset itself is a group, and has order $3! = 6$, as a subset of S_4 and compatible with the group operation(composition of maps) in S_4. Therefore this S_3-like subset is the subgroup we are searching.

(or write the subset as $\{\text{id}, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\} \subset S_4$).
(Note: Later on you will see the only order 6 groups are $\mathbb{Z}_6$-like or S_3-like. by noticing no element in S_4 has order 6 as in $\mathbb{Z}_6$, you look for S_3-like subgroups.)

Problem 4. Let σ and τ denote the transpositions (12) and (23) in S_8. Prove that $\sigma\tau\sigma = \tau\sigma\tau$.

Proof. We note that $\sigma^2 = \text{id} = \tau^2$ and $\sigma\tau = (2\ 3\ 1)$, which has order 3. Hence we have

$$\sigma\tau\sigma\tau\sigma\tau = (\sigma\tau)^3 = 1$$

Multiplying $\tau\sigma\tau$ on the right, and note that $\sigma^2 = \text{id} = \tau^2$, we get

$$\sigma\tau\sigma = \tau\sigma\tau$$

as required. □

Problem 5. Let G be an abelian group, prove that $H = \{a \in G \,|\, a^3 = e\}$ is a subgroup of G.

Proof. We use the criterion proved in tutorial. Let $x, y \in H$. By definition, $x^3 = y^3 = e$.
Note $y^3 = e$ implies $(y^{-1})^3 = e$, by multiplying y^{-1} three times from the left or right.
So $(xy^{-1})^3 = xy^{-1}xy^{-1}xy^{-1} = xxxy^{-1}y^{-1}y^{-1} = x^3(y^{-1})^3 = ee = e$
Hence $xy^{-1} \in H$, and hence H is a subgroup of G.

□