Solution for Homework No.4

Problem 1. Determine if the following maps are homomorphisms of groups (No reasons needed).

(1). $\Phi : \mathbb{R}^* \to \mathbb{R}^*, \quad \Phi(a) = 2019a$

No. $\Phi(ab) = 2019ab$ but $\Phi(a)\Phi(b) = (2019a)(2019b) = 2019^2 ab$

(2). $\Phi : \mathbb{R}^* \to \mathbb{R}^*, \quad \Phi(a) = a^{2019}$

Yes.

(3). $\Phi : \mathbb{R} \to \mathbb{R}, \quad \Phi(a) = 2019a$

Yes.

(4). $\Phi : \mathbb{R} \to \mathbb{R}, \quad \Phi(a) = a^2$

No. $\Phi(a+b) = (a+b)^2 = a^2 + 2ab + b^2$ but $\Phi(a) + \Phi(b) = a^2 + b^2$

(5). $\Phi : GL(n, \mathbb{R}) \to \mathbb{R}^*, \quad \Phi(A) = Det(A)^{10}$.

Yes. Det is multiplicative

(6). $\Phi : \mathbb{R} \to \mathbb{R}^*, \quad \Phi(a) = 10^a$.

Yes.

(7). $\Phi : \mathbb{R}^* \to \mathbb{R}, \quad \Phi(a) = 10^a$.

No. $\Phi(ab) = 10^{ab}$ but $\Phi(a) + \Phi(b) = 10^a + 10^b$

(8). $\Phi : \mathbb{R}^* \to \mathbb{R}, \quad \Phi(a) = \log_{10}(a^2)$.

Yes.

(9). $\Phi : S_3 \to S_4$,

$$\Phi(\sigma) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ i & j & k & 4 \end{pmatrix}, \quad \text{where } \sigma = \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}.$$

Yes.

Problem 2. Find a homomorphism $\Phi : \mathbb{C}^* \to \mathbb{C}^*$ such that $Ker(\Phi) = U_5$ (no reasons needed).

$Ker(\Phi) = U_5$ means that $\Phi(U_5) = 1 \in \mathbb{C}^*$. One convenient choice is $\Phi(a) = a^5$ where $a \in \mathbb{C}^*$. It is easily verified as homomorphism, and $a^5 = 1$ precisely when $a \in U_5$.

Problem 3. Let A and B be groups. Find an isomorphism $\Phi : A \times B \to B \times A$.

The required isomorphism is $\Phi((a, b)) = (b, a)$, where $a \in A$, $b \in B$. We need to check it is an isomorphism (As in tutorial, a simple operation like the inverse operation is NOT a homomorphism)

Let $(a, b), (a', b') \in A \times B$. Then $\Phi((a, b)(a', b')) = \Phi((aa', bb')) = (bb', aa') = (b, a)(b', a') = \Phi((a, b))\Phi((a', b'))$. Hence it is a homomorphism.

Let $(b, a) \in B \times A$. Then we notice that $(a, b) \in A \times B$ and $\Phi((a, b)) = (b, a)$. Hence Φ is surjective.

Finally let $(a, b), (a', b') \in A \times B$ such that $\Phi((a, b)) = \Phi((a', b'))$. Then $(b, a) = (b', a')$ and hence $b = b', a = a'$. It follows that $(a, b) = (a', b)$ and hence Φ is injective.

So Φ is an isomorphism as required.

Problem 4. Let G be a finite group, H_1 and H_2 be subgroups of G. (1). Prove that $H_1 \cap H_2$ is a subgroup of G. (2). If $|H_1|$ and $|H_2|$ are relatively prime, prove that $H_1 \cap H_2 = \{e\}$.

(1) This is a routine verification. We use the condense form of subgroup criterion as before. Trivially $H_1 \cap H_2 \subset G$. Suppose $g, h \in H_1 \cap H_2$. Then $g, h \in H_1$ and $g, h \in H_2$ by definition. By the condensed form of the criterion, $gh^{-1} \in H_1$ and $gh^{-1} \in H_2$, hence $gh^{-1} \in H_1 \cap H_2$. By the criterion, $H_1 \cap H_2 \leq G$.

(2) In fact $H_1 \cap H_2 \subset H_1$ and $H_1 \cap H_2 \subset H_2$. By the same verification as (1), we have $H_1 \cap H_2$ is a subgroup for both H_1 and H_2. By Lagrange's theorem, $|H_1 \cap H_2| \mid |H_1|$ and $|H_1 \cap H_2| \mid |H_2|$, and hence $|H_1 \cap H_2| \mid gcd(|H_1|, |H_2|) = 1$. So $|H_1 \cap H_2| = 1$ but we certainly have $e \in H_1 \cap H_2$. Hence $H_1 \cap H_2 = \{e\}$.

Problem 5. Let G and G' be finite groups, suppose that $|G|$ and $|G'|$ are relatively prime, prove that a homomorphism $\phi : G \to G'$ must be trivial, i.e., $\phi(a) = e'$ for all $a \in G$

Proof. This is actually a corollary to tutorial 10 question 1. (Also from the textbook Exercise 13.44 and 13.45). From these two exercises we have $\phi(G) \mid |G|$ and $\phi(G) \mid |G'|$, Hence $\phi(G) \mid gcd(|G|, |G'|) = 1$. But we certainly have $\phi(e) = e'$ for any homorphism ϕ, where e and e' are identity elements of G and G' respectively. So $\phi(G) = \{e'\}$, i.e. the homomorphism is trivial. $\square$