Homework No.5 for Math 3121

No need to submit your work. The solution will be posted on my webpage before Dec 8.

Problem 1. Multiple choice (each problem has only one correct answer, no reasons needed).

(1). Let G be a finite group with $|G| = n$, which is the following statement is **false** ?

(a). For each divisor d of n, there exists a subgroup H of G such that $|H| = d$.

(b). For every $a \in G$, $a^n = e$.

(c). If H is a subgroup of G, then $|H|$ is a divisor of n.

(d). The order of every element $a \in G$ is a divisor of n.

(2). Which of the following rings is an integral domain?

(a). $\mathbb{Z} \times \mathbb{Z}$. (b). $\mathbb{Z}_{20}$. (c) $\mathbb{Z}$. (d). None of above

(3). The maximal possible order of an element in S_7 is

(a) 7, (b). 7!, (c). 12 (d). None of above

(4). If R is a commutative ring, $a \in R$, $a \neq 0$ is NOT a 0-divisor, which of the following is NOT correct?

(a) $ab = 0$ implies that $b = 0$. (b) a^{2019} is not a 0-divisor.

(c) a^{-1} exists. (d) $-a$ is not a 0-divisor.

(5). Which of the following groups is isomorphic to the factor group $\mathbb{R}/\mathbb{Z}$?

(a). $\mathbb{C}$. (b). $\mathbb{R}^*$. (c). $GL_2(\mathbb{R})$.

(d). $\{z \in \mathbb{C}^* \mid |z| = 1\}$, the operation is the multiplication

(6). Which of the following sets is a subring of $M_2(\mathbb{R})$?

(a). $S = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R}\}$. (b). $S = \{\begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} \mid x \in \mathbb{R}\}$

(c). $S = \{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a, b, c \in \mathbb{R}\}$ (d). None of the above.

(7). Which of the following elements in $\mathbb{Z}_{100}$ has a multiplicative inverse? (recall that a' is the multiplicative inverse of a if $aa' = 1$).
(a). 55 , (b). 9, (c). 40 (d). None of above.

Problem 2. Multiple choice. Find the kernel Ker(Φ) of the following group homomorphisms Φ (no reasons needed).

(1). $\Phi : \mathbb{R}^* \to \mathbb{R}^*$, $\Phi(a) = a^{2018}$, Ker(Φ) is

(a) U_{2018} (b) $\{1, -1\}$ (c) $\{1\}$.

(2). $\Phi : GL(n, \mathbb{R}) \to \mathbb{R}^*$, $\Phi(A) = Det(A)$, Ker(Φ) is

(a) $\{e\}$ (b) $\{A \in GL(n, \mathbb{R}) \mid Det(A) = 1\}$ (c) None of above.

(3). $\Phi : \mathbb{R} \to \mathbb{C}^*$, $\Phi(a) = e^{2\pi i a}$, Ker(Φ) is

(a) $\mathbb{Z}$ (b) $\{0\}$ (c) None of above.

(4). $\Phi : \mathbb{C}^* \to \mathbb{C}^*$, $\Phi(z) = z^6$, Ker(Φ) is

(a) U_6 (b) $\{1, -1\}$ (c) None of above.

(5). $\Phi : S_3 \to S_4$, $\Phi(\sigma) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ i & j & k & 4 \end{pmatrix}$, where $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$,

Ker(Φ) is

(a) $\{e\}$ (b) S_3 (c). None of above.

Problem 3. Determine if the following maps are homomorphisms of groups (No reasons needed).

(1). $\Phi : \mathbb{R}^* \to \mathbb{R}^*$, $\Phi(a) = 19a$

(2). $\Phi : \mathbb{R}^* \to \mathbb{R}^*$, $\Phi(a) = a^{19}$

(3). $\Phi : \mathbb{R} \to \mathbb{R}$, $\Phi(a) = 19a$

(4). $\Phi : \mathbb{R} \to \mathbb{R}$, $\Phi(a) = -a$

(5). $\Phi : GL(n, \mathbb{R}) \to \mathbb{R}^*$, $\Phi(A) = Det(A)^{2019}$.

(6). $\Phi : \mathbb{R} \to \mathbb{R}^*$, $\Phi(a) = 2^a$.

(7). $\Phi : \mathbb{R}^* \to \mathbb{R}$, $\Phi(a) = 2^a$.

(8). $\Phi : \mathbb{R}^* \to \mathbb{R}$, $\Phi(a) = \log_6(a^2)$.

Problem 4. Determine whether each of the following maps is a ring ho-

momorphism (no reasons needed)
(1). $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $\phi((a, b)) = b$.
(2). $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $\phi(a, b) = ab$.
(3). $\phi : \mathbb{R} \to M_2(\mathbb{R})$ given by $\phi(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$.
(4). $\Phi : \mathbb{R} \to M_2(\mathbb{R})$ given by $\phi(a) = \begin{pmatrix} a & -a \\ 0 & 0 \end{pmatrix}$.
(5). $\Phi : \mathbb{R} \to M_2(\mathbb{R})$ given by $\phi(a) = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$.
(6). $\phi : \mathbb{C} \to \mathbb{C}$ given by $\phi(a + bi) = a - bi$.
(7). $\phi : \mathbb{C} \to \mathbb{C}$ given by $\phi(z) = 2z$.
(8). Let g be given 2×2 invertible matrix, $\phi : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ given by $\phi(X) = gXg^{-1}$.

Problem 5. (no reasons needed) (1) Find a subring of $M_2(\mathbb{R})$ that is isomorphisc to $\mathbb{R}$.
(2) Find a subring R of $\mathbb{Q}$ such that R contains $\mathbb{Z}$ but $R \neq \mathbb{Z}$ and $R \neq \mathbb{Q}$.

Problem 6. Let R be a ring with unity 1. Suppose $a \in R$ has multiplicative inverse $a^{-1} \in R$, that is $aa^{-1} = a^{-1}a = 1$. Prove that the map $\phi : R \to R$ given by $\phi(x) = axa^{-1}$ is a ring homomorphism. Which of the following proofs is correct ?

Proof 1. For arbitrary $x \in R$, $\phi(x) = axa^{-1} = aa^{-1}x = x$, so $\phi(x + y) = x + y = \phi(x) + \phi(y)$, $\phi(xy) = xy = \phi(x)\phi(y)$. This proves ϕ is a ring homomorphism.

Proof 2. For arbitrary $x, y \in R$, $\phi(x+y) = a(x+y)a^{-1} = axa^{-1}+aya^{-1} = \phi(x) + \phi(y)$, $\phi(xy) = axya^{-1} = (axa^{-1})(aya^{-1}) = \phi(x)\phi(y)$. This proves ϕ is a ring homomorphism.

Proof 3. For arbitrary $x, y \in R$, $\phi(x+y) = a(x+y)a^{-1} = axa^{-1} + aya^{-1} = \phi(x) + \phi(y)$, This proves ϕ is a ring homomorphism.

Problem 7. Let R be a finite commutative ring with unity 1. Suppose $a \in R, a \neq 0$ and a is **not** a 0 divisor. Prove that there exists $a' \in R$, that is $aa' = 1$. Which of the following proofs is correct?

Proof 1. Consider the infinite list $a, a^2, a^3, \ldots,$ since R is finite, there exists $m > n \geq 1$ such that $a^m = a^n$. So $a^m - a^n = 0$, $a^n(a^{m-n} - 1) = 0$. Because a is not 0-divisor, so $a^{m-n} - 1 = 0$, so $a^{m-n} = 1$, so $aa^{m-n-1} = 1$. $a' = a^{m-n-1}$.

Proof 2. Because $a \neq 0$ and a is not a 0 divisor, so a^{-1} exists, so $a' = a^{-1}$.

Proof 3. Because R is finite, let $n = |R|$, by Lagrange theorem, $a^n = 1$, so $a' = a^{n-1}$.