Solutions to Homework No.5 for Math 3121

Final Exam will take place on Dec 12, 9:30-11:30am in online format. The exam paper will be sent to you by email around 9:25 am.

Problem 1. Multiple choice (each problem has only one correct answer, no reasons needed).

(1). Let G be a finite group with $|G| = n$, which is the following statement is **false** ?
(a). For each divisor d of n, there exists a subgroup H of G such that $|H| = d$.
(b). For every $a \in G$, $a^n = e$.
(c). If H is a subgroup of G, then $|H|$ is a divisor of n.
(d). The order of every element $a \in G$ is a divisor of n.

Answer: (a) is false. Reason 1: $|A_4| = 12$, A_4 has no subgroup with order 4. Reason 2: The other three statements are true.

(2). Which of the following rings is an integral domain?
(a). $\mathbb{Z} \times \mathbb{Z}$. (b). $\mathbb{Z}_{20}$. (c) $\mathbb{Z}$. (d). None of above

Answer: (c)

(3). The maximal possible order of an element in S_7 is
(a) 7, (b). 7!, (c). 12 (d). None of above

Answer: (c) Reason: (123)(4567) has order 12; no element has order bigger than 12.

(4). If R is a commutative ring, $a \in R$, $a \neq 0$ is NOT a 0-divisor, which of the following is NOT correct?
(a) $ab = 0$ implies that $b = 0$. (b) a^{2019} is not a 0-divisor.
(c) a^{-1} exists. (d) $-a$ is not a 0-divisor.

Answer: (c). Reason: In ring $\mathbb{Z}$, 2 is not a 0-divisor, but 2^{-1} doesn't exist in $\mathbb{Z}$.

(5). Which of the following groups is isomorphic to the factor group $\mathbb{R}/\mathbb{Z}$?
(a). $\mathbb{C}$. (b). $\mathbb{R}^*$. (c). $GL_2(\mathbb{R})$.
(d). $\{z \in \mathbb{C}^* \mid |z| = 1\}$, the operation is the multiplication

Answer: (d). Reason: the map $\phi : \mathbb{R} \to \{z \in \mathbb{C}^* \mid |z| = 1\}$ given by

$\phi(x) = e^{2\pi ix}$ is a surjective group homomorphism with $Ker(\phi) = \mathbb{Z}$, apply the homomorphism theorem (Theorem 14.11) to ϕ.

(6). Which of the following sets is a subring of $M_2(\mathbb{R})$?
(a). $S = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R}\}$. (b). $S = \{\begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} \mid x \in \mathbb{R}\}$
(c). $S = \{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a, b, c \in \mathbb{R}\}$ (d). None of the above.

Answer: (c).

(7). Which of the following elements in $\mathbb{Z}_{100}$ has a multiplicative inverse? (recall that a' is the multiplicative inverse of a if $aa' = 1$).
(a). 55 , (b). 9, (c). 40 (d). None of above.

Answer: (b). Reason: because 9 is relatively prime to 100 and Theorem 20.6.

Problem 2. Multiple choice. Find the kernel Ker(Φ) of the following group homomorphisms Φ (no reasons needed).
(1). $\Phi : \mathbb{R}^* \to \mathbb{R}^*$, $\Phi(a) = a^{2018}$, Ker(Φ) is
(a) U_{2018} (b) $\{1, -1\}$ (c) $\{1\}$.
(2). $\Phi : GL(n, \mathbb{R}) \to \mathbb{R}^*$, $\Phi(A) = Det(A)$, Ker(Φ) is
(a) $\{e\}$ (b) $\{A \in GL(n, \mathbb{R}) \mid Det(A) = 1\}$ (c) None of above.
(3). $\Phi : \mathbb{R} \to \mathbb{C}^*$, $\Phi(a) = e^{2\pi ia}$, Ker(Φ) is
(a) $\mathbb{Z}$ (b) $\{0\}$ (c) None of above.
(4). $\Phi : \mathbb{C}^* \to \mathbb{C}^*$, $\Phi(z) = z^6$, Ker(Φ) is
(a) U_6 (b) $\{1, -1\}$ (c) None of above.
(5). $\Phi : S_3 \to S_4$, $\Phi(\sigma) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ i & j & k & 4 \end{pmatrix}$, where $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$, Ker($\Phi$) is
(a) $\{e\}$ (b) S_3 (c). None of above.

Answer: (1) b. (2) b. (3)a. (4)a. (5)a.

Problem 3. Determine if the following maps are homomorphisms of groups (No reasons needed).
(1). $\Phi : \mathbb{R}^* \to \mathbb{R}^*$, $\Phi(a) = 19a$
(2). $\Phi : \mathbb{R}^* \to \mathbb{R}^*$, $\Phi(a) = a^{19}$
(3). $\Phi : \mathbb{R} \to \mathbb{R}$, $\Phi(a) = 19a$
(4). $\Phi : \mathbb{R} \to \mathbb{R}$, $\Phi(a) = -a$
(5). $\Phi : GL(n, \mathbb{R}) \to \mathbb{R}^*$, $\Phi(A) = Det(A)^{2019}$.

(6). $\Phi : \mathbb{R} \to \mathbb{R}^*$, $\quad \Phi(a) = 2^a$.
(7). $\Phi : \mathbb{R}^* \to \mathbb{R}$, $\quad \Phi(a) = 2^a$.
(8). $\Phi : \mathbb{R}^* \to \mathbb{R}$, $\quad \Phi(a) = \log_6(a^2)$.

Answer: (1) No. (2) Yes. (3) Yes. (4) Yes. (5) Yes. (6)Yes (7) No (8) Yes

Problem 4. Determine whether each of the following maps is a ring homomorphism (no reasons needed)
(1). $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $\phi((a, b)) = b$.
(2). $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $\phi(a, b) = ab$.
(3). $\phi : \mathbb{R} \to M_2(\mathbb{R})$ given by $\phi(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$.
(4). $\Phi : \mathbb{R} \to M_2(\mathbb{R})$ given by $\phi(a) = \begin{pmatrix} a & -a \\ 0 & 0 \end{pmatrix}$.
(5). $\Phi : \mathbb{R} \to M_2(\mathbb{R})$ given by $\phi(a) = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$.
(6). $\phi : \mathbb{C} \to \mathbb{C}$ given by $\phi(a + bi) = a - bi$.
(7). $\phi : \mathbb{C} \to \mathbb{C}$ given by $\phi(z) = 2z$.
(8). Let g be given 2×2 invertible matrix, $\phi : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ given by $\phi(X) = gXg^{-1}$.

Answer: (1) Yes (2) No. (3) Yes. (4) Yes. (5) No. (6)Yes (7) No (8) Yes

Problem 5. (no reasons needed) (1) Find a subring of $M_2(\mathbb{R})$ that is isomorphisc to $\mathbb{R}$.
(2) Find a subring R of $\mathbb{Q}$ such that R contains $\mathbb{Z}$ but $R \neq \mathbb{Z}$ and $R \neq \mathbb{Q}$.

Answer: (1) $\{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{R}\}$ (the answer is not unique).
(2) $\{\frac{m}{2^n} \mid m \in \mathbb{Z}, n = 0, 1, 2, \dots\}$(the answer is not unique).

Problem 6. Let R be a ring with unity 1. Suppose $a \in R$ has multiplicative inverse $a^{-1} \in R$, that is $aa^{-1} = a^{-1}a = 1$. Prove that the map $\phi : R \to R$ given by $\phi(x) = axa^{-1}$ is a ring homomorphism. Which of the following proofs is correct ?

Proof 1. For arbitrary $x \in R$, $\phi(x) = axa^{-1} = aa^{-1}x = x$, so $\phi(x+y) = x+y = \phi(x) + \phi(y)$, $\phi(xy) = xy = \phi(x)\phi(y)$. This proves ϕ is a ring homomorphism.

Proof 2. For arbitrary $x, y \in R$, $\phi(x+y) = a(x+y)a^{-1} = axa^{-1} + aya^{-1} = \phi(x) + \phi(y)$, $\phi(xy) = axya^{-1} = (axa^{-1})(aya^{-1}) = \phi(x)\phi(y)$. This proves ϕ is a ring homomorphism.

Proof 3. For arbitrary $x, y \in R$, $\phi(x+y) = a(x+y)a^{-1} = axa^{-1} + aya^{-1} = \phi(x) + \phi(y)$, This proves ϕ is a ring homomorphism.

Answer: Proof 2 is correct.

Problem 7. Let R be a finite commutative ring with unity 1. Suppose $a \in R, a \neq 0$ and a is **not** a 0 divisor. Prove that there exists $a' \in R$, that is $aa' = 1$. Which of the following proofs is correct?

Proof 1. Consider the infinite list $a, a^2, a^3, \ldots$, since R is finite, there exists $m > n \geq 1$ such that $a^m = a^n$. So $a^m - a^n = 0$, $a^n(a^{m-n} - 1) = 0$. Because a is not 0-divisor, so $a^{m-n} - 1 = 0$, so $a^{m-n} = 1$, so $aa^{m-n-1} = 1$. $a' = a^{m-n-1}$.

Proof 2. Because $a \neq 0$ and a is not a 0 divisor, so a^{-1} exists, so $a' = a^{-1}$.

Proof 3. Because R is finite, let $n = |R|$, by Lagrange theorem, $a^n = 1$, so $a' = a^{n-1}$.

Answer: Proof 1 is correct.