Math 4991, Lecture on April 3, 2020

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Today's Plan.

(1). Review of Complex Analysis (continued).

(2). Elliptic Functions.

§ 1. Review of Complex Analysis (continued).

Let D be an open domain in the complex plane $\mathbb{C}$.

Let $f(z)$ be a meromporphic function on D, for any $a \in D$, $f(z)$ has a Laurent power series expansion at a

$$f(z) = c_m(z-a)^m + \cdots + c_{m+1}(z-a)^{m+1} + \text{higher terms}$$

where $c_m \neq 0$.

If $m < 0$, then a is a pole of f of order $-m$.

The residue of f at a is defined by

$$\mathrm{res}_a(f) = c_{-1} = \text{coefficent of } (z-a)^{-1}.$$

Residue Theorem . Let $f(z)$ be a meromorhic function on a simply connected domain D, C be a simple counter-clockwise closed contour in D that doesn't contains any poles of f, then

$$\frac{1}{2\pi i}\int_C f(z)dz = \sum_{a\text{:poles of f } \textit{enclosed by } C} \mathrm{res}_a f$$

Example of Application of Residue Theorem .

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1} dx$$

We compute

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1} dx = \int_{-\infty}^{\infty} \frac{\cos\ x}{x^2+1} dx + i \int_{-\infty}^{\infty} \frac{\sin\ x}{x^2+1} dx$$

Consider contour integral

$$\int_{C(R)} \frac{e^{iz}}{z^2+1} dZ = \int_{-R}^{R} \frac{e^{ix}}{x^2+1} dx + \int_{S(R)} \frac{e^{iz}}{z^2+1} dZ$$

where $S(R)$ is the upper semi-circle centered at the origin with radius R.

We have

$$\lim_{R\to\infty}\int_{S(R)}\frac{e^{iz}}{z^2+1}dZ=0$$

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{e^{ix}}{x^2+1}dx=\int_{-\infty}^{\infty}\frac{e^{ix}}{x^2+1}dx$$

Thus

$$\int_{-\infty}^{\infty}\frac{e^{ix}}{x^2+1}dx=\lim_{R\to\infty}\int_{C(R)}\frac{e^{iz}}{z^z+1}dz$$

For $R > 1$, the contour $C(R)$ contains only one pole of the integrand $\frac{e^{iz}}{z^2+1}$, that is i. So by Residue Theorem,

$$\int_{C(R)} \frac{e^{iz}}{z^2+1} dz = 2\pi i \operatorname{res}_i \frac{e^{iz}}{z^2+1} = \pi e^{-1}$$

So we have

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1} dx = \pi e^{-1}$$

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1} dx = \pi e^{-1}, \quad \int_{-\infty}^{\infty} \frac{\sin x}{x^2+1} dx = 0$$

Theorem . Let $f(z)$ be a meromorhic function on a simply connected domain D, C be a simple counter-clockwise closed contour in D that doesn't contains any poles or zeros of f, then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = \text{number of zeros and poles enclosed by } C$$

where the zeros are counted with multiplicity, and the poles are counted with negative multiplicity.

Proof. The poles of $\frac{f'(z)}{f(z)}$ are precisely the zeros or poles of $f(z)$

$$\mathrm{res}_a \frac{f'(z)}{f(z)} = \text{order of } \ a \text{ as a zero of } f(z)$$

Elliptic Functions

Definition. Let ω_1 and ω_2 be complex numbers that are linearly independent over $\mathbb{R}$. An **elliptic function** with periods ω_1 and ω_2 is a meromorphic function $f(z)$ on $\mathbb{C}$ such that

$$f(z) = f(z + \omega_1), \quad f(z) = f(z + \omega_2)$$

for all $z \in \mathbb{C}$.

Denoting the "lattice of periods" by

$$\Lambda = \{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}\}.$$

It is clear that the condition

$$f(z) = f(z + \omega_1), \quad f(z) = f(z + \omega_2)$$

is equivalent to

$$f(z) = f(z + \omega)$$

for all $\omega \in \Lambda$.

We denote $\mathcal{M}(\Lambda)$ the space of all elliptic functions with lattice of periods Λ.

Proposition 2.1. $\mathcal{M}(\Lambda)$ *is a field.*

Consider the domain

$$D \stackrel{\text{def}}{=} \{t_1\omega_1 + t_2\omega_2 \,|\, 0 \leq t_1, t_2 \leq 1\}.$$

Then D satisfies the conditions that

(1). For every z, there exists $\omega \in \Lambda$ such that $z - \omega \in D$.

(2). If $z_1, z_2 \in D$ and $z_1 - z_2 \in \Lambda$, then z_1 and z_2 are in the boundary of D.

This is an analog of the following: $\mathbb{Z} \subset \mathbb{R}$,

(a) Every $x \in \mathbb{R}$, there exists $r \in [0, 1]$ such that $z - r \in \mathbb{Z}$.

(b) If $x_1, x_2 \in [0, 1]$ and $x_1 - x_2 \in \mathbb{Z}$, then x_1 and x_2 are in the boundary of $[0, 1]$, i.e., $x_1, x_2 \in \{0, 1\}$.

Any domain with properties (1) (2) is called a **fundamental domain** for Λ.

Any translation of D, $a + D$ is also a fundamental domain for Λ.

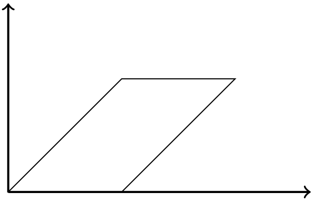


Figure: The domain $D = \{t_1 + t_2(1+i) \mid 0 \leq t_1, t_2 \leq 1\}$.

Theorem

If an elliptic function $f(z)$ with period lattice Λ is analytic, then it is a constant function.

Proof. $|f(z)|$ is a real valued continuous function with periods ω_1 and ω_2. Since D is a compact domain, so $|f(z)|$, considered as a function on D, achieves a maximum at $z_0 \in D$. For any $z \in \mathbb{C}$, $z + \omega \in D$ for some $\omega \in \Lambda$, so

$$|f(z)| = |f(z + \omega)| \leq |f(z_0)|$$

So $|f(z_0)|$ is the maximum of $|f(z)|$ on $\mathbb{C}$, by the maximum principle, $f(z)$ is a constant.

the Weierstrass elliptic function $\wp(z)$ for a lattice Λ is defined as

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

we notice that on any compact disk defined by $|z| \leq R$, all but possibly finitely many $\omega \in \Lambda$ satisfies $|\omega| > 2R$. For such ω, one has

$$\left|\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right| = \left|\frac{2\omega z - z^2}{\omega^2 (\omega - z)^2}\right| = \left|\frac{z\left(2 - \frac{z}{\omega}\right)}{\omega^3 \left(1 - \frac{z}{\omega}\right)^2}\right| \leq \frac{10R}{|\omega|^3}$$

This implies that the series converges uniformly on $|z| \leq R$, so we have a meromorphic function on $\mathbb{C}$ with poles on the lattice Λ.

Theorem. If a series of analytic functions on a domain D

$$f_1(z) + f_2(z) + \dots$$

converges uniformly, then the limit $S(z)$ is also an analytic function on D. And

$$f_1'(z) + f_2'(z) + \dots$$

also converges on D and the convergence is uniform on every compact subsets in D, the limit is $S'(z)$.

By the above theorem, $\wp(z)$ is a meromorphic function on $\mathbb{C}$.

And we have

$$\wp'(z) = -2\sum_{\omega\in\Lambda}\frac{1}{(z-\omega)^3}$$

has periods Λ, so we have

$$\wp(z+\omega) - \wp(z) = C$$

is a constant, put $z = -\frac{\omega}{2}$, we see that $\wp(\frac{\omega}{2}) - \wp(-\frac{\omega}{2}) = C$, it is obvious that $\wp(z)$ is even function, so $C = 0$. This proves $\wp(z)$ is an elliptic function with period Λ.

Theorem.

$$\left(\wp'(z)\right)^2 = 4\left(\wp(z)\right)^3 - g_2\wp(z) - g_3 \tag{1}$$

where

$$g_2 = 60 \sum_{\omega\in\Lambda\setminus\{0\}} \frac{1}{\omega^4}$$

and

$$g_3 = 140 \sum_{\omega\in\Lambda\setminus\{0\}} \frac{1}{\omega^6}.$$

Theorem

The field $\mathcal{M}(\Lambda)$ is generated by $\wp(z)$ and $\wp'(z)$ over $\mathbb{C}$ subject to the relation (1)

For $f \in \mathcal{M}(\Lambda)$ and $f \neq 0$, if $a \in \mathbb{C}$ is a zero (or pole) of f of order k, then for any $\omega \in \Lambda$, $a + \omega$ is also a zero (pole, resp.) of f with the same order k.

Theorem.

Let $f \in \mathcal{M}(\Lambda)$ and $f \neq 0$, let $a_1, \dots, a_m$ be the zeros of f (modulo Λ) with orders $k_1, \dots, k_m$; and $b_1, \dots, b_n$ be the poles of f (modulo Λ) with orders $l_1, \dots, l_n$. Then

$$k_1 + \cdots + k_m - (l_1 + \cdots + l_n) = 0$$

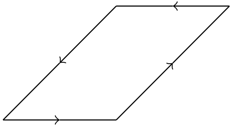
and

$$k_1 a_1 + \cdots + k_m a_m - (l_1 b_1 + \cdots + l_n b_n) \in \Lambda.$$

Proof.

Let $a + D$ be a fundamental parallelogram of the period lattice Λ such that the four boundary edges of D contains no zeros nor poles.

Let C the contour that is the boundary of D oriented counter-clock wisely (see figure below). $C = C_1 \cup C_2 \cup C_3 \cup C_4$, where C_1, C_3 are parallel (horizontal in the figure) and C_2, C_4 are parallel.



By Residue Theorem

$$I \stackrel{\text{def}}{=} \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = k_1 + \cdots + k_m - (l_1 + \cdots + l_n)$$

On the other hand,

$$I = \frac{1}{2\pi i}\left(\int_{C_1}\frac{f'(z)}{f(z)}dz + \int_{C_2}\frac{f'(z)}{f(z)}dz + \int_{C_3}\frac{f'(z)}{f(z)}dz + \int_{C_4}\frac{f'(z)}{f(z)}dz\right)$$

Since the values of f'/f are equal on C_1 and C_3, but orientations on C_1 and C_3 are opposite, so $\int_{C_1} + \int_{C_3} = 0$. Similarly $\int_{C_2} + \int_{C_4} = 0$. So $I = 0$.

This proves

$$k_1 + \cdots + k_m - (l_1 + \cdots + l_n) = 0$$

For the second identity, we consider the contour integral

$$I' \stackrel{\text{def}}{=} \frac{1}{2\pi i} \int_C \frac{zf'(z)}{f(z)} dz.$$

By Residue Theorem,

$$I' = k_1 a_1 + \cdots + k_m a_m - (l_1 b_1 + \cdots + l_n b_n)$$

On the other hand side,

$$
\begin{aligned}
I' &= \frac{1}{2\pi i}\left(\int_{C_1}\frac{zf'(z)}{f(z)}dz + \int_{C_3}\frac{zf'(z)}{f(z)}dz\right)\\
&+ \frac{1}{2\pi i}\left(\int_{C_2}\frac{zf'(z)}{f(z)}dz + \int_{C_4}\frac{zf'(z)}{f(z)}dz\right)
\end{aligned}
$$

$$\frac{1}{2\pi i}\left(\int_{C_1}\frac{zf'(z)}{f(z)}dz+\int_{C_3}\frac{zf'(z)}{f(z)}dz\right)=-\omega_2\frac{1}{2\pi i}\int_{C_1}\frac{f'(z)}{f(z)}dz$$

$$\frac{1}{2\pi i}\left(\int_{C_2}\frac{zf'(z)}{f(z)}dz+\int_{C_4}\frac{zf'(z)}{f(z)}dz\right)=-\omega_1\frac{1}{2\pi i}\int_{C_2}\frac{f'(z)}{f(z)}dz$$

The Second identity follows from the following:

Claim:

$$\frac{1}{2\pi i}\int_{C_1}\frac{f'(z)}{f(z)}dz \in \mathbb{Z}$$

$$\frac{1}{2\pi i}\int_{C_2}\frac{f'(z)}{f(z)}dz \in \mathbb{Z}$$

Proof of Claim.

Since $f(z)$ has no zeros nor poles on C_1, it has no zeros nor poles in an simply connected open neighborhood U of C_1. There exists an analytic function $h(z)$ on U such that $h'(z) = \frac{f'(z)}{f(z)}$ and $h(a+\omega_1) - h(a) \in 2\pi i\mathbb{Z}$ in fact $h(z) = \log f(z)$ (a branch of $\log f(z)$)

$$\frac{1}{2\pi i}\int_{C_1} \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}(h(a+\omega_1) - h(a)) \in \mathbb{Z}.$$

The end