Math 4991, Lecture on April 6, 2020

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Today's Plan.

(1). Elliptic Functions (Review).

(2). Theta Functions.

Elliptic Functions

Definition. Let ω_1 and ω_2 be complex numbers that are linearly independent over $\mathbb{R}$. An **elliptic function** with periods ω_1 and ω_2 is a meromorphic function $f(z)$ on $\mathbb{C}$ such that

$$f(z) = f(z + \omega_1), \quad f(z) = f(z + \omega_2)$$

for all $z \in \mathbb{C}$.

Denoting the "lattice of periods" by

$$\Lambda = \{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}\}.$$

It is clear that the condition

$$f(z) = f(z + \omega_1), \quad f(z) = f(z + \omega_2)$$

is equivalent to

$$f(z) = f(z + \omega)$$

for all $\omega \in \Lambda$.

We denote $\mathcal{M}(\Lambda)$ the space of all elliptic functions with lattice of periods Λ.

Proposition 2.1. $\mathcal{M}(\Lambda)$ *is a field.*

Theorem. If an elliptic function $f(z)$ with period lattice Λ is analytic, then it is a constant function.

the Weierstrass elliptic function $\wp(z)$ for a lattice Λ is defined as

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

Theorem.

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3 \tag{1}$$

where

$$g_2 = 60 \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^4}$$

and

$$g_3 = 140 \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^6}.$$

Theorem. The field $\mathcal{M}(\Lambda)$ is generated by $\wp(z)$ and $\wp'(z)$ over $\mathbb{C}$ subject to the relation

$$\wp'(z)^2 = 4\,\wp(z)^3 - g_2\,\wp(z) - g_3 \tag{2}$$

for g_2, g_3 as above.

Theorem.

Let $f \in \mathcal{M}(\Lambda)$ and $f \neq 0$, let $a_1, \ldots, a_m$ be the zeros of f (modulo Λ) with orders $k_1, \ldots, k_m$; and $b_1, \ldots, b_n$ be the poles of f (modulo Λ) with orders $l_1, \ldots, l_n$. Then

$$k_1 + \cdots + k_m - (l_1 + \cdots + l_n) = 0$$

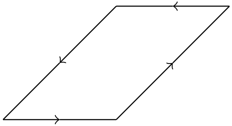
and

$$k_1 a_1 + \cdots + k_m a_m - (l_1 b_1 + \cdots + l_n b_n) \in \Lambda.$$

Sketch of Proof. Apply Residue Theorem the integrals

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz, \quad \frac{1}{2\pi i}\int_C \frac{zf'(z)}{f(z)}dz.$$

where C is the contour:



3. Theta Functions

The Jacobi theta function (named after Carl Gustav Jacobi) is a function defined for two complex variables z and τ, where z can be any complex number and τ is confined to the upper half-plane, which means it has positive imaginary part. It is given by the formula

$$\vartheta(z;\tau) = \sum_{n=-\infty}^{\infty} \exp(\pi i n^2 \tau + 2\pi i n z)$$

Each term is an analytic function of variables z and τ, and the convergence is uniform in each compact subset in $\mathbb{C} \times H$ (here H is the upper half place), so $\vartheta(z;\tau)$ is an analytic function on $\mathbb{C} \times H$.

Clearly we have

$$\vartheta(z+1;\tau) = \vartheta(z;\tau).$$

And we have

$$\vartheta(z+\tau;\tau) = \exp(-\pi i\tau - 2\pi iz)\,\vartheta(z;\tau).$$

Proof.

$$
\begin{aligned}
&\vartheta(z+\tau;\tau) \\
&= \sum_{n=-\infty}^{\infty} \exp(\pi i n^2 \tau + 2\pi i n(z+\tau)) \\
&= \sum_{n=-\infty}^{\infty} \exp(\pi i n^2 \tau + 2\pi i n\tau + 2\pi i n z) \\
&= \sum_{n=-\infty}^{\infty} \exp(-\pi i \tau)\exp(\pi i (n+1)^2 \tau + 2\pi i n z) \\
&= \sum_{n=-\infty}^{\infty} \exp(-\pi i \tau)\exp(-2\pi i z)\exp(\pi i (n+1)^2 \tau + 2\pi i (n+1) z) \\
&= \exp(-\pi i \tau - 2\pi i z) \sum_{n=-\infty}^{\infty} \exp(\pi i (n+1)^2 \tau + 2\pi i (n+1) z) \\
&= \exp(-\pi i \tau - 2\pi i z)\,\vartheta(z;\tau).
\end{aligned}
$$

The above two identities imply that for arbitrary integers a, b,

$$\vartheta(z + a + b\tau; \tau) = \exp(-\pi i b^2 \tau - 2\pi i b z)\, \vartheta(z; \tau).$$

Proposition. Given a fixed τ with $\mathrm{Im}\,\tau > 0$, if an analytic function $f(z)$ on $\mathbb{C}$ satisfies

$$f(z+1) = f(z), \quad f(z+\tau) = \exp(-\pi i \tau - 2\pi i z)\, f(z)$$

Then

$$f(z) = C\,\vartheta(z;\tau).$$

for some scalar C.

Sketch of Proof. Because $f(z+1) = f(z)$,

$$f(z) = \sum_{n \in \mathbb{Z}} c_n \exp(2\pi i n z)$$

Using the condition

$$f(z+\tau) = \exp(-\pi i \tau - 2\pi i z) f(z)$$

we find a recurve relations between c_n and c_{n+1}.

Theorem 3.1.

If $4n$ real numbers a_k, b_k, c_k, d_k $(k = 1, 2, \ldots, n)$ satisfy the conditions $\sum_{k=1}^n a_i - \sum_{k=1}^n c_i \in \mathbb{Z}$ and $\sum_{k=1}^n b_k = \sum_{k=1}^n d_k$, then

$$\frac{\vartheta(z + a_1 + b_1\tau;\tau)\,\vartheta(z + a_2 + b_2\tau;\tau)\cdot\cdots\cdot\vartheta(z + a_n + b_n\tau;\tau)}{\vartheta(z + c_1 + d_1\tau;\tau)\,\vartheta(z + c_2 + d_2\tau;\tau)\cdot\cdots\cdot\vartheta(z + c_n + d_n\tau;\tau)}$$

is an elliptic function of period lattice $\mathbb{Z} + \mathbb{Z}\tau$.

Proof. It is clear that the function is periodic with period 1. It is enough to prove that is periodic with period τ.

Proof (continued). We denote the function by $f(z) = \frac{A(z)}{B(z)}$.

$$A(z) = \vartheta(z + a_1 + b_1\tau; \tau) \cdots \vartheta(z + a_n + b_n\tau; \tau)$$
$$B(z) = \vartheta(z + c_1 + d_1\tau; \tau) \cdots \vartheta(z + c_n + d_n\tau; \tau)$$

Proof (continued).

$$\begin{aligned}
&A(z+\tau) \\
&= \vartheta(z+a_1+b_1\tau+\tau;\tau)\cdots\vartheta(z+a_n+b_n\tau+\tau;\tau) \\
&= \exp(-\pi i\tau - 2\pi i(z+a_1+b_1\tau))\vartheta(z+a_1+b_1\tau;\tau)\cdots \\
&\quad \exp(-\pi i\tau - 2\pi i(z+a_n+b_n\tau))\vartheta(z+a_n+b_n\tau;\tau) \\
&= \exp(-n\pi i\tau - 2n\pi iz - 2\pi i(a_1+\cdots+a_n) - 2\pi i(b_1+\cdots+b_n)\tau)A(z)
\end{aligned}$$

Similarly,

$$B(z+\tau) = \exp(-n\pi i\tau - 2n\pi iz - 2\pi i(c_1+\cdots+c_n) - 2\pi i(d_1+\cdots+d_n)\tau)B(z)$$

So $\frac{A(z+\tau)}{B(z+\tau)} = \frac{A(z)}{B(z)}$.

Theorem. Every non-zero elliptic function with period lattice $\mathbb{Z} + \mathbb{Z}\tau$ can be written as a nonzero scaler times a function as in Theorem 3.1.

Lemma. For fixed τ, the one variable function $\vartheta(z;\tau)$ has only one zero in $\mathbb{C}$ modulo Λ. Each zero has multiplicity 1.

Proof. Let $D = \{t_1 + t_2\tau \mid 0 \le t_1, t_2 \le 1\}$ be a fundamental parallelogram. Choose $r \in \mathbb{C}$ such that the boundary of $r + D$ contains no zeros of $\vartheta(z;\tau)$. Let C be the boundary of $a + D$ with anti-clockwise orientation. It is enough to prove

$$\frac{1}{2\pi i}\int_C \frac{\vartheta'(z;\tau)}{\vartheta(z;\tau)} dz = 1.$$

Set $h(z) \stackrel{\text{def}}{=} \frac{\vartheta'(z;\tau)}{\vartheta(z;\tau)}$.

Formally $h(z) = \frac{d}{dz} \log \vartheta(z;\tau)$.

$\vartheta(z+1;\tau) = \vartheta(z;\tau)$ implies that

$$h(z+1) = h(z).$$

The functional equation

$$\vartheta(z+\tau;\tau) = \exp(-\pi i\tau - 2\pi iz)\,\vartheta(z;\tau)$$

implies that

$$h(z+\tau) = -2\pi i + h(z)$$

Proof (continued).

$$\begin{aligned}
&\frac{1}{2\pi i}\int_C \frac{\vartheta'(z;\tau)}{\vartheta(z;\tau)}dz \\
&= \frac{1}{2\pi i}\int_{C_1} \frac{\vartheta'(z;\tau)}{\vartheta(z;\tau)}dz + \frac{1}{2\pi i}\int_{C_3} \frac{\vartheta'(z;\tau)}{\vartheta(z;\tau)}dz \\
&+ \frac{1}{2\pi i}\int_{C_2} \frac{\vartheta'(z;\tau)}{\vartheta(z;\tau)}dz + \frac{1}{2\pi i}\int_{C_4} \frac{\vartheta'(z;\tau)}{\vartheta(z;\tau)}dz +
\end{aligned}$$

By the quasi-periodicity θ, we have $\int_{C_2} + \int_{C_4} = 0$.
$\int_{C_1} + \int_{C_3} = 2\pi i$.

Lemma. For fixed τ, the zeros of the one variable function $\vartheta(z;\tau)$ are

$$\frac{1}{2}+\frac{1}{2}\tau+m+n\tau$$

$m, n \in \mathbb{Z}$. Each zero has multiplicity 1.

Proof. One checks directly $\frac{1}{2}+\frac{1}{2}\tau$ is a zero.

Let α_i be the zeros of f $(i = 1, \ldots, n)$ and β_i be the poles of f $(i = 1, \ldots, n)$. (Everything modulo Λ).
Then we have

$$\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i$$

On the other hand side, we can construct an elliptic function $g(z)$ of type

$$\frac{\vartheta(z+a_1+b_1\tau;\tau)\,\vartheta(z+a_2+b_2\tau;\tau)\cdot\cdots\cdot\vartheta(z+a_n+b_n\tau;\tau)}{\vartheta(z+c_1+d_1\tau;\tau)\,\vartheta(z+c_2+d_2\tau;\tau)\cdot\cdots\cdot\vartheta(z+c_n+d_n\tau;\tau)}$$

that has the same zeros and poles as $f(z)$.

$f(z)/g(z)$ has no zeros nor poles. So $f(z)/g(z) = C$ constant.

$f(z) = C\,g(z)$.

We have studied the relations of theta function $\vartheta(z;\tau)$ with the elliptic functions with periods 1 and τ.

Lemma. If $f(z)$ is an elliptic function with periods ω_1 and ω_2, for arbitrary $a \neq 0$, the function $f(az)$ is an elliptic function with periods $a^{-1}\omega_1$ and $a^{-1}\omega_2$

Proof. We have

$$f(a(z + a^{-1}\omega_1)) = f(az + \omega_1) = f(az).$$

End