Math 4991, Lecture on March 30, 2020

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Plan.

In the lectures on March 30, April 3 and April 6, we will discuss the topic "**Elliptic Functions and Theta Functions.**"

(1). Review of Complex Analysis.

(2). Elliptic Functions.

(3). Theta Functions.

§ 1. Review of Complex Analysis.

Let D be a **connected** open set in $\mathbb{C}$, a continuous complex valued $f(z)$ defined on D is called an **analytic function** if $f'(z)$ exists everywhere in D.

Recall $f'(z)$ is the complex derivative defined by

$$\lim_{\delta \to 0} \frac{f(z+\delta) - f(z)}{\delta},$$

where δ goes to 0 at all the directions in $\mathbb{C}$.

Analytic functions have good properties that general smooth functions don't have.

Theorem 1.1. Let $f(z)$ be an analytic function on D, C be a simple counter-clockwise closed contour in D, if the domain enclosed by C is in D, then

$$\int_C f(z)dz = 0.$$

For C as above, a in in the domain enclosed by C, then

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz = f(a).$$

Theorem 1.1 imply the the Theorems 1.2, 1.3, 1.4, 1.5 below.

Theorem 1.2. If $f(z)$ is an analytic function on D, if $|f(z)|$ has a local maximal at some point in D, then $f(z)$ is a constant function.

Theorem 1.3. The derivative $f^{(n)}(z)$ of arbitrary order n exists, and

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{1}{n!}f^{(n)}(a).$$

In real variable functions, the similar result fail to hold.

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{n^3}$$

has continuous first order derivative, but $f^{(3)}(x)$ doesn't exist.

Theorem 1.4. If $f(z)$ is an analytic function on D, for every $a \in D$, the Taylor expansion at a

$$f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \ldots$$

converges absolutely to $f(z)$ uniformly on any closed disc $|z-a| \leq r$ inside D.

Let $f(x)$ be the real variable function defined by

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Then $f^{(n)}(x)$ exists for all x, and $f^{(n)}(0) = 0$.
The Taylor series at 0 is 0, so it doesn't converge to $f(x)$ for $x > 0$.

Theorem 1.5. If the zero points $\{a \mid f(a) = 0\}$ has a limit point in D, then $f(z) = 0$.

The similar result does not hold for real variable functions.

A **meromorphic function** on D is a map $f : D \to \mathbb{C} \cup \{\infty\}$ such that
(1). If $f^{-1}(\infty)$ is discrete subset in D (this means, if $a \in f^{-1}(\infty)$, there is an open neighborhood U of a such that $U \cap f^{-1}(\infty) = \{a\}$). For each $a \in f^{-1}(\infty)$, there exists a positive integer n, such that $\lim_{z \to a}(z-a)^n f(z)$ exists and is non-zero. (Such a is called the pole of $f(z)$, n is called the order of the pole).
(2). By (1), $D - f^{-1}(\infty)$ is an open set, $f(z)$ is analytic on $D - f^{-1}(\infty)$.

For a meromporphic function $f(z)$ on D, if $a \in D$ is a pole of order $n \geq 1$, $f(z)$ has a Laurent power series expansion at a

$$c_{-n}(z-a)^{-n} + \cdots + c_{-1}(z-a)^{-1} + \sum_{k=0}^{\infty} c_k (z-a)^k$$

where $c_{-n} \neq 0$. The coefficient c_{-1} is called the **residue of f at a** and is denoted by

$$\mathrm{res}_a f = c_{-1}.$$

Theorem 1.6. Let $f(z)$ be a meromorhic function on a simply connected domain D, C be a simple counter-clockwise closed contour in D that doesn't contains any poles of f, then

$$\frac{1}{2\pi i}\int_C f(z)dz = \sum_{a\text{:poles of f } \textit{in the region enclosed by C}} \mathrm{res}_a f$$

Example. Let C be the unit circle oriented counter-clock wisely,

$$\frac{1}{2\pi i}\int_C \frac{1}{\sin\ z} dz = 1.$$

Examples of Analytic and Meromorphic Functions.

Almost all the high school functions are analytic functions on certain domain in $\mathbb{C}$.

Polynomial functions $a_n z^n + \cdots + a_1 z + a_0$ are analytic functions on $\mathbb{C}$.

The exponential functions e^z defined by, for $z = x + iy$,

$$e^z = e^x(\cos y + i \sim y)$$

is an analytic function on $\mathbb{Z}$.

Trigonometry functions and exponential functions are unified in complex analysis

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

$$\sin z = \frac{1}{2}(e^{iz} + e^{-iz})$$

If $f(z), g(z)$ are analytic functions on D, and $g(z) \neq 0$, then $\frac{f(z)}{g(z)}$ is a meromorphic function on D.

Conversely every meromorphic function $h(z)$ on D is locally a quotient of analytic functions.

That is, for every $a \in D$, there is an open neighborhood U of a such that

$$h(z) = \frac{f(z)}{g(z)}$$

for some analytic functions f, g on U, $g \neq 0$.

Example 1. A **rational function** is a meromorphic function on $\mathbb{C}$ of the form

$$f(z) = \frac{p(z)}{q(z)}$$

where $p(z)$ and $q(z)$ are polynomials, we may assume $p(z)$ and $q(z)$ have no common zeros. a is pole of $f(z)$ iff $q(a) = 0$, its order is the multiplicity of a as a zero of $q(z)$.

Example 2. $f(z) = \frac{1}{e^z - 1}$ is a meromorphic function on $\mathbb{C}$, whose poles are $2\pi i\mathbb{Z}$.

Example 3. $\log z$, as a real function, is defined on the half real line $z > 0$. It can be extended to an analytic function on

$$D = \mathbb{C} - \mathbb{R}_{\leq 0}$$

by

$$\log z = \log |z| + i \arg z$$

where $\arg z$ is the angle from the real axis to the ray from the origin to z, and we require

$$-\pi < \arg z < \pi.$$

Example 4. Let s be a complex number, z^s is an analytic function on

$$D = \mathbb{C} - \mathbb{R}_{\leq 0}$$

by

$$z^s = e^{s \log z}$$

where $\log z$ is defined in Example 3 above.

Example 5. If $f(z)$ is an analytic function on D and $g(w)$ is an analytic function on D'. Suppose $g(D') \subset D$, then the composition $(f \circ g)(w) = f(g(w))$ is an analytic function on D' and we have the chain rule:

$$\frac{d}{dw} f(g(w)) = f'(g(w))\, g'(w).$$

The space of analytic functions on D, $\mathcal{O}(D)$, is a commutative algebra over $\mathbb{C}$, as a ring, it is an integral domain.

Proposition 1.7. *The space of all analytic functions on D is an integral domain. The space of all meromorphic functions on D is a field.*

Elliptic Functions

Definition. Let ω_1 and ω_2 be complex numbers that are linearly independent over $\mathbb{R}$. An **elliptic function** with periods ω_1 and ω_2 is a meromorphic function $f(z)$ on $\mathbb{C}$ such that

$$f(z) = f(z + \omega_1), \quad f(z) = f(z + \omega_2)$$

for all $z \in \mathbb{C}$.

Denoting the "lattice of periods" by

$$\Lambda = \{ m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z} \} .$$

It is clear that the condition

$$f(z) = f(z + \omega_1), \quad f(z) = f(z + \omega_2)$$

is equivalent to

$$f(z) = f(z + \omega)$$

for all $\omega \in \Lambda$.

The end