Math 6170 C, Lecture on Feb 24, 2020

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Today's Plan:

(1) Review and Examples of Chapter one.

(2). Section 1 of Chapter two.

The complex number field $\mathbb{C}$ below can be replaced by any algebraically closed field.

Example 1.

Principal ideal $I = (X^2 + Y^2 - 1) \subset \mathbb{C}[X, Y]$ is a prime ideal,

because $X^2 + Y^2 - 1$ is an irreducible polynomial in $\mathbb{C}[X, Y]$,

Proof. Because $X^2 + Y^2 - 1$ is irreducible one variable polynomial over field $\mathbb{C}(Y)$.

$$V \overset{\text{def}}{=} \{(x, y) \in \mathbb{C}^2 \mid x^2 + y^2 - 1 = 0\}$$

is an affine variety (in $\mathbb{A}^2(\mathbb{C})$).

The coordinate ring of V is $\mathbb{C}[V] = \mathbb{C}[X, Y]/(X^2 + Y^2 - 1)$

The field of rational functions on V is

$$\mathbb{C}(V) = \mathrm{Frac}\,\mathbb{C}[V].$$

It is the algebraic extension of $\mathbb{C}(X)$ by the $Y^2 + (X^2 - 1) = 0$.
So the transcendental degree of $\mathbb{C}(V)$ is the same as that of $\mathbb{C}(Y)$, which is 1. So $\dim V = 1$.

This proves $\dim V = 1$. It is an affine curve.

Does V have any singular points?

Recall that if an affine variety V variety in $\mathbb{A}^n(\mathbb{C})$ has the prime ideal $I(V)$ generated by $f_1, \ldots, f_m \in \mathbb{C}[X_1, \ldots, X_n]$. A point $x = (x_1, \ldots, x_n) \in V$ is called a **regular point** (smooth point) if the rank of the $n \times m$ matrix

$$\partial_i f_j(x_1, \ldots, x_n)$$

is $n - \dim V$.

x is called a **singular point** if it is NOT regular. The set of singular points of V is a proper closed subset of V.

For V given by

$$V = \{(x, y) \in \mathbb{C}^2 \mid x^2 + y^2 - 1 = 0\}$$

Its ideal is generated by $X^2 + Y^2 - 1$.
A point (x, y) is a singular point on V iff

$$x^2 + y^2 - 1 = 0 \text{ and } (2x, 2y) \text{ has rank } 0$$

equivalently

$$x^2 + y^2 - 1 = 0, 2x = 0, 2y = 0.$$

So V has no singular point. V is a smooth affine curve.

We embed $\mathbb{A}^2(\mathbb{C})$ into

$$\mathbb{P}^2(\mathbb{C}) = \{[x, y, z] \mid x, y, z \in \mathbb{C} \text{ not all } 0\}$$

by

$$(x, y) \mapsto [x, y, 1]$$

Recall that $[x, y, z] = [x', y', z']$ iff there is $\lambda \neq 0$ such that $x = \lambda x', y = \lambda y', z = \lambda z'$.

The projective closure of V is given as the zero set of homogenization of $X^2 + Y^2 - 1$:

$$X^2 + Y^2 - Z^2 = 0$$

$$\bar{V} = \{[x, y, z] \in \mathbb{P}^2(\mathbb{C}) \mid x^2 + y^2 - z^2 = 0\}$$

The points $[x, y, z]$ in $\bar{V}$ with $z \neq 0$ is identified with the points in V.

$$\bar{V} - V = \{[x, y, z] \mid x^2 + y^2 - z^2 = 0, z = 0\} = \{[1, i, 0], [1, -i, 0]\}.$$

Example 2.

$X^2 - Y^3$ is an irreducible polynomial in $\mathbb{C}[X, Y]$.

The zero set

$$V = \{(x, y) \mid x^2 - y^3 = 0\}$$

is an affine variety in $\mathbb{A}^2(\mathbb{C})$.

The coordinate ring of V is

$$\mathbb{C}[V] = \mathbb{C}[X, Y]/(X^2 - Y^3).$$

The field $\mathbb{C}(V)$ of rational functions of V is the algebraic extension of $\mathbb{C}(Y)$ by $X^2 - Y^3 = 0$, it has transcendental degree 1 over $\mathbb{C}$, so

$$\dim V = 1.$$

The singular points of V are the solution set of

$$x^2 - y^3 = 0,\ 2x = 0, -3y^2 = 0.$$

The only solution is $(0, 0)$. So all the points in V except $(0, 0)$ are regular points.

The projective closure of V is the solution set (in $\mathbb{P}^2(\mathbb{C})$) of the homogeneous equation

$$\bar{V} = \{[x, y, z] \in \mathbb{P}^2(\mathbb{C}) \mid x^2 z - y^3 = 0\}.$$

Notice that $X^2 Z - Y^3$ is the homogenization of $X^2 - Y^3$.

$$\bar{V} - V = \{[x, y, z] \mid x^2 z - y^3 = 0, z = 0\} = \{[1, 0, 0]\}.$$

Is this new point $[1, 0, 0]$ a regular point in $\bar{V}$?

We do the dehomogenization of $X^2Z - Y^3$ with respect X, i.e., we set $X = 1$, we get $Z - Y^3$. $[1, 0, 0]$ corresponds to $(0, 0)$ in the affine curve

$$V' = \{(y, z) \mid z - y^3 = 0\}.$$

One checks that $(0, 0)$ is regular in V'.

The projective curve $\bar{V}$ has only one singular point.

For an affine variety V with coordinate ring

$$\mathbb{C}[V] = \mathbb{C}[X_1, \ldots, X_n]/I(V).$$

Every $f(X_1, \ldots, X_n) + I(V)$ defines a function

$$V \to \mathbb{C},\ (x_1, \ldots, x_n) \mapsto f(x_1, \ldots, x_n).$$

What is about a rational function $\frac{f+I}{g+I} \in \mathbb{C}(V)$? this element is often written as $\frac{f}{g}$ $(f, g \in \mathbb{C}[X_1, \ldots, X_n])$, it defines a function on

$$V - \{x \in V \mid g(x) = 0\}, \quad x \mapsto \frac{f(x)}{g(x)}.$$

The largest possible domain of f/g may be larger than $V - \{x \in V \mid g(x) = 0\}$. Since f/g may have a different expression $\tilde{f}/\tilde{g}$. Then it defines on $V - \{x \in V \mid \tilde{g}(x) = 0\}$.

The union of all such domains is the largest domain of the rational function f/g.

For a projective variety $V \subset \mathbb{P}^n(\mathbb{C})$ with the ideal $I(V) \subset \mathbb{C}[X_0, X_1, \ldots, X_n]$. The graded ring

$$\mathbb{C}[X_0, X_1, \ldots, X_n]/I(V)$$

are NOT functions on V. Because the points in $\mathbb{P}^n(\mathbb{C})$ is an equivalence class.

$$[x_0, x_1, \ldots, x_n] = [\lambda x_0, \lambda x_1, \ldots, \lambda x_n].$$

For any polynomial $f(X_0, X_1, \ldots, X_n)$, usually

$$f(x_0, x_1, \ldots, x_n) \neq f(\lambda x_0, \lambda x_1, \ldots, \lambda x_n).$$

The graded ring

$$\mathbb{C}[X_0, X_1, \ldots, X_n]/I(V) = \oplus_{k=0}^{\infty} R_k$$

is interpreted as the sum of sections of line bundles on V:

$$R_k = \Gamma(\mathcal{L}^k).$$

We consider the degree 0 elements in

$$\operatorname{Frac} \mathbb{C}[X_0, X_1, \dots, X_n]/I(V),$$

that is, the elements that can be written as

$$\frac{f + I(V)}{g + I(V)}, \quad f \text{ and } g \text{ are homogeneous, } \deg f = \deg g.$$

The quotient is often written as $\frac{f}{g}$.
The collection of all such elements is a subfield, we denote it by

$$\mathbb{C}(V).$$

The set of all the homogenous elements of degree 0 is a subfield of $\mathrm{Frac}\,\mathbb{C}[X]/I(V)$.

It is called the **function field** of projective variety V and is denoted by $\mathbb{C}(V)$.

For

$$\frac{f}{g} \in \mathbb{C}(V), \quad f, g \in \mathbb{C}[X], \ \deg f = \deg g = d$$

if $[x_0, x_1, \ldots, x_n] \in V$ with $g(x_0, x_1, \ldots, x_n) \neq 0$. Then

$$\frac{f(x_0, \ldots, x_n)}{g(x_0, \ldots, x_n)}$$

is independent of the homogeneous coordinates, because

$$f(\lambda x_0, \ldots, \lambda x_n) = \lambda^d f(x_0, \ldots, x_n), \ \ g(\lambda x_0, \ldots, \lambda x_n) = \lambda^d g(x_0, \ldots, x_n).$$

So $\frac{f}{g} \in \mathbb{C}(V)$ defines a function on

$$V - \{[x] \in V \mid g(x) = 0\}$$

which is an open set in V.

The largest domain of $\frac{f}{g}$ is the union of all

$$V - \{[x] \in V \mid \tilde{g}(x) \neq 0\}$$

where $\frac{f}{g} = \frac{\tilde{f}}{\tilde{g}}$.

If $V \subset \mathbb{A}^n(\mathbb{C})$ is an affine variety, $\bar{V} \subset \mathbb{P}^n(\mathbb{C})$ is the projective closure. Then $\mathbb{C}(V) = \mathbb{C}(\bar{V})$ are isomorphic.

Prove for Examples 1, $V = \{(x, y) \mid x^2 + y^2 - 1 = 0\}$ first, then you try to prove the general case.

Morphism between Affine Varieties.

If $V \subset \mathbb{A}^n(\bar{K})$ and $W \subset \mathbb{A}^m(\bar{K})$ are affine varieties. A map $\phi : V \to W$ is called a morphism of affine varieties if there is a polynomial map

$$\Phi : \mathbb{A}^n(\bar{K}) \to \mathbb{A}^m(\bar{K})$$

such that

$$\Phi(V) \subset W$$

and

$$\Phi|_V = \phi : V \to W$$

A polynomial map $\Phi : \mathbb{A}^n(\bar{K}) \to \mathbb{A}^m(\bar{K})$ is a map given by

$$\begin{aligned} y_1 &= p_1(x_1, \ldots, x_n) \\ y_2 &= p_2(x_1, \ldots, x_n) \\ \ldots \\ y_m &= p_m(x_1, \ldots, x_n). \end{aligned}$$

where $p_1, \ldots, p_m \in \bar{K}[X_1, \ldots, X_n]$.

Such a Φ induces a ring homomorphism

$$\Phi^* : \bar{K}[Y] = \bar{K}[Y_1, \ldots, Y_m] \mapsto \bar{K}[X_1, \ldots, X_n] = \bar{K}[X].$$

$$\Phi^*(Y_i) = p_i(X_1, \ldots, X_n).$$

The condition $\Phi(V) \subset W$ implies that

$$\Phi^*(I(W)) \subset I(V).$$

(Prove it! it is just a bit abstract, but not hard)

So we have a $\bar{K}$-algebra homomorphism

$$\phi^* : \bar{K}[Y]/I(W) \to \bar{K}[X]/I(V)$$

that is,

$$\phi^* : \bar{K}[W] \to \bar{K}[V].$$

Conversely if we have $\bar{K}$-algebra homomorphism

$$\psi : \bar{K}[W] \to \bar{K}[V]$$

then we have a morphism

$$\phi : V \to W$$

such that $\phi^* = \psi$.

Proposition.

The set of morphisms $V \to W$ are in one-to-one correspondence with the set of $\bar{K}$-algebra homomorphisms $\bar{K}[W] \to \bar{K}[V]$.

V and W are isomorphic iff $\bar{K}[W]$ and $\bar{K}[V]$ are isomorphic as $\bar{K}$-algebras.

The category of affine varieties over $\bar{K}$ is anti-equivalent to the category of finitely generated commutative $\bar{K}$-algebras that are integral domains.

Morphism between Projective Varieties.

We first discuss the morphism between $\mathbb{P}^n(\bar{K})$ and $\mathbb{P}^m(\bar{K})$.
Given $m+1$ homogeneous polynomials $f_0(X_0,\ldots,X_n),\ldots,f_m(X_0,\ldots,X_n)$.
Assume all f_i have the **same degree** d. Then it defines a map from $\phi:\mathbb{P}^n(\bar{K})\to\mathbb{P}^m(\bar{K})$

$$\phi([x_0,x_1,\ldots,x_n]) = [f_0(x),f_1(x),\ldots,f_m(x)]$$

Because the condition f_i's are homogeneous with the equal degree, so ϕ is well-defined.

The domain of ϕ is the open set

$$D_f \stackrel{\text{def}}{=} \mathbb{P}^n(\bar{K}) - \{[x_0,\ldots,x_n] \mid f_0(x)=f_1(x)=\cdots=f_m(x)=0\}$$

We call such a map a **rational map** from $\mathbb{P}^n(\bar{K})$ to $\mathbb{P}^m(\bar{K})$.

If we have another set of $m+1$ homogeneous polynomials of equal degree

$$g_0(X_0, \ldots, X_n), \ldots, g_m(X_0, \ldots, X_n)$$

So we have a map from

$$D_g \stackrel{\text{def}}{=} \mathbb{P}^n(\bar{K}) - \{[x_0, \ldots, x_n] \mid g_0(x) = g_1(x) = \cdots = g_m(x) = 0\}$$

to $\mathbb{P}^m(\bar{K})$.

Suppose $f_i g_j = f_j g_i$ for all $0 \leq i, j \leq m$, the on the intersection $D_f \cap D_g$, two maps are equal.

These two maps are considered equal. The domain of a rational map is the union of all D_f's.

Definition.

Let $V \subset \mathbb{P}^n(\bar{K})$, $W \subset \mathbb{P}^m(\bar{K})$ be projective varieties. A **rational map** $\phi : V_1 \to V_2$ is a map of the form

$$\phi([x_0, \ldots, x_n]) = [\phi_0(x), \phi_1(x), \ldots, \phi_m(x)]$$

where ϕ_i's are homogeneous polynomials of equal degree, and for every $f \in I(W)$,

$$f(\phi_0(X), \ldots, \phi_m(X)) \in I(V_1).$$

The domain of ϕ includes

$$D_\phi \stackrel{\text{def}}{=} V - \{[x] \,|\, \phi_0(x) = \phi_1(x) = \cdots = \phi_m(x) = 0\}.$$

If we have another rational map defined by another set of homogeneous polynomials $\psi_0, \ldots, \psi_m$ of equal degree and satisfies $\phi_i \psi_j = \phi_j \psi_i$ for all $0 \leq i, j \leq m$, then two maps are equal on

$$D_\phi \cap D_\psi.$$

We consider as the same rational map. The domain of the rational map is the union of all D_ϕ's.

A **morphism** of projective varieties V to W is a rational map $\phi : V \to W$ that has domain V.

V and W are isomorphic if there are morphisms $\phi : V \to W$ and $\psi : W \to V$ such that $\phi \circ \psi = Id_W$ and $\psi \circ \phi = Id_V$.

Exercise 1. Prove the composition of morphisms is a morphism.

Exercise 2. Prove the map $\phi : \{X^2 + Y^2 - Z^2 = 0\} \to \mathbb{P}^1$ given by

$$[x, y, z] \mapsto [xy, z^2]$$

is an isomorphism.

The word "curve" in this Chapter means a projective variety of dimension 1 over $\bar{K}$.

Recall that K is a separable field, $\bar{K}$ is the algebraic closure of K.

§II.1 Curves.

Definition. For a projective variety V with rational function field $\bar{K}(V)$ and a point P, the local ring of P is

$$\bar{K}[V]_P = \{\frac{f}{g} \in \bar{K}(V) \mid g(P) \neq 0\}.$$

If V is an affine variety, $P \in V$, $\bar{V}$ is a projective closure of V, then the local rings

$$\bar{K}[V]_P = \bar{K}[\bar{V}]_P.$$

The function fields $\bar{K}(V)$ and $\bar{K}(\bar{V})$ are equal.

$\bar{K}[V]_P$ is a subring of $\bar{K}(V)$. It has unique maximal ideal given by the kernel of homomorphism

$$ev : \bar{K}[V]_P \to \bar{K}, \quad \frac{f}{g} \mapsto \frac{f(P)}{g(P)}.$$

One proves that every element in $\bar{K}[V]_P - Ker(ev)$ is invertible in $\bar{K}[V]_P$.

Proposition.

Let C be a curve and $P \in C$ a smooth point. Then $\bar{K}[C]_P$ is a discrete value ring.

Proof. We denote $Ker(ev) = M_P$, which is a maximal ideal of $\bar{K}[C]_P$. Because P is smooth so

$$\dim_{\bar{K}} M_P/M_P^2 = \dim C = 1$$

The standard result in commutative algebra implies that $\bar{K}[C]_P$ is a discrete valuation ring.

The valuation is given by

$$\mathrm{ord}_P : \bar{K}[C]_P \to \{0, 1, 2, \dots\} \cup \{\infty\}$$

$$\mathrm{ord}_P(0) \stackrel{\mathrm{def}}{=} \infty$$

$$\text{For } f \notin M_P, \ \ \mathrm{ord}_P(f) \stackrel{\mathrm{def}}{=} 0$$

$$\text{For } f \in M_P, \ \ \mathrm{ord}_P(f) \stackrel{\mathrm{def}}{=} \max(d \mid f \in M_P^d)$$

For any $t \in M_P - M_P^2$, every non-zero element in $\bar{K}[C]_P$ can be written as $t^k u$, where u is a unit, $k \geq 0$. $\mathrm{ord}_P(t^k u) = k$.

Definition.

Let R be an integral domain, a surjective map $e : R \to \{0, 1, 2, \dots\} \cup \{\infty\}$ is called a **discrete valuation** if
(1). $e(a) = \infty$ iff $a = 0$.
(2). $e(fg) = e(f) + e(g)$ for $f, g \neq 0$.
(3). $e(f + g) \geq \min(e(f), e(g))$.
(4). $e(f) = 0$ iff f is a unit in R.

Then R is called a **discrete valuation ring**.

e can be extended to a map $e : \mathrm{Frac}\, R \to \mathbb{Z} \cup \{\infty\}$ by

$$e(\frac{f}{g}) = e(f) - e(g).$$

A discrete valuation ring is a local ring with the unique maximal

$$M = \{a \in R \mid e(a) > 0\}.$$

It is a PID with only two prime ideals $\{0\}$ and M.

Example.

Given a prime p,

$$\mathbb{Z}_{(p)} = \{\frac{n}{m} \mid p \nmid m\}$$

is a discrete valuation ring with valuation

$$e(p^k \frac{a}{b}) = k$$

for a, b relatively prime to p.

Example.

Let k be a field, the formal power series ring $k[[t]]$ is a discrete valuation ring. Notice that $\sum_{i=0}^{\infty} c_i t^i$ is a unit iff $c_0 \neq 0$. Every non-zero element in $k[[t]]$ can be uniquely written as

$$t^k u, \quad k \in \mathbb{Z}_{\geq 0}, u \text{ is a uint.}$$

The evaluation is defined as

$$e(t^k u) = k, \quad e(0) = \infty$$

For example:

$$e(2t^2 + t^4 + \dots) = 2, e(1 + t^2 + \dots) = 0$$

The completion of $\bar{K}[C]_P$ is $\bar{K}[[t]]$.

The field of rational functions $\bar{K}(C)$ is $\operatorname{Frac} \bar{K}[C]_P$. So ord_P extends to

$$\operatorname{ord}_P : \bar{K}(C) \to \mathbb{Z} \cup \{\infty\}.$$

$\operatorname{ord}_P(f)$ is called **the order of f at P**.

Proposition 1.2.

Let C be a smooth curve, $f \in \bar{K}(C)$ and $f \neq 0$, then for all $P \in C$ except finitely many points, $\mathrm{ord}_P(f) = 0$.

Let C be a curve, $P \in C$, $t \in \bar{K}(C)$ is called a **uniformizer at** P if $\mathrm{ord}_P(t) = 1$.

Proposition 1.2'. Assume $\bar{K}$ is of characteristic 0 or positive characteristic p such every $a \in \bar{K}$ is a p-power, i.e., $x^p = a$ is solvable in $\bar{K}$. Let C be a curve over $\bar{K}$, $t \in \bar{K}(C)$ be a uniformizer at some point, then $\bar{K}(C)$ is a finite separable extension of $\bar{K}(t)$.

Proposition 1.2. Assume K is of characteristic 0 or positive characteristic p such every $a \in K$ is a p-power, i.e., $x^p = a$ is solvable in K. Let C be a curve over K, $t \in K(C)$ be a uniformizer at some point, then $K(C)$ is a finite separable extension of $K(t)$.

End