# Math 6170 C, Lecture on Feb 26, 2020

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# Plan.

(1) Review of the concept of rational map and morphisms between projective varieties (I § 3)

(2). Review of local ring at a smooth point of a curve (II. § 1)

(3). Chapter II, § 2.

# Morphisms between Projective Varieties

Let $V_1 \subset \mathbb{P}^n(\bar{K})$, $V_2 \subset \mathbb{P}^m(\bar{K})$ be projective varieties. A rational map from $V_1$ to $V_2$ is a map of the form

$$\phi(P) = [f_0(P), \ldots, f_m(P)]$$

where $f_0, \ldots, f_m \in \bar{K}(V_1)$ have the property that for every $P \in V_1$ at which $f_0, \ldots, f_m$ are all defined, and $f_i(P)$ are not all 0,

$$\phi(P) = [f_0(P), \ldots, f_m(P)] \in V_2 \tag{1}$$

The domain of $\phi$ includes the set

$$\{P \in V_1 \mid \text{all } f_0(P), \ldots, f_1(P) \text{ are defined and at least one is not } 0\}.$$

# Definition.

A rational map $V_1 \to V_2$ given by $[f_0, \ldots, f_m]$ is regular at a point $Q$, if there is a function $g \in \bar{K}(V_1)$ such that each $gf_i$ is regular at $Q$, and for some $i$, $(gf_i)(Q) \neq 0$
If such a $g$ exists, we set

$$\phi(Q) = [(gf_0)(Q), \ldots, (gf_m)(Q)].$$

Exercise: $\phi(Q)$ is well-defined. That is, if there is another set $\tilde{g} \in \bar{K}(V_1)$ satisfying the similar condition, then

$$[(gf_0)(Q), \ldots, (gf_m)(Q)] = [(\tilde{g}f_0)(Q), \ldots, (\tilde{g}f_m)(Q)].$$

Hint: Prove that $\tilde{g}g^{-1}$ and $\tilde{g}^{-1}g$ are both regular at $Q$.

# Local Ring of a Curve at a Smooth Point: An Example

For a curve $C$ over $\bar{K}$, $P \in C$ a smooth point, the local ring $\bar{K}[C]_P$ is a discrete valuation.

We have a surjective valuation map

$$\mathrm{ord}_P : \bar{K}[C]_P \to \{0, 1, \dots\} \cup \{\infty\}$$

It extends to a map

$$\mathrm{ord}_P : \bar{K}(C) \to \mathbb{Z} \cup \{\infty\}$$

$t \in \bar{K}[C]_P$ is called a **uniformizer** if

$$\mathrm{ord}_P(t) = 1.$$

Every nonzero element $f \in \bar{K}(C)$ can be written as

$$f = t^k u, \qquad k \in \mathbb{Z}, u \text{ is a unit in } \bar{K}[C]_P.$$

$$V = \{(x, y) \in \mathbb{A}^2(\bar{K}) \mid y^2 = x(x-1)(x-\lambda)\}$$

Assume $\lambda \neq 0, 1$, every point in $V$ is regular. Its projective closure

$$\bar{V} = \{[x, y, z] \in \mathbb{P}^2(\bar{K}) \mid y^2 z = x(x-z)(x-\lambda z)\}$$

has only one point at infinity: $[0, 1, 0]$, which is also regular.

So $\bar{V}$ is a smooth curve. This is an example of Legendre curve, which is an elliptic curve. Its function field is

$$\bar{K}(V) = \bar{K}(\bar{V}) = \mathrm{Frac}\,\bar{K}[X, Y]/(Y^2 - X(X-1)(X-\lambda))$$

$P = (0, 0) \in V$. The local ring at $P$ is

$$\bar{K}[V]_P = \{\frac{f}{g} \in \bar{K}(V) \mid g(P) \neq 0\}.$$

It is easy to see that $\bar{K}[V]_P$ consists of the elements of the form

$$\frac{f(X,Y)}{g(X,Y)}$$

such that $g(0,0) \neq 0$, i.e., the constant term of $g$ is non-zero. The maximal ideal $M_P$ of $\bar{K}[V]_P$ consists of the elements of the form

$$\frac{f(X,Y)}{g(X,Y)}$$

such that $f(0,0) = 0, g(0,0) \neq 0$.

One finds that $\dim_{\bar{K}} M_P/M_P^2 = 1$, $Y + M_P^2$ is the generator of $M_P/M_P^2$. So

$$\mathrm{ord}_P(Y) = 1$$

Because $X - 1$ and $X - \lambda$ are units in $\bar{K}[V]_P$, so

$$\mathrm{ord}_P(X-1) = \mathrm{ord}_P(X-\lambda) = 0.$$

By relation $Y^2 = X(X-1)(X-\lambda)$, we have

$$\mathrm{ord}_P(X) = 2$$

Exercise (1) Find $\mathrm{ord}_P(Y^2 - X)$

(2) Prove that $\frac{Y^2 - X}{Y^2 + (1-\lambda)X}$ is a regular at $P$

**Proposition 2.1.** Let $C$ be a curve, $V \subset \mathbb{P}^N(\bar{K})$ a variety, $P \in C$ a smooth point, and $\phi : C \to V$ a rational map. Then $\phi$ is regular at $P$. In particular, if $C$ is smooth, then $\phi$ is regular at all points in $C$, so it is a morphism.

*Proof.* Suppose $\phi = [f_0, f_1, \ldots, f_N]$, $f_i \in \bar{K}(C)$, Not all $f_0, \ldots, f_N$ are 0.

Choose a uniformizer $t \in \bar{K}(C)$ at $P$. Then each non-zero $f_i$ can be written as $f_i = t^{k_i} u_i$, $k_i \in \mathbb{Z}$, $u_i \in \bar{K}[C]$ is a unit.

We assume $f_j \neq 0$ and $k_j$ is the smallest among all $k_i$'s then $t^{-k_j} f_0, t^{-k_j} f_1, \ldots, t^{-k_j} f_N$ are all regular at $P$ and $(t^{-k_j} f_j)(P) = u_i(P) \neq 0$.

This proves $\phi$ is regular at $P$.

# Example 2.2.

Let $C/K$ be a smooth curve and $f \in K(C)$ a non-zero rational function. Then $f$ defines a rational map (also denoted by $f$ for simplicity)

$$f : C \to \mathbb{P}^1(\bar{K}), \quad P \mapsto [f(P), 1]$$

The formula makes sense for $P$'s with $\mathrm{ord}_P(f) \geq 0$.

For $P$ a pole of $f$, we take a uniformizer $t$ at $P$, then $f = t^{-k}u$, $k < 0$, $u \in \bar{K}[C]_P$ is a unit, so $u(P) \neq 0$.

$$f : P \mapsto [(t^k f)(P), t^k(P)] = [u(P), 0] = [1, 0].$$

The above map is not the non-constant map $\infty$. Conversely, any morphism that is not constant map $\infty$ is given by a unique non-zero $f \in \bar{K}(C)$.

# Theorem 2.3.

Let $\phi : C_1 \to C_2$ be a morphism of curves. Then $\phi$ is either constant or surjective.

*Proof.* Because $C_1$ is projective, so $\mathrm{Im}(\phi)$ is a closed subset of $C_2$. And $\mathrm{Im}(\phi)$ is connected. Since $C_2$ is a curve, a closed connected subset is either a point or $C_2$ itself.

# Rationality Question.

If a projective variety $V$ is defined over $K$, this means that $V \subset \mathbb{P}^n(\bar{K})$ has the property that its ideal $I(V)$ can be generated by homogeneous polynomials $f_1, \dots, f_m$ in $K[X_0, X_1, \dots, X_n]$.

The function field $K(V)$ over $K$ is defined to be the subfield of $\mathrm{Frac}\, K[X]/(f_1, \dots, f_m)$ that consists of elements of degree 0. If $V_1, V_2$ are both defined over $K$, then one can define the concept of a rational map or a morphism from $V_1$ to $V_2$ **defined over** $K$.

Let $C_1/K$ and $C_2/K$ be curves and $\phi : C_1 \to C_2$ a non-constant rational map defined over $K$. Then $\phi$ induces an field extension

$$\phi^* : K(C_2) \to K(C_1)$$

# Theorem 2.4.

Let $C_1/K$ and $C_2/K$ be curves. **Assume both are smooth** (i.e., all $\bar{K}$-points are smooth. Then

(a). Let $\phi : C_1 \to C_2$ be a non-constant morphism defined over $K$. Then $K(C_1)$ is a finite extension of $K(C_2)$.

(b). Let $\iota : K(C_2) \to K(C_1)$ be an injection of fields fixing $K$. Then there is unique non-constant morphism $\phi : C_1 \to C_2$ defined over $K$ that induces the $\iota$.

(c). If $\mathbb{K}$ is finitely generated extension of $K$ of transcendental degree 1 satisfying $\mathbb{K} \cap \bar{K} = K$, then there exists a unique smooth curve $C$ over $K$ such that $K(C) = \mathbb{K}$.

Proof of (a): Because $K(C_1)$ and $K(C_2)$ have transcendental degree 1 over $K$, and both are finitely generated field extensions of $K$.

The following two categories are anti-equivalent:

Geometric Category: Objects are smooth curves defined over $K$. Morphisms are non-constant rational maps defined over $K$.

Algebaic Category: Objects are finitely generated field extensions $\mathbb{K}$ of $K$ with $\mathbb{K} \cap \bar{K} = K$ and $\mathrm{Tr}\deg \mathbb{K}/K = 1$. Morphisms are field homomorphisms over $K$.

Another example of anti-equivalence of categories:

Geometric Category: objects are compact Hausdorff topological spaces, morphisms are continuous maps.

Algebaic Category: objects are unital commutative $C^*$-algebras, morphisms are $C^*$-homomorphisms.

An easy example of anti-equivalence of categories:

Geometric Category: objects are finite sets, morphisms are maps.

Algebaic Category: objects are unital finite dimensional commutative $\mathbb{C}$-algebras with no non-zero nilpotent algebras, morphisms are $\mathbb{C}$-algebra homomorphisms.

# Definition.

Let $\phi : C_1 \to C_2$ be a non-constant map of curves over $K$, we define the **degree** of $f$ by

$$\deg \phi = [K(C_1) : K(C_2)].$$

# Definition.

Let $\phi : C_1 \to C_2$ be a non-constant map of smooth curves, and let $P \in C_1$. The **ramification index** of $\phi$ at $P$, denoted by $e_\phi(P)$, is given by

$$e_\phi(P) = \mathrm{ord}_P(\phi^* t_{\phi(P)}),$$

where $t_{\phi(P)} \in K(C_2)$ is a uniformizer at $\phi(P)$.

Note that $e_\phi(P) \geq 1$, because

$$C_1 \xrightarrow{\phi} C_2 \xrightarrow{t_{\phi(P)}} \mathbb{P}^1, \quad P \mapsto \phi(P) \mapsto 0$$

We say that $\phi$ is **unramified at** $P$ if $e_\phi(P) = 1$, $\phi$ is unramified if $\phi$ is unramified at every point of $C_1(\bar{K})$.

Complex analytic analog of ramification:

$$f : \mathbb{C} \to \mathbb{C}, \quad f(z) = z^5$$

$f$ is ramified at $z = 0$, the ramification index is 5. $f$ is unramified at all other points: for any $c \neq 0$, any root of the equation $z^5 - c = 0$ has multiplicity 1.

# Proposition 2.6.

Let $\phi$ be a non-constant map of smooth curves.
(a) For every $Q \in C_2$,

$$\sum_{P \in \phi^{-1}(Q)} e_\phi(P) = \deg(\phi).$$

(b) For all but finitely many $Q \in C_2$,

$$|\phi^{-1}(Q)| = \deg_s(\phi).$$

where $\deg_s(\phi)$ is the separable degree of the field extension $\phi^*$.
(c) Let $\psi : C_2 \to C_3$ be another non-constant map. Then for all $P \in C_1$,

$$e_{\psi\circ\phi}(P) = e_\phi(P) e_\psi(\phi P)$$

# *Sketch of Proof of (a).*

Consider the field extension $\bar{K}(C_2) \subset \bar{K}(C_1)$ induced by $\phi$. Let $R$ be the integral closure of the local ring $\bar{K}[C_2]_Q$ in $\bar{K}(C_1)$, then $R$ is a free module of $\bar{K}[C_2]_Q$ with rank $\deg(\phi)$. $R$ has exactly $|\phi^{-1}(Q)|$ maximal ideals, each corresponds a point in $\phi^{-1}(Q)$. Then consider $R/RM_Q$ as a vector space of $\bar{K}[C_2]_Q/M_Q = \bar{K}$ with dimension $\deg(\phi)$. Then prove

$$R/RM_Q = \oplus_{P \in \phi^{-1}(Q)} (\bar{K})^{e_\phi(P)}.$$

This will prove (a).

# Corollary.

A map $\phi : C_1 \to C_2$ is unramified iff $|\phi^{-1}(Q)| = \deg(\phi)$ for all $Q \in C_2$.

# The Frobenius Map.

If $\mathrm{char}(K) = p > 0$, and let $q = p^r$. For any $n$-variable polynomial $f \in K[X]$, let $f^{(q)}$ be the polynomial obtained from $f$ by raising each coefficient of $f$ to the $q$-th power.

Then for any curve $C/K$ we can define a new curve $C^{(q)}/K$ by describing its homogeneous ideal as

$$I(C^{(q)}) = \text{ideal generated by } f^{(q)}, \;\; f \in I(C)$$

There is a natural map from $C$ to $C^{(q)}$, called the $q$-power Frobenius morphism, given by

$$\phi([x_0, \ldots, x_n]) = [x_0^q, \ldots, x_n^q]$$

Then

$$\phi^* K(C^{(q)}) = \{f^q \mid f \in K(C)\}$$

so the field extension $K(C^{(q)}) \subset K(C)$ is a purely inseparable of degree $q$.

Conversely of $\phi : C \to C'$ is a non-constant morphism of smooth curves over $K$ such that $\phi^* : K(C') \to K(C)$ is a purely inseparable extension of degree $q$ , then $C' = C^{(q)}$ and $\phi$ is the $q$-power Frobenius map.

Let $C$ be a curve over $\bar{K}$. A divisor of $C$ is a formal finite $\mathbb{Z}$-linear combination of points in $C$:

$$n_1(P_1) + \cdots + n_k(P_k).$$

This sum can be regarded as the sum over all the points $P$ in $C$

$$\sum_{P \in C} n_P(P)$$

such that for $P = P_i$, $n_P = n_i$ and all other $k_P$ are 0.

The set of all divisors is denoted by

$$\mathrm{Div}(C)$$

which has a group structure under the obvious addition.

The degree of $D = \sum_{P \in C} n_P(P)$ is

$$\deg D = \sum_{P \in C} n_P.$$

# End