Math 6170 C, Lecture on March 2, 2020

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Plan.

(1) Example of Curve: $\mathbb{P}^1$

(2). Chapter II. § 3. Divisors

(3). Chapter II, § 4. Differentials

Curve $\mathbb{P}^1$

$C = \mathbb{P}^1(\bar{K}) \subset \mathbb{P}^1(\bar{K})$. Its ideal is $I(C) = \{0\}$. Its function field $\bar{K}(C)$ is the degree 0 elements in $\mathrm{Frac}\ \bar{K}[X, Y]$, i.e.,

$$\bar{K}(C) = \{\frac{f(X,Y)}{g(X,Y)} \mid f,\ g \text{ homogeneous},\ \text{and } \deg f = \deg g\}.$$

$$\frac{f(X,Y)}{g(X,Y)} \mapsto \frac{f(X,1)}{g(X,1)}$$

identify $\bar{K}(C)$ with

$$\mathrm{Frac}\ \bar{K}[X]$$

This agrees with the fact that the projective closure of $\mathbb{A}^1(\bar{K})$ is $\mathbb{P}^1(\bar{K})$ The function field of $\mathbb{A}^1(\bar{K})$ is $\mathrm{Frac}\ \bar{K}[X] = \bar{K}(X)$. As a set

$$C = \mathbb{P}^1(\bar{K}) = \mathbb{A}^1(\bar{K}) \cup \{\infty\} = \bar{K} \cup \{\infty\}$$

where ∞ has homogeneous coordinate $[1,0]$.

If $P = a \in \bar{K}$ is a point in C, the local ring at P is

$$\bar{K}[C]_P = \{\frac{f(X)}{g(X)} \mid g(a) \neq 0\}.$$

A uniformizer at P is $X - a$.

What is the local ring of ∞? what is a uniformizer at ∞?

Recall $\infty = [1,0]$, we identify the function field of $C = \mathbb{P}^1(\bar{K})$ with $\mathrm{Frac}\,\bar{K}[Y] = \bar{K}(Y)$ by

$$\frac{f(X,Y)}{g(X,Y)} \mapsto \frac{f(1,Y)}{g(1,Y)}$$

The identification of $\bar{K}(X) \sim \bar{K}(Y)$ is given by

$$f(X) \mapsto f(Y^{-1}).$$

The local ring of ∞ in $\bar{K}(Y)$ is

$$\{\frac{h(Y)}{g(Y)} \mid g(0) \neq 0\}$$

A uniformizer at ∞ is Y.

In the setting $\bar{K}(X)$, $\frac{1}{X}$ is a uniformizer at ∞ and the local ring of ∞ is

$$\{\frac{f(X)}{g(X)} \mid \deg f \leq \deg g\}.$$

$$\mathrm{ord}_\infty(\frac{f(X)}{g(X)}) = \deg g - \deg f$$

$\mathbb{P}^1(\bar{K})$ is defined on K, its function field over K is

$$K(C) = K(X).$$

The Galois group $G_{\bar{K}/K}$ acts on $\mathbb{P}^1(\bar{K})$, the fixed point is the K-rational points $\mathbb{P}^1(K) = K \cup \{\infty\}$.

The group $GL_2(\bar{K})$ acts on $\mathbb{P}^1(\bar{K})$ as automorphisms of projective varieties:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [x, y] = [ax + by, cx + dy]$$

In the identification $\mathbb{P}^1(\bar{K}) = \bar{K} \cup \{\infty\}$, the above action is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax+b}{cx+d}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \infty = \frac{a}{c}$$

$C_1 = \mathbb{P}^1(\bar{K})$ and $C_2 = \mathbb{P}^1(\bar{K})$.

Let X_1 be a uniformizer of $0 \in \mathbb{A}^1 \subset \mathbb{P}^1(\bar{K})$, so $\bar{K}(C_1) = \bar{K}(X_1)$.

Similarly let X_2 be a uniformizer of $0 \in \mathbb{A}^1 \subset \mathbb{P}^1(\bar{K})$, so $\bar{K}(C_2) = \bar{K}(X_2)$.

Let $h(X_1) = a_n X_1^n + \cdots + a_1 X_1 + a_0$ be a degree n polynomial, $n \geq 1$. h gives a morphism $\phi : C_1 \to C_2$ given by

$$\phi(a) = h(a) \quad \text{for } a \in \bar{K}; \qquad \phi(\infty) = \infty.$$

It induces the field extension

$$\phi^* : \bar{K}(X_2) \to \bar{K}(X_1)$$

given by

$$\frac{f(X_2)}{g(X_2)} \mapsto \frac{f(h(X_1))}{g(h(X_1))}$$

What is $\deg \phi$?

$\deg \phi = n$.

Method 1. Compute $[\bar{K}(X_1) : \bar{K}(X_2)]$,

$$\bar{K}(X_1) = \bar{K}(X_2)[X_1]$$

The minimal polynomial of X_1 over $\bar{K}(X_2)$ is

$$a_n Y^n + \cdots + a_1 Y + a_0 - X_2$$

Method 2 (Assume $\mathrm{Char}\, K = 0$). For almost all $a \in \bar{K}$, the equation

$$\phi(X_1) = a$$

equivalently

$$a_n X_1^n + \cdots + a_1 X_1 + a_0 = a$$

has exactly n solutions.

$\phi(\infty) = \infty$. $\phi^{-1}(\infty) = \{\infty\}$.

$e_\phi(\infty) = n$

Exercise: find all the ramified points and the ramification indices.

Let C be a curve over $\bar{K}$. A divisor of C is a formal $\mathbb{Z}$-linear combination of points in C:

$$\sum_{P \in C} n_P (P)$$

such that almost all coefficients n_P are 0

The set of all divisors is denoted by

$$\mathrm{Div}(C)$$

which has a group structure under the obvious addition

$$D_1 = 2(P_1) - 3(P_2), \quad D_2 = -(P_2) + 5(P_3)$$

Then

$$D_1 + D_2 = 2(P_1) - 4(P_2) + 5(P_3)$$

The degree of $D = \sum_{P \in C} n_P(P)$ is

$$\deg D = \sum_{P \in C} n_P.$$

Assume C is smooth, $f \in \bar{K}(C)$, $f \neq 0$.

$$\mathrm{div}(f) = \sum_{P \in C} \mathrm{ord}_P(f)\,(P).$$

This defines a group homomorphism

$$\mathrm{div} : \bar{K}(C)^* \to \mathrm{Div}(C).$$

because

$$\mathrm{ord}_P(fg) = \mathrm{ord}_P(f) + \mathrm{ord}_P(g)$$

Proposition 3.1.

Let C be a smooth curve over $\bar{K}$ and $f \in \bar{K}(C)^*$. Then
(a) $\mathrm{div}(f) = 0$ iff $f \in \bar{K}^*$.
(b) $\deg(\mathrm{div}(f)) = 0$.

Proof of (a). If $f \in \bar{K}^*$, then $\mathrm{ord}_P(f) = 0$ for all $P \in C$, so $\mathrm{div}(f) = 0$. Conversely, if $\mathrm{div}\, f = 0$. f gives a morphism $f : C \to \mathbb{P}^1(\bar{K})$, this maps doesn't take value ∞ as $\mathrm{div}\, f = 0$. So f is a constant. SO $f \in \bar{K}^*$.

(b) will be proved later.

A divisor of the form $\mathrm{div}(f)$ is called a **principal divisor**.

The set of all principal divisors is a subgroup of $\mathrm{Div}\, C$. It is the image of the group homomorphism:

$$\mathrm{div} : \bar{K}(C)^* \to \mathrm{Div}(C)$$

The **divisor class group** or **Picard group** of C, denoted by $\mathrm{Pic}(C)$, is defined to be the quotient

$$\mathrm{Div}(C)/\{\mathrm{div}(f) \mid f \in \bar{K}(C)^*\}$$

Definition.

The **degree** 0 **part of the divisor class group** of C, which we denote by $\mathrm{Pic}^0(C)$, is the quotient of $\mathrm{Div}^0(C)$ by $\{\mathrm{div}(f) \mid f \in \bar{K}(C)^*\}$.

We have the exact sequence

$$1 \to \bar{K}^* \to \bar{K}(C)^* \to \mathrm{Div}^0(C) \to \mathrm{Pic}^0(C) \to 0.$$

The analogous exact sequence for a number field F with the ring of integers R is

$$1 \to \mathrm{units}(R) \to F^* \to \text{fractional ideals} \to \text{ideal class group of } F \to 1$$

Idea: Number fields and function fields $\bar{K}(C)$ have lots in common.

A better analog involves Arakelov theory for number fields (arithmetic curves).

Let $\phi : C_1 \to C_2$ be a non-constant morphism of smooth curves

ϕ induces a field embedding

$$\phi^* : \bar{K}(C_2) \to \bar{K}(C_1)$$

and

$$\phi_* : \bar{K}(C_1) \to \bar{K}(C_2)$$

which is the norm map of the embedding ϕ^*.

We define a group homomorphism

$$\phi^* : \mathrm{Div}(C_2) \to \mathrm{Div}(C_1)$$

$$\phi^*(Q) = \sum_{P \in \phi^{-1}(Q)} e_\phi(P)(P).$$

and a group homomorphism

$$\phi_* : \mathrm{Div}(C_1) \to \mathrm{Div}(C_2)$$

$$\phi_*(P) = (\phi P)$$

Proposition 3.6.

Let $\phi : C_1 \to C_2$ be a non-constant morphism of smooth curves. Then

(a) $\deg(\phi^* D) = (\deg \phi)(\deg D)$ for $D \in \mathrm{Div}(C_2)$.

(b) $\phi^*(\mathrm{div}\, f) = \mathrm{div}(\phi^* f)$ for $f \in \bar{K}(C_2)^*$.

(c) $\deg(\phi_* D) = \deg D$ for $D \in \mathrm{Div}(C_1)$.

(d) $\phi_*(\mathrm{div}\, f) = \mathrm{div}(\phi_* f)$ for $f \in \bar{K}(C_1)$.

(e) $\phi_* \circ \phi^* : \mathrm{Div}(C_2) \to \mathrm{Div}(C_2)$ is the multiplication by $\deg \phi$.

(f) If $\psi : C_2 \to C_3$ is another such map, then

$$(\psi \circ \phi)^* = \phi^* \circ \psi^*, \quad (\psi \circ \phi)_* = \psi_* \circ \phi_*$$

From the Proposition, we see that ϕ^* and ϕ_* take divisors of degree 0 to divisors of degree 0, and the principal divisors to principal divisors. So they induces maps

$$\phi^* : \mathrm{Pic}^0(C_2) \to \mathrm{Pic}^0(C_1), \qquad \phi_* : \mathrm{Pic}^0(C_1) \to \mathrm{Pic}^0(C_2)$$

Proof of $\deg \operatorname{div}(f) = 0$ for $f \in \bar{K}(C)^*$.

$f : C \to \mathbb{P}^1(\bar{K})$, it is easy to see that

$$\operatorname{div}(f) = f^*((0) - (\infty)) = f^*(0) - f^*(\infty)$$

So

$$\deg \operatorname{div}(f) = \deg f^*(0) - \deg f^*(\infty) = \deg f - \deg f = 0.$$

Rationality Issues:

If C is defined over K, then the Galois group $G_{\bar{K}/K}$ acts on C, so $G_{\bar{K}/K}$ acts on $\mathrm{Div}(C)$: for $\sigma \in G_{\bar{K}/K}$,

$$(n_1(P_1) + \cdots + n_k(P_k))^\sigma \stackrel{\text{def}}{=} n_1(P_1^\sigma) + \cdots + n_k(P_k^\sigma).$$

A divisor

$$D = n_1(P_1) + \cdots + n_k(P_k) \in \mathrm{Div}(C)$$

is **defined over** K if D is fixed by all $\sigma \in G_{\bar{K}/K}$.

This doesn't mean each $P_i \in C(K)$.

For example, $K = \mathbb{R}$, $\bar{K} = \mathbb{C}$, $G_{\mathbb{C}/\mathbb{R}} = \{e, \tau\}$.

$$(2 + i) + (2 - i) \in \mathrm{Div}(\mathbb{P}^1)$$

is defined over $\mathbb{R}$, but $2 \pm i \notin \mathbb{P}^1(\mathbb{R})$

The set of divisors defined over K is denoted by $\mathrm{Div}_K(C)$.

Similarly $G_{\bar{K}/K}$ acts on $\mathrm{Pic}^0(C)$, the fixed point set is denoted by

$$\mathrm{Pic}^0_K(C)$$

Definition. Let C be a curve. The space of meromorphic differential forms on C, denoted by Ω_C, is the $\bar{K}(C)$-vector space generated by symbols of the form df for $f \in \bar{K}(C)$, subject to the following three relations:

(1). $d(f+g) = df + dg$

(2) $d(fg) = gdf + fdg$

(3) $da = 0$ for $a \in \bar{K}$.

Let $\phi : C_1 \to C_2$ be a non-constant map of curves, $\phi^* : \bar{K}(C_2) \to \bar{K}(C_1)$ is the corresponding field extension. It induces a $\bar{K}$-linear map on differentials:

$$\phi^* : \Omega_{C_2} \to \Omega_{C_1}$$

$$\phi^*(fdg) = (\phi^* f)d(\phi^* g)$$

Proposition 4.2.

Let C be a curve.

(a) Ω_C is a 1-dimensional $\bar{K}(C)$-vector space.

(b) Let $x \in \bar{K}(C)$. Then dx is a $\bar{K}(C)$ basis for Ω_C iff $\bar{K}(C)/\bar{K}(x)$ is a finite separable extension.

(c) Let $\phi : C_1 \to C_2$ be a non-constant morphism. Then ϕ is separable (equivalently $\bar{K}(C_1)/\bar{K}(C_2)$ is a separable extension) iff

$$\phi^* : \Omega_{C_2} \to \Omega_{C_1}$$

is injective.

The proof uses the following formulas:
For $y \in \bar{K}(C)$, $c \in \bar{K}$,

$$d(ky) = kdy$$

That is, the symbol is $\bar{K}$-linear.

For $y \in \bar{K}(C)$,

$$d(y^n) = ny^{n-1}dy$$

For $P(y) \in \bar{K}[y]$,

$$dP(y) = P'(y)dy.$$

Proposition 4.3.

Let C be a smooth curve, $P \in C$, $t \in \bar{K}(C)$ be a uniformizer at P.

(a) For every $\omega \in \Omega_C$, there exists a unique $g \in \bar{K}(C)$ such that

$$\omega = g\, dt.$$

We denote g by ω/dt.

(b) Let $f \in \bar{K}(C)$ be regular at P, then df/dt is also regular at P.

(c) The number $\mathrm{ord}_P(\omega/dt)$ does not depends the choice of uniformizer t. We call this value the order of ω at P, and also denote it by $\mathrm{ord}_P(\omega)$.

End