Math 6170 C, Lecture on March 4, 2020

Yongchang Zhu

Plan.

(1) Computation of ramification index: an example

(2). Review of Chapter II, § 4. Differentials

(3). Chapter II, § 5. The Riemann-Roch Theorem

Computation of ramification index: an example

$C_1 = \mathbb{P}^1(\bar{K})$ with the function field $\bar{K}(C_1) = \bar{K}(X)$;

$C_2 = \mathbb{P}^1(\bar{K})$ with the function field $\bar{K}(C_2) = \bar{K}(Y)$

A morphism

$$\phi : C_1 \to C_2, \qquad Y = \phi(X) = f(X)$$

where $f(X) = \frac{M(X)}{N(X)} \in \mathbb{C}(X)$,
where $M(X), N(X) \in \bar{K}[x]$ are relatively prime polynomials.

If $a \in \bar{K}$ satisfies $N(a) \neq 0$, then $\phi(a) = \frac{M(a)}{N(a)}$.

If $a \in \bar{K}$ satisfies $N(a) = 0$, then $\phi(a) = \infty$.

If $a = \infty$, $\deg M > \deg N$, then $\phi(a) = \infty$

If $a = \infty$, $\deg M < \deg N$, then $\phi(a) = 0$

If $a = \infty$, $\deg M = \deg N$, then $\phi(a) = \frac{c}{d}$,
where c is the leading coefficient of M and d is the leading coefficient of N.

So formally we have

$$\phi(\infty) = \lim_{a\to\infty} \frac{M(a)}{N(a)}.$$

The corresponding field embedding $\phi^* : \bar{K}(C_2) = \bar{K}(Y) \to \bar{K}(C_1) = \bar{K}(X)$ is

$$Y \mapsto \frac{M(X)}{N(X)}.$$

$$h(Y) \mapsto h\left(\frac{M(X)}{N(X)}\right)$$

How do we compute ramification indices $e_\phi(a)$?

Assume $a \in \bar{K}$ and $\phi(a) \in \bar{K}$.
Take a uniformizer at $\phi(a)$, say $Y - \phi(a) = Y - \frac{M(a)}{N(a)}$,

By the definition of $e_\phi(a)$, we have

$$e_\phi(a) = \mathrm{ord}_a \left(\phi^*(Y - \phi(a))\right)$$

$$\begin{aligned}\phi^*(Y-\phi(a)) &= \frac{M(X)}{N(X)} - \frac{M(a)}{N(a)} \\ &= \frac{N(a)M(X)-M(a)N(X)}{N(a)N(X)}\end{aligned}$$

$$\begin{aligned}e_\phi(a) &= \mathrm{ord}_a\left(\phi^*(Y-\phi(a))\right) \\ &= \mathrm{ord}_a\left(N(a)M(X)-M(a)N(X)\right) - \mathrm{ord}_a(N(a)N(X)) \\ &= \mathrm{ord}_a\left(N(a)M(X)-M(a)N(X)\right)\end{aligned}$$

How to compute $e_\phi(\infty)$?

Case 1. $\phi(\infty) \in b \in \bar{K}$
Case 2. $\phi(\infty) = \infty \in \bar{K}$

Case 1. A uniformizer of b is $Y - b$, compute $\mathrm{ord}_\infty(\frac{M(X)}{N(X)} - b)$

Case 2. A uniformizer of ∞ is Y^{-1}, compute $\mathrm{ord}_\infty(\frac{N(X)}{M(X)})$

Definition. Let C be a curve. The space of meromorphic differential forms on C, denoted by Ω_C, is the $\bar{K}(C)$-vector space generated by symbols of the form df for $f \in \bar{K}(C)$, subject to the following three relations:

(1). $d(f+g) = df + dg$

(2) $d(fg) = gdf + fdg$

(3) $da = 0$ for $a \in \bar{K}$.

The following formula is useful:
suppose $X, Y \in \bar{K}(C)$,

$$d(X^m Y^n) = mX^{m-1}Y^n dX + nX^m Y^{n-1} dY$$

More generally, for a polynomial $P(X, Y)$ of X, Y, we have

$$dP(X, Y) = \partial_X P(X, Y) dX + \partial_Y P(X, Y) dy.$$

The same formula holds for $P(X, Y)$ a rational expression of X and Y.

Proposition 4.2.

Let C be a curve.

(a) Ω_C is a 1-dimensional $\bar{K}(C)$-vector space.

Proposition 4.2 (continued).

(b) Let $x \in \bar{K}(C)$. Then dx is a $\bar{K}(C)$ basis for Ω_C iff $\bar{K}(C)/\bar{K}(x)$ is a finite separable extension.

Proposition 4.2 (continued).

(c) Let $\phi : C_1 \to C_2$ be a non-constant morphism. Then ϕ is separable (equivalently $\bar{K}(C_1)/\bar{K}(C_2)$ is a separable extension) iff

$$\phi^* : \Omega_{C_2} \to \Omega_{C_1}$$

is injective.

Proposition 4.3.

Let $P \in C$, t be a uniformizer at P.

(a) For any $\omega \in \Omega_C$, there exists a unique $g \in \bar{K}(C)$ such that

$$\omega = g dt$$

We denote g by $\frac{\omega}{dt}$.

Proposition 4.3 (continued).

(b) If $f \in \bar{K}(C)$ is regular at P, then $\frac{df}{dt}$ is regular at P.

Proposition 4.3 (continued).

(c) The quantity $\mathrm{ord}_P(\omega/dt)$ is independent of t. We call it the order of ω at P and denote it by $\mathrm{ord}_P(\omega)$.

Proposition 4.3 (continued).

(d) Let $x \in \bar{K}(C)$ such that $\bar{K}(C)/\bar{K}(x)$ is separable and $x(P) = 0$. Then for all $f \in \bar{K}(C)$,

$$\mathrm{ord}_P(fdx) = \mathrm{ord}_P(f) + \mathrm{ord}_P(x) - 1.$$

(e) For all but finitely many $P \in C$,

$$\mathrm{ord}_P(\omega) = 0.$$

Definition.

Let $\omega \in \Omega_C$, $\omega \neq 0$. **The divisor associated to** ω is

$$\operatorname{div}(\omega) = \sum_{P \in C} \operatorname{ord}_P(\omega)(P) \in \operatorname{Div}(C)$$

Definition.

A differential $\omega \in \Omega_C$ is **regular (or holomorphic)** if

$$\mathrm{ord}_P \omega \geq 0 \quad \text{for all } P \in C.$$

It is clear that

$$\mathrm{div}(f\omega) = \mathrm{div}(f) + \mathrm{div}(\omega).$$

If ω_1, ω_2 are nonzero elements in Ω_C, then

$$\mathrm{div}(\omega_1) - \mathrm{div}(\omega_2)$$

is a principal divisor. The image of $\mathrm{div}(\omega)$ in $\mathrm{Pic}(C)$ is independent of the choice of nonzer0 $\omega \in \Omega_C$, we call the **canonical divisor class** on C. Any divisor in this divisor class is called a **canonical divisor**.

Example: $C = \mathbb{P}^1(\bar{K})$, $\bar{K}(C) = \bar{K}(X)$.

$$dX \in \Omega_C$$

For $a \in \bar{K} \subset \mathbb{P}^1(\bar{K})$, $X - a$ is a uniformizer at a,

$$d(X - a) = dX - da = dX$$

$$\mathrm{ord}_a(dX) = 0$$

$t \stackrel{\text{def}}{=} \frac{1}{X}$ is a uniformizer at ∞, so

$$dt = d(\frac{1}{X}) = -\frac{1}{X^2}dX, \quad dX = -X^2 dt$$

$$\mathrm{ord}_\infty(dX) = \mathrm{ord}_\infty(-X^2) = \mathrm{ord}_\infty(X^2) = -2.$$

$-2(\infty)$ is a canonical divisor of $\mathbb{P}^1(\bar{K})$.

Let C be a curve. We define a partial order on $\mathrm{Div}(C)$:

$$D = \sum n_P(P) \geq 0 \quad \text{iff all } n_P \geq 0$$

$$D_1 \geq D_2 \quad \text{iff } D_1 - D_2 \geq 0.$$

Example. $D_1 = 2(P_1) - 3(P_2) + 10(P_3)$

$D_2 = (P_1) - 5(P_2) + 9(P_3)$

$D_3 = 3(P_1) + 9(P_3)$

$$D_1 \geq D_2, \quad D_3 \geq D_2,$$

D_1 and D_3 are not comparable.

Definition.

For $D \in \mathrm{Div}(C)$, we associate to D the space of functions

$$\mathcal{L}(D) \stackrel{\mathrm{def}}{=} \{f \in \bar{K}(C)^* \mid \mathrm{div}(f) \geq D\} \cup \{0\}$$

equivalently

$$\mathcal{L}(D) \stackrel{\mathrm{def}}{=} \{f \in \bar{K}(C) \mid \mathrm{div}(f) \geq D\}$$

here we use the convention that $\mathrm{div}(0) = \sum_P \infty(p) > D$ for any $D \in \mathrm{Div}(C)$.

$\mathcal{L}(D)$ is a vector space over $\bar{K}$,

because $\operatorname{div}(cf) = \operatorname{div}(f)$ for $c \in \bar{K}^*$.

$$\operatorname{ord}_P(f+g) \geq \min(\operatorname{ord}_P(f), \operatorname{ord}_P(g))$$

implies that $\mathcal{L}(D)$ is closed under $+$.

Example. $C = \mathbb{P}^1(\bar{K})$, $\quad D = 3(1) - (2)$

$$\mathcal{L}(D) = \{f \in \bar{K}(X) \,|\, \mathrm{ord}_1(f) \geq -3,\ \mathrm{ord}_2(f) \geq 1,\ \mathrm{ord}_P(f) \geq 0 \mathrm{for}\ P \neq 1, 2\}$$

$$\mathcal{L}(D) = \{\frac{(X-2)(aX^2 + bX + c)}{(X-1)^3} \mid a, b, c \in \bar{K}\}$$

Proposition 5.2.

Let $D \in \mathrm{Div}(C)$. $l(D) = \dim_{\bar{K}} \mathcal{L}(D)$

(a) If $\deg D < 0$, then $\mathcal{L}(D) = \{0\}$, $l(D) = 0$.

Proof. If not, there is $f \in \bar{K}(C)^*$, $\mathrm{div}(f) \geq -D$, this implies

$$0 = \deg(\mathrm{div}(f)) \geq \deg(-D) = -\deg(D) > 0$$

Contradiction.

Proposition 5.2 (continued).

(b) $\dim_{\bar{K}} \mathcal{L}(D) < \infty$

Proposition 5.2 (continued).

(c) D_1 and D_2 are linearly equivalent, i.e., $D_1 = D_2 + \mathrm{div}(g)$ for some $f \in \bar{K}(C)^*$, then

$$\mathcal{L}(D_1) \sim \mathcal{L}(D_2), \quad l(D_1) = l(D_2)$$

Proof. The linear map

$$\mathcal{L}(D_1) \to \mathcal{L}(D_2), \; ; f \mapsto fg$$

is an isomorphism.

Let $\omega \in \Omega_C$, $\omega \neq 0$,
so $K = \mathrm{div}(\omega)$ is a canonical divisor.

$$\begin{aligned} \mathcal{L}(K) &= \{f \in \bar{K}(C) \mid \mathrm{div}(f) \geq -\mathrm{div}(\omega)\} \\ &= \{f \in \bar{K}(C) \mid \mathrm{div}(f\omega) \geq 0\} \end{aligned}$$

So $\mathcal{L}(K)\omega$ is the space of holomorphic differentials on C.

Definition

Let C be a smooth curve, the **genus of** C, denoted by g, is defined to be

$$g \stackrel{\text{def}}{=} l(K)$$

The genus is equal to the dimension of the space of holomorphic differentials on C.

Theorem 5.4. Riemann-Roch Theorem.

Let C be a smooth curve and K be a canonical divisor on C.Then for every divisor D,

$$l(D) - l(K - D) = \deg D + 1 - g$$

Corollary 5.5.

Let K be a canonical divisor of C, then

$$\deg K = 2g - 2$$

If $\deg D > 2g - 2$, then

$$l(D) = \deg D + 1 - g.$$

Proof.

In $l(D) - l(K - D) = \deg D + 1 - g$
we take $D = K$, we get

$$g - 1 = \deg K + 1 - g, \quad \deg K = 2g - 2$$

For $C = \mathbb{P}^1$, $K = -2(\infty)$ is a canonical divisor, so

$$\deg K = -2$$

The formula $\deg K = 2g - 2$ implies that $g = 0$.

The genus of $\mathbb{P}^1$ is 0.

Proposition 5.8.

If the smooth curve C is defined over K, $D \in Div_K(C)$, then $\mathcal{L}(D)$ has a basis consisting of functions on $K(C)$.

End