Math 6170 C, Lecture on May 13 , 2020

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Plan

(1) Brief Review of Concept of New Forms

(1) XI (Knapp). Eichler-Shimura Theory (continued)

A cusp form f in $S_k(\Gamma_0(N))$ is called an eigenform if $T(n)f = \lambda_n f$ for all positive integers n with $gcd(n, N) = 1$.

A new form $f \in S_k(\Gamma_0(N))$ is an eigenform that is orthogonal to every eigenform constructed from $S_k(\Gamma_0(N/r))$, where $r > 1$ is a divisor of N.

Theorem. A new form f is also an eigenvector for all $T(n)$. If f has q-expansion

$$f = \sum_{n=1}^{\infty} c_n q^n$$

$c_1 = 1$, then $T(n)f = c_n f$.

XI. Eichler-Shimura Theory.

The theory gives for each new form $f \in S_2(\Gamma_0(N))$ with q-expansion

$$f = \sum_{n=1}^{\infty} c_n q^n, \quad c_1 = 1, \ c_n \in \mathbb{Z}$$

an elliptic curve E over $\mathbb{Q}$. The L function of E and f coincide as Euler products.

Definition. An element $\gamma \in \Gamma_0(N)$ is called an **elliptic element** if $|\mathrm{Tr}\,\gamma| < 2$. An element $\gamma \in \Gamma_0(N)$ is called a **parabolic element** if $|\mathrm{Tr}\,\gamma| = 2$.

Lemma (1) $\gamma \in \Gamma_0(N)$ is an elliptic element iff $\gamma \neq \pm 1$ and is of finite order .

(2) If $\gamma \in \Gamma_0(N)$ is a parabolic element, then γ fixes an element in $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \sqcup \{\infty\}$.

Fix a base point $\tau_0 \in \mathcal{H}$. For $f \in S_2(\Gamma_0(N))$, we define the contour integral

$$F(\tau) = \int_{\tau_0}^{\tau} f(z)dz$$

where we take any contour from τ_0 to τ.

We let, for $\gamma \in \Gamma_0(N)$,

$$\Phi_f(\gamma) = \int_{\tau_0}^{\gamma(\tau_0)} f(z)dz$$

Recall the formula for a change of variable for contour integrals:

$$\int_C f(z)dz = \int_{C'} f(\phi(w))\phi'(w)dw$$

where $z = \phi(w)$ is a bi-holomorphic map that transform the contour C' to C.

Lemma. $\Phi_f(\gamma)$ is independent of the choice of the base point τ_0.

Proof. We need to prove

$$\int_{\tau_0}^{\gamma(\tau_0)} f(z)dz - \int_{\tau_1}^{\gamma(\tau_1)} f(z)dz = 0$$

it is equivalent to prove

$$\int_{\tau_0}^{\tau_1} f(z)dz = \int_{\gamma(\tau_0)}^{\gamma(\tau_1)} f(z)dz$$

which is true by a change of variable and the invariance of $f(z)dz$ under γ.

Proposition 11.1

For $f \in S_2(\Gamma_0(N))$, Φ_f is a homomorphism of $\Gamma_0(N)$ into the additive group of $\mathbb{C}$. If γ is elliptic or parabolic, then $\Phi_f(\gamma) = 0$.

Proof.

$$\Phi_f(\gamma_1\gamma_2) = \int_{\tau_0}^{\gamma_1\gamma_2(\tau_0)} f(z)dz$$
$$= \int_{\tau_0}^{\gamma_1(\tau_0)} f(z)dz + \int_{\gamma_1(\tau_0)}^{\gamma_1\gamma_2(\tau_0)} f(z)dz$$
$$= \int_{\tau_0}^{\gamma_1(\tau_0)} f(z)dz + \int_{\tau_0}^{\gamma_2(\tau_0)} f(z)dz$$
$$= \Phi_f(\gamma_1) + \Phi_f(\gamma_2)$$

Let Λ be the image of Φ_f, when $\dim S_2(\Gamma_0(N))$, Λ is a lattice in $\mathbb{C}$.

The map $F : \mathcal{H} \to \mathbb{C}$

$$F(\tau) = \int_{\tau_o}^{\tau} f(z)dz$$

induce a map

$$\Gamma_0(N)\backslash\mathcal{H} \to \mathbb{C}/\Lambda.$$

By adding cusps to $\Gamma_0(N)\backslash\mathcal{H}$, we obtain a compact Riemann surface $X_0(N)$.

We first construct $X_0(N)$ as a topological space.

Put

$$\mathcal{H}^* = \mathcal{H} \sqcup \mathbb{Q} \sqcup \{\infty\}$$

and topologize $\mathcal{H}^*$ as follows: A basic open neighborhood about $\tau \in \mathcal{H}$ is an open set wholly within $\mathcal{H}$, A basic open neighborhood about ∞ is

$$\{\tau \in \mathcal{H} \mid \mathrm{Im}\,\tau > r\} \sqcup \{\infty\}.$$

If $r \in \mathbb{Q}$, a basic open neighborhood about r is $D \sqcup \{r\}$ where D is an open disc of center $r + iy$ and radius $y > 0$.

A subset $U \subset \mathcal{H}^*$ is an open set iff for every $P \in U$, a basic open neighborhood defined above about P lies in U.

Exercise. Prove that if $g \in GL(2, \mathbb{Q})_+$, $r \in \mathbb{Q} \sqcup \{\infty\}$, then g send a basic open neighborhood of r to a basic open neighborhood of $g(r)$.

$GL(2, \mathbb{Q})_+$ acts on $\mathcal{H}^*$ as homeomorphisms.

$\Gamma_0(N)$ acts on $\mathcal{H}^*$ as homeomorphisms. We consider the orbit space

$$X_0(N) = \Gamma_0(N) \backslash \mathcal{H}^*$$

We given $X_0(N)$ the quotient topology: a set $U \subset X_0(N)$ is open iff $\pi^{-1}(U)$ is an open set in $\mathcal{H}^*$, where

$$\pi : \mathcal{H}^* \to \Gamma_0(N) \backslash \mathcal{H}^* = X_0(N)$$

is the quotient map.

Definition. A Riemann surface is a Hausdorff topological space X with an open cover $X = \cup_{i \in I} U_i$ and each open set U_i has a coordinate map

$$z_i : U_i \to V_i$$

where V_i is an open set in $\mathbb{C}$ such that

(1) z_i is a homeomorphism, i.e., an isomorphism of topological spaces;

(2) for every i, j,

$$z_j \circ z_i^{-1} : z_i(U_i \cap U_j) \to z_j(U_i \cap U_j)$$

is bi-holomorphic.

If X is a Riemann surface, $U \subset X$ is an open subset, a function $f : U \to \mathbb{C}$ is called an analytic function (a holomorphic function) if for every i, the composition map

$$z_i(U_i \cap U) \xrightarrow{z_i^{-1}} U_i \cap U \xrightarrow{f} \mathbb{C}$$

is analytic.

A map $f : X \to \mathbb{C} \cup \{\infty\}$ is called a meromorphic function if it is locally quotient of two analytic functions.

The space of meromorphic functions on a connected Riemann surface is a field.

If the Riemann surface X is compact, the field of meromorphic functions on X has transcendental degree one over $\mathbb{C}$.

$X_0(N) = \Gamma_0(N)\backslash\mathcal{H}^*$ has a Riemann surface structure.

It is compact.

Every non-singular projective curve over $\mathbb{C}$ is a compact Riemann surface. Conversely every compact Riemann surface X has a unique structure of non-singular projective curve over $\mathbb{C}$.

The concepts analytic maps between Riemann surfaces and morphisms between algebraic curves coincide.

The field of meromorphic functions on X is the same as the field of rational functions on the corresponding curve.

Proposition 11.6 The space of holomorphic differentials on $X_0(N)$ is isomorphic to $S_2(\Gamma_0(N))$.

Let X be a compact Riemann surface, let $\Omega_{\mathrm{hol}}(X)$ be the space of holomorphic differentials on X. $\dim_{\mathbb{C}}\Omega_{\mathrm{hol}}(X) = g$ is called the **genus of** X.

Then the fist homology group $H_1(X, \mathbb{Z})$ is a free $\mathbb{Z}$-module of rank $2g$.

Suppose the genus of X $g \geq 1$. let $c_1, \ldots, c_{2g}$ be a basis of $H_1(X, \mathbb{Z})$, and $\omega_1, \ldots, \omega_g$ be a basis of $\Omega_{\mathrm{hol}}(X)$.

Then $2g$ vectors in $\mathbb{C}^g$ given by

$$(\int_{c_i} \omega_1, \int_{c_i} \omega_2, \ldots, \int_{c_i} \omega_g)$$

$i = 1, 2, \ldots, 2g$ are linearly independent over $\mathbb{R}$. Let $\Lambda(X)$ be the lattice spanned the above vectors.

The **Jacobi variety** of X is defined as $J(X) = \mathbb{C}^g / \Lambda(X)$ (as a complex manifold, but it is a variety)

We fix a base point $x_0 \in X$, then we have a map

$$\Phi : X \to J(X)$$

given by

$$\Phi(x) = (\int_{x_0}^{x} \omega_1, \ldots, \int_{x_0}^{x} \omega_g)$$

Because $J(X)$ is a group, we extend Φ to a map

$$\Phi : \mathrm{Div}(X) \to J(X)$$

$$\Phi(\sum_{i=1}^{m} k_i[x_i]) = \sum_{i=1}^{m} k_i \Phi(x_i)$$

Theorem. Φ induces a group isomorphism from $\mathrm{Pic}^0(X)$ to $J(X)$. Recall that $\mathrm{Pic}^0(X)$ is the quotient group of degree 0 divisors by the principal divisors.

Hecke Operators on Integral Homology.

For every $\tau_0 \in \mathcal{H}^*$, $\gamma \in \Gamma_0(N)$, for any path $[\tau_0, \gamma\tau_0]$ in $\mathcal{H}^*$ that starts at τ_0 and ends at $\gamma\tau_0$,

The image of $[\tau_0, \gamma\tau_0]$ under $\pi : \mathcal{H}^* \to X_0(N) = \Gamma_0(N)\backslash\mathcal{H}^*$ is a 1-cycle in $X_0(N)$. Its class in $H_1(X_0(N), \mathbb{Z})$ depends only on γ.

It is easy to see that the map $\Gamma_0(N) \to H_1(X_0(N), \mathbb{Z})$ given by

$$\gamma \mapsto \pi[\tau_0, \gamma\tau_0]$$

is a group homomorphism. This map induces

Proposition 11.22.

$$H_1(X_0(N), \mathbb{Z}) \cong \Gamma_0(N)^{ab}/\Gamma_{ep}^{ab}$$

where Γ_{ep} denotes the subgroup generated by elliptic and parabolic subgroups.

Recall that

$$M(n,N)=\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in M_2(\mathbb{Z})\,|\,ad-bc=n, c\equiv 0 \bmod N, \ \ gcd(a,N)=1\}$$

$$M(n,N)=\sqcup_{i=1}^{K}\Gamma_0(N)\alpha_i$$

For $[\gamma] \in \Gamma_0(N)^{ab}/\Gamma_{ep}^{ab}$ represented by $\gamma \in \Gamma_0(N)$,

$$\alpha_i \gamma = \gamma_i \alpha_{\sigma(i)}$$

where σ is the permutation $\{\alpha_1, \ldots, \alpha_K\}$ corresponding the action of γ on the cosets

$$\Gamma_0 \alpha_1, \ldots, \Gamma_o \alpha_K.$$

We define

$$T(n)[\gamma] = \sum_{i=1}^{K} [\gamma_i]$$

This defines an action of Hecke operator on $H_1(X_0(N), \mathbb{Z})$. This action is compatible with the $T(n)$-action on $S_2(\Gamma_0(N)) = \Omega_{\mathrm{hol}}(X_0(N))$:

Proposition 11.23

$$\int_{T(n)c} \omega = \int_c T(n)\omega.$$

The eigenvalues of $T(n)$ on $S_2(\Gamma_0(N))$ are algebraic integers.

Definition. A meromorphic function f on $\mathcal{H}$ is said to be automorphic of weight 0 and level N if

$$f \circ [g]_0 = f(\frac{a\tau + b}{c\tau + d}) = f(\tau)$$

for all

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$$

and for every $r \in SL(2, \mathbb{Z})$, the q-expansion of $f \circ [r]$ has only finitely many negative terms.

We denote by $A_0(\Gamma_0(N))$ the space of all automorphic functions of weight 0 and level N. Then $A_0(\Gamma_0(N))$ is a field, and

$$A_0(\Gamma_0(N)) = K(X_0(N))$$

here $K(X_0(N))$ is the field of rational functions on $X_0(N)$.

$K(X_0(1)) = \mathbb{C}(j)$,
where j is the unique automorphic function for $\Gamma_0(1) = SL(2, \mathbb{Z})$ with q-expansion

$$j = q^{-1} + 744 + \sum_{n=1}^{\infty} c_n q^n.$$

Recall $j(\tau) = 1728 g_2(\tau)/\Delta(\tau)$.

Theorem 11.33 $A_0(\Gamma_0(N)) = K(X_0(N)) = \mathbb{C}(j, j_N)$, where

$$j_N(\tau) = j(N\tau).$$

Corollary 11.49. There is a non-singular projective curve $C/\mathbb{Q}$ which has functions field (over $\mathbb{Q}$) as

$$\mathbb{Q}(j, j_N).$$

$C(\mathbb{C})$ is $X_0(N)$.

For every new form $f \in S_2(\Gamma_0(N))$ with q-expansion

$$f = q + \sum_{n=2}^{\infty} c_n q^n$$

with $c_n \in \mathbb{Z}$, there exists an elliptic curve E over $\mathbb{Q}$ with L-function same as $L(f, s)$.

E is a quotient of $J(X_0(N))$ by a codimension 1 sub-abelian variety.

End