Math 6170 C, Lecture on May 4 , 2020

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Plan

(1) VIII (Knapp). Modular Forms for $SL(2, \mathbb{Z})$.

(2) IX (Knapp). Modular Forms for Hecke Subgroups.

VIII Modular Forms For $SL(2,\mathbb{Z})$.

Equivalence classes of lattices in $\mathbb{C}$ $\Leftrightarrow$ Elliptic Curves over $\mathbb{C}$.

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \quad \mapsto \quad \mathbb{C}/\Lambda$$

$$\mathbb{C}/\Lambda \simeq \mathbb{C}/c\Lambda, \quad z + \Lambda \mapsto cz + c\Lambda$$

Every lattice is equivalent to $\mathbb{Z} + \mathbb{Z}\tau$ for $\mathrm{Im}\,\tau > 0$.

Two lattices $\mathbb{Z} + \mathbb{Z}\tau$ and $\mathbb{Z} + \mathbb{Z}\tau'$ are equivalent iff

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

for some

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).$$

Set the upper half space to be

$$\mathcal{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im} \tau > 0\}.$$

The group $SL(2, \mathbb{R})$ acts on $\mathcal{H}$ by

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}.$$

The set of isomorphisms classes of elliptic curves over $\mathbb{C}$ can be identified with

$$\mathcal{H}/SL(2, \mathbb{Z})$$

the set of orbits.

Definition.

A **unrestricted modular** form of weight k for $SL(2,\mathbb{Z})$ is an analytic function on $\mathcal{H}$ that satisfies

$$f(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^k f(\tau) \tag{1}$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$.

It is enough to check (1) for

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

For reasons that
(1) S, T generates $SL(2,\mathbb{Z})$.

(2) For each k, the group

$$GL(2,\mathbb{R})_+ = \{g \in GL(2,\mathbb{R}) \,|\, \det g > 0\}$$

acts from the right on the space of analytic functions on $\mathcal{H}$ by

$$(f \circ [g]_k)(\tau) = \det(g)^{\frac{k}{2}}(c\tau + d)^{-k} f(\frac{a\tau + b}{c\tau + d}).$$

The modular from condition (1) is equivalent to

$$f \circ [g]_k = f$$

for all $g \in SL(2,\mathbb{Z})$.

Examples. $k \geq 2$,

$$G_{2k}(\tau) = \sum_{(m,n)\in\mathbb{Z}^2,(m,n)\neq(0,0)} \frac{1}{(m\tau+n)^{2k}}$$

is a unrestricted modular form of weight $2k$ for $SL(2,\mathbb{Z})$.

If f is a unrestricted modular form of weight k, take $\gamma = T$, we have

$$f(\tau+1) = f(\tau)$$

So f has a series expansion

$$f(\tau) = \sum_{n \in \mathbb{Z}} c_n q^n$$

with $q = e^{2\pi i \tau}$. This is called the q **expansion** of f.

We say an unrestricted modular form f is **holomorphic at** ∞ and is a **modular form** if its q-expansion has $c_n = 0$ for $n < 0$. If also $c_0 = 0$, we call f a **cusp form**.

If f is a modular form of weight k,

$$f(\tau) = c_0 + c_1 e^{2\pi i\tau} + \dots$$

For each $n > 0$,

$$\lim_{\tau \to i\infty} e^{2\pi i n\tau} = \lim_{\tau \to i\infty} e^{2\pi i n x} e^{-2\pi n y} = 0$$

So

$$\lim_{\tau \to i\infty} f(\tau) = c_0$$

Let M_k (S_k) be the space of modular forms (cusp forms) of weight k for $SL(2,\mathbb{Z})$.

Then $M_k = 0$ for k odd.
Apply the condition

$$f(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^k f(\tau)$$

to

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

we get

$$f(\tau) = -f(\tau)$$

So $f(\tau) = 0$.

$M_k = 0$ for $k = -l < 0$.

Proof. If $f \in M_k$, and $f \neq 0$
$f(\tau) = c_0 + c_1 e^{2\pi i \tau} + \dots$, $\lim_{\tau \to i\infty} f(\tau) = c_0$.
The function $g(q) = c_0 + c_1 q + \dots$ is analytic on $|q| < 1$, if $f(\tau)$ is not constant, then $|g(0)| = |f(i\infty)|$ is not a maximum.

$$f(-\frac{1}{\tau}) = (-\tau)^{-l} f(\tau)$$

Take $\lim_{\tau \to i\infty}$, we get $\lim_{\tau \to i0} f(\tau) = 0$.
This implies that $|f(\tau)|$ takes a maximum at some point in $\mathcal{H}$, so $f(\tau)$ is a constant by the maximum principle.

$$M_0 = \mathbb{C}$$

$$M_0 + M_2 + \cdots + M_{2k} + \dots$$

is a graded ring,

$$S_2 + S_4 + \dots$$

is an ideal of $\oplus_{k=0}^{\infty} M_{2k}$.

Example. For $k \geq 2$,

$$G_{2k}(\tau) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$$

where
$\sigma_l(n) = \sum_{d|n} d^l$.

So G_{2k} is a modular form of weight $2k$ for $SL(2, \mathbb{Z})$.

Theorem

The commutative algebra $M = \oplus_{k=0}^{\infty} M_{2k}$ *is generated by* G_4 *and* G_6 *over* $\mathbb{C}$*, and* G_4 *and* G_6 *are algebraically independent over* $\mathbb{C}$*, so the monomials*

$$\{G_4^m G_6^n \mid 4m + 6n = N\}$$

is a basis for M_N*.*

$$M_{2k} = \mathbb{C} G_{2k} \oplus S_{2k}$$

Let $g_2 = 60G_4$, $g_3 = 140G_6$,

$$\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2$$

is a cusp modular form of weight 12, $\Delta(\tau) \neq 0$ for all $\tau \in \mathcal{H}$.

$$j(\tau) = 1728g_2(\tau)^3/\Delta(\tau)$$

has weight 0.

$$j(\tau) = \frac{1}{q} + 744 + \sum_{n=1}^{\infty} c_n q^n$$

All $c_n \in \mathbb{Z}_{>0}$.

A fundamental domain for $SL(2,\mathbb{Z})$-action on $\mathcal{H}$ is

$$R = \{\tau \in \mathcal{H} \mid -\frac{1}{2} \leq \mathrm{Re}\,\tau \leq \frac{1}{2}, |\tau| \geq 1\}$$

Every point in $\mathcal{H}$ is $SL(2,\mathbb{Z})$-equivalent to some point in R, An interior point in R is not equivalent to any other point in R. Any boundary point except $\tau = i$ has exactly another boundary point that is $SL(2,\mathbb{Z})$-equivalent to it.

Another proof of that fact that $\Delta(\tau)$ never vanishes.

$$\Delta(\tau) = (2\pi)^{12} q \Pi_{n=1}^{\infty}(1-q^n)^{24}.$$

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \Pi_{n=1}^{\infty}(1-q^n)$$

Then

$$\eta(\tau+1) = e^{\frac{\pi i}{1}2}\eta(\tau),\ \eta(-\frac{1}{\tau}) = (-i\tau)^{\frac{1}{2}}\eta(\tau)$$

where $(-i\tau)^{\frac{1}{2}}$ satisfies $\mathrm{re}\,(-i\tau)^{\frac{1}{2}} > 0$.

Hecke Operators.

For every positive integer $n \geq 2$, let

$$M(n) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z},\ ad - bc = n\}.$$

$M(n)$ are closed by left multiplication and right multiplication by elements in $SL(2, \mathbb{Z})$.

$M(n)$ is equal to a disjoint union of finitely many right cosets of $SL(2, \mathbb{Z})$.

$$M(n) = \bigcup_{i=1}^{\nu(n)} SL(2, \mathbb{Z})\alpha_i.$$

We have

Lemma. The integral matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $ad = n$, $a > 0, d > 0$ and $0 \leq b < d$ are a complete set of representatives for the right coset of $SL(2, \mathbb{Z})$ on $M(n)$.

Notice that

$$\begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL(2, \mathbb{Z}).$$

Lemma. Let

$$M(n) = \bigcup_{i=1}^{\nu(n)} SL(2,\mathbb{Z})\alpha_i.$$

For a unrestricted modular form f of weight k for $SL(2,\mathbb{Z})$, then

$$f \circ [\alpha_1]_k + \cdots + f \circ [\alpha_{\nu(n)}]_k$$

is independent of the choices of α_i and it is also a unrestricted modular form f of weight k for $SL(2,\mathbb{Z})$.

Proof. The set $M(n)$ is closed under the right multiplication by elements in $SL(2,\mathbb{Z})$. So $SL(2,\mathbb{Z})$ acts on the set of rights cosets of $SL(2,\mathbb{Z})$ in $M(n)$:

$$SL(2,\mathbb{Z})\alpha \cdot g = SL(2,\mathbb{Z})(\alpha g)$$

Therefore $\alpha_i g = h_i \alpha_{\sigma(i)}$ for $i = 1, \ldots, \nu(n)$, $h_i \in SL(2, \mathbb{Z})$, σ is a permutation of $\{1, 2, \ldots, \nu(n)\}$

$$
\begin{aligned}
&\left(f \circ [\alpha_1]_k + \cdots + f \circ [\alpha_{\nu(n)}]_k\right) \circ [g]_k \\
&= f \circ [\alpha_1]_k \circ [g]_k + \cdots + f \circ [\alpha_{\nu(n)}]_k \circ [g]_k \\
&= f \circ [\alpha_1 g]_k + \cdots + f \circ [\alpha_{\nu(n)} g]_k \\
&= f \circ [h_1 \circ \alpha_{\sigma(1)}]_k + \cdots + f \circ [h_{\nu(n)} \alpha_{\sigma(\nu(n))}]_k \\
&= f \circ [\alpha_{\sigma(1)}]_k + \cdots + f \circ [\alpha_{\sigma(\nu(n))}]_k \\
&= f \circ [\alpha_1]_k + \cdots + f \circ [\alpha_{\nu(n)}]_k
\end{aligned}
$$

We define nth Hecke operator $T_k(n)$ on M_k by

$$T_k(n)f = n^{\frac{k}{2}-1} \sum_{i=1}^{\nu(n)} f \circ [\alpha_i]_k.$$

Proposition 8.16. Let $f \in M_k$, if f has q-expansion $f(\tau) = \sum_{n=0}^{\infty} c_n q^n$. Then $T_k(m)f$ has q-expansion

$$T_k(m)f = \sum_{n=0}^{\infty} b_n q^n$$

where

$$b_n = \begin{cases} c_0 \sigma_{k-1}(m) & \text{if } n = 0 \\ c_m & \text{if } n = 1 \\ \sum_{a|gcd(n,m)} a^{k-1} c_{nm/a^2} & \text{if } n > 1 \end{cases}$$

$T_k(m)$ carries S_k to S_k.

Theorem 8.19. On the space M_k, the Hecke operators satisfy
(a) For a prime power p^r with $r \geq 1$,

$$T_k(p^r)T_k(p) = T_k(p^{r+1}) + p^{k-1}T_k(p^{r-1})$$

(b) $T_k(m)T_k(n) = T_k(mn)$ if $gcd(m,n) = 1$.

(c) The algebra generated by $T_k(n)$ for $n = 2, 3, \dots$ is generated by $T_k(p)$ with p prime and is commutative.

Other results:

$$T_{2k}(n)G_{2k} = \sigma_{2k-1}(n)\, G_{2k}$$

On S_k, we define an inner product (Peterson inner product) by

$$(f,h) = \int_R f(\tau)\overline{h(\tau)}y^{k-2}dxdy$$

here $\tau = x + iy$. The $T_k(n)$ are self-dual operators with respect to the inner product. So S_k has an eigen-basis.

IX. Modular Forms For Hecke Subgroups.

The **principal congruence subgroup** $\Gamma(N)$ (N is a positive integer) is defined by

$$\Gamma(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N\}$$

It is the kernel of the group homomorphism $SL(2, \mathbb{Z}) \to SL(2, \mathbb{Z}/N\mathbb{Z})$ induced from the ring homomorphism $\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$.

A subgroup H in $SL(2,\mathbb{Z})$ is called a **congruence subgroup** if $H \supset \Gamma(N)$ for some N.

$$\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \mid c \equiv 0 \bmod N\}$$

is a congruence subgroup. The groups $\Gamma_0(N)$ are called the **Hecke subgroups**.

Lemma. If $H \subset SL(2,\mathbb{Z})$ is a congruence subgroup, then for every $g \in SL(2,\mathbb{Q})$, $gHg^{-1} \cap SL(2,\mathbb{Z})$ is also a congruence subgroup.

Proof. It is enough to prove that for every $g \in SL(2,\mathbb{Q})$, $g\Gamma(N)g^{-1}$ contains some $\Gamma(M)$. Since $SL(2,\mathbb{Q})$ is generated by

$$\begin{pmatrix} 1 & \pm\frac{1}{L} \\ 0 & 1 \end{pmatrix}, L \in \mathbb{Z}_{>0}, S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

it is enough to prove this for $g = \begin{pmatrix} 1 & \pm\frac{1}{L} \\ 0 & 1 \end{pmatrix}$ or $g = S$.

The case $g = S$ is obvious, as $S \in SL(2,\mathbb{Z})$ and $\Gamma(N)$ is normal subgroup of $SL(2,\mathbb{Z})$.

Proof (continued). For $g = \begin{pmatrix} 1 & \pm\frac{1}{L} \\ 0 & 1 \end{pmatrix}$,

$$g^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} g = \begin{pmatrix} a + \frac{c}{L} & b + \frac{dL - aL - c}{L^2} \\ c & d - \frac{c}{L} \end{pmatrix}$$

we see that

$$g^{-1}\Gamma(NL^2)g \subset \Gamma(N)$$

so

$$\Gamma(NL^2) \subset g\Gamma(N)g^{-1}$$

□

Let $\mathbb{P}^1(\mathbb{Q})$ be the set of 1-dimensional $\mathbb{Q}$-subspaces in $\mathbb{Q}^2$. $GL(2,\mathbb{Q})$ acts on $\mathbb{P}^1(\mathbb{Q})$. So $SL(2,\mathbb{Z})$ acts on $\mathbb{P}^1(\mathbb{Q})$.

$$\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$$

Lemma. (1). $SL(2,\mathbb{Z})$ acts on $\mathbb{P}^1(\mathbb{Q})$ transitively, i.e., there is only one orbit.
(2). If H is a congruence subgroup, then there are only finitely many H-orbits in $\mathbb{P}^1(\mathbb{Q})$. Each orbit is called a cusp for H.

Let R be the usual fundamental domain in $\mathcal{H}$ for $SL(2,\mathbb{Z})$. Let H be a congruence subgroup, let

$$SL(2,\mathbb{Z}) = \bigcup_{i=1}^{n} H\alpha_i$$

then $\cup_{i=1}^{n} \alpha_i F$ is a fundamental domain for H.

Definition. Let H be a congruence subgroup, an **unrestricted modular form of weight** $k \in \mathbb{Z}$ **for** H is an analytic function f on $\mathcal{H}$ with

$$f(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^k f(\tau)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H$.

Definition. An **unrestricted modular form of weight** $k \in \mathbb{Z}$ **and level** $N \geq 1$ is a unrestricted modular form of weight k for $\Gamma_0(N)$.

Lemma. If f is an unrestricted modular form of weight k for a congruence subgroup H, for every $g \in SL(2, \mathbb{Q})$, $f \circ [g]_k$ is an unrestricted modular form of weight k for congruence subgroup $g^{-1}Hg \cap SL(2, \mathbb{Z})$.

Proof. For $u = g^{-1}hg \in g^{-1}Hg \cap SL(2, \mathbb{Z})$, so $h \in H$,

$$(f \circ [g]_k) \circ [u]_k = f \circ [gu]_k = f \circ [hg]_k = (f \circ [h]_k) \circ [g]_k = f \circ [g]_k.$$

If f is an unrestricted modular form of weight k and level N, take

$$\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N),$$

we have

$$f(\tau + 1) = f(\tau).$$

So f has q-expansion

$$f(\tau) = \sum_{n \in \mathbb{Z}} c_n q^n.$$

We say f is **holomorphic** at ∞ if $c_n = 0$ for $n < 0$ and f **vanishes at** ∞ if $c_n = 0$ for $n \leq 0$.

For f as in the previous page, $r \in \mathbb{Q}$, let $g \in SL(2,\mathbb{Z})$ satisfy $g \cdot \infty = r$. Then $f \circ [g]_k$ is an unrestricted modular form of weight k for $g^{-1}\Gamma_0(N)g$, which is a congruence subgroup, i.e., $g^{-1}\Gamma_0(N)g \supset \Gamma(M)$ for some M, for

$$\gamma = \begin{pmatrix} 1 & M \\ 0 & 1 \end{pmatrix} \in \Gamma(M),$$

$$(f \circ [g]_k) \circ [\gamma]_k = f \circ [g]_k$$

implies that

$$(f \circ [g]_k)(\tau + M) = (f \circ [g]_k)(\tau)$$

So $f \circ [g]_k$ has q-expansion

$$(f \circ [g]_k)(\tau) = \sum_{n \in \mathbb{Z}} c_n e^{2\pi i n \frac{1}{M} \tau}$$

f is called to be holomorphic at r if $c_n = 0$ for $n < 0$ and vanishes at r if $c_n = 0$ for $n \leq 0$.

It can proved that the above definition is independent of the choice of g. And if f is holomorphic (vanishes) at r, then f is holomorphic (vanishes) at αr for $\alpha \in \Gamma_0(N)$.

Definition. A modular form of weight k and level N is an analytic function on $\mathcal{H}$ that satisfies
(1) f is an unrestricted modular form of weight k and level N.
(2) f is holomorphic at all the cusps.

f is called a cusp form of weight k and level N if (2) above is replaced by "f vanishes at all the cusps".

End