Math 6170 C, Lecture on May 6 , 2020

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Plan

(1) IX (Knapp). Modular Forms for Hecke Subgroups (continued).

IX. Modular Forms For Hecke Subgroups (continued).

The **principal congruence subgroup** $\Gamma(N)$ (N is a positive integer) is defined by

$$\Gamma(N) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \}$$

A subgroup H in $SL(2,\mathbb{Z})$ is called a **congruence subgroup** if $H \supset \Gamma(N)$ for some N.

$$\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \mid c \equiv 0 \mod N\}$$

is a congruence subgroup. The groups $\Gamma_0(N)$ are called the **Hecke subgroups**.

Definition. Let H be a congruence subgroup, an **unrestricted modular form of weight** $k \in \mathbb{Z}$ **for** H is an analytic function f on $\mathcal{H}$ with

$$f(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^k f(\tau)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H$.

If f is an unrestricted modular form of weight $k \in \mathbb{Z}$ for a congruence subgroup H, for every $g \in SL(2, \mathbb{Z})$,

$$f \circ [g]_k$$

is an unrestricted modular form of weight $k \in \mathbb{Z}$ for the congruence subgroup $g^{-1}Hg$.

More generally, for every $g \in SL(2, \mathbb{Q})$,

$$f \circ [g]_k$$

is an unrestricted modular form of weight $k \in \mathbb{Z}$ for the congruence subgroup $g^{-1}Hg \cap SL(2, \mathbb{Z})$.

Definition. An **unrestricted modular form of weight** $k \in \mathbb{Z}$ **and level** $N \geq 1$ is a unrestricted modular form of weight k for $\Gamma_0(N)$.

If f is an unrestricted modular form of weight k for a congruence subgroup $H \supset \Gamma(N)$, take

$$\gamma = \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \in \Gamma(N) \subset H,$$

we have

$$f(\tau + N) = f(\tau).$$

So f has q-expansion

$$f(\tau) = \sum_{n \in \mathbb{Z}} c_n q^{\frac{n}{N}}, \quad q = e^{2\pi i \tau}.$$

We say f is **holomorphic** at ∞ if $c_n = 0$ for $n < 0$ and f **vanishes at** ∞ if $c_n = 0$ for $n \leq 0$.

An unrestricted modular form f of weight k for congruence subgroup H is called a **modular form (cusp form)** of weight k for H if for every $g \in SL(2, \mathbb{Z})$,

$$f \circ [g]_k$$

is holomorphic at ∞ (vanishes at ∞).

Write $SL(2, \mathbb{Z}) = \sqcup_i H\alpha_i$, it is enough to check the conditions for α_i's.

If $H = \Gamma_0(N)$, we call f a modular form (cusp form) of weight k and level N.

If f is a unrestricted modular form of weight k for a congruence subgroup H, then real valued function

$$\phi(\tau) = |f(\tau)| (\mathrm{Im}\tau)^{\frac{k}{2}}$$

is H-invariant.

Proof. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H$,

$$\begin{aligned}
&\phi(\frac{a\tau + b}{c\tau + d}) \\
&= |f(\frac{a\tau + b}{c\tau + d})| \left(\mathrm{Im}\frac{\mathrm{a}\tau + \mathrm{b}}{\mathrm{c}\tau + \mathrm{d}}\right)^{\frac{k}{2}} \\
&= |f(\tau)| \, |c\tau + d|^k \left(\mathrm{Im}\frac{\mathrm{a}\tau + \mathrm{b}}{\mathrm{c}\tau + \mathrm{d}}\right)^{\frac{k}{2}} \\
&= |f(\tau)| \, \mathrm{Im}\tau^{\frac{k}{2}} = \phi(\tau)
\end{aligned}$$

We denote the space of modular forms (cusp forms) of weight k and level N by by $M_k(\Gamma_0(N))$ ($S_k(\Gamma_0(N))$).

Lemma 9.6. Let $f \in S_k(\Gamma_0(N))$ with q-expansion $f(\tau) = \sum_{n=1}^{\infty} c_n q^n$, then
(a). $\phi(\tau) = |f(\tau)| \,(\mathrm{Im}\tau)^{\frac{k}{2}}$ is invariant under $\Gamma_0(N)$ and is bounded.
(b). $|c_n| \leq C' \, n^{\frac{k}{2}}$ for some constant C'.

Proof of (b).

$$c_n = \int_0^1 f(\tau)e^{-2\pi i n\tau}dx.$$

Because

$$|f(\tau)y^{\frac{k}{2}}| \leq C, \ \text{so} \ |f(\tau)| \leq C\, y^{-\frac{k}{2}}.$$

$$|c_n| \leq C\, e^{2\pi n y} y^{-\frac{k}{2}}.$$

Take $y = \frac{1}{n}$, $|c_n| \leq Ce^{2\pi i}n^{\frac{k}{2}}$.

Let

$$\alpha_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$$

Then α_N^{-1} is in the normalizer of $\Gamma_0(N)$.

Proof.

$$\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$$
$$= \begin{pmatrix} d & -\frac{c}{N} \\ -Nb & a \end{pmatrix}$$

Proposition 9.7. If $f \in M_k(\Gamma_0(N))$ $(S_k(\Gamma_0(N)))$, then so is $f \circ [\alpha_N]_k$

We denote by

$$\omega_N : S_k(\Gamma_0(N)) \to S_k(\Gamma_0(N))$$

the map $f \mapsto f \circ [\alpha_N]_k$.

We have

$$\omega_N^2 = Id$$

Proof.

$$\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} = \begin{pmatrix} -N & 0 \\ 0 & -N \end{pmatrix}.$$

L Functions of a Cusp Form.

Theorem 9.8. Let $f = \sum_{n=1} c_n q^n \in S_k(\Gamma_0(N))$. Suppose it satisfies $\omega_N f = \epsilon f$, where $\epsilon = \pm 1$. Then
(1)

$$L(s, f) = \sum_{n=1}^{\infty} \frac{c_n}{n^s}$$

converges on $\mathrm{Re}\, s > \frac{k}{2} + 1$.
(2) $L(s, f)$ has analytic continuation to whole $\mathbb{C}$.
(3) Let

$$\Lambda(s, f) = N^{\frac{s}{2}} (2\pi)^{-s} \Gamma(s) L(s, f)$$

then we the functional equation

$$\Lambda(s, f) = \epsilon(-1)^{\frac{k}{2}} \Lambda(k - s, f).$$

Proof. of (1).

$$|\frac{c_n}{n^s}| \leq C' \frac{n^{\frac{k}{2}}}{n^{\mathrm{re}\, s}} = C' \frac{1}{n^{\mathrm{re}\, s\, -\, \frac{k}{2}}},$$

where we used Lemma 9.6 (b).
The result now follows from tha fact that

$$\sum_{n=1}^{\infty} \frac{1}{n^r}$$

converges on $r > 1$.

Proof of (2) and (3). $\omega_N f = \epsilon f$ implies that for $y > 0$,

$$f(\frac{i}{Ny}) = \epsilon N^{\frac{k}{2}} i^k y^k f(iy)$$

$$\begin{aligned}
&\int_0^\infty f(iy) y^{s-1} dy \\
&= \sum_{n=1}^{\infty} \int_0^\infty c_n e^{-2\pi n y} y^{s-1} dy \quad (2\pi n y \to t) \\
&= \sum_{n=1}^{\infty} c_n (2\pi n)^{-s} \int_0^\infty e^{-t} t^{s-1} dt \\
&= \sum_{n=1}^{\infty} c_n (2\pi n)^{-s} \Gamma(s) \\
&= N^{-\frac{s}{2}} \Lambda(s, f)
\end{aligned}$$

Proof of (2) and (3) (continued).

This proves

$$\Lambda(s,f) = N^{\frac{s}{2}} \int_0^{\infty} f(iy) y^{s-1} dy.$$

$$\Lambda(s,f) = N^{\frac{s}{2}} \int_0^{\frac{1}{\sqrt{N}}} f(iy) y^{s-1} dy + N^{\frac{s}{2}} \int_{\frac{1}{\sqrt{N}}}^{\infty} f(iy) y^{s-1} dy$$

$$= \epsilon N^{\frac{1}{2}(k-s)} i^k \int_{\frac{1}{\sqrt{N}}}^{\infty} f(iy) y^{k-s-1} dy + N^{\frac{s}{2}} \int_{\frac{1}{\sqrt{N}}}^{\infty} f(iy) y^{s-1} dy$$

Theorem The spaces $M(\Gamma_0(N))$ and $S(\Gamma_0(N))$ are finite dimensional.

Recall that $M(n)$ is the set of 2×2 matrices over $\mathbb{Z}$ with determinant n. Let

$$M(n, N) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(n) \,|\, c \equiv 0 \mod N \text{ and } gcd(a, N) = 1 \}$$

$M(n, N)$ is closed under left and right multiplication by elements in $\Gamma_0(N)$.

Theorem 9.12. Let $M(n, N) = \sqcup_{i=1}^{m} \Gamma_0(N)\alpha_i$, If $f \in M_k(\Gamma_0(N))$, then $T_k(N)f$ given by

$$T_k(n)f = n^{\frac{k}{2}-1} \sum_{i=1}^{m} f \circ [\alpha_i]_k$$

is a modular form of weight k and level N. It f is a cusp form, so is $T_k(n)f$

The invariance of $T_k(n)f$ under the right action of $[g]_k$ for $g \in \Gamma_0(N)$ follows from the following general lemma.

Lemma. Let a group G acts on a vector space V from the right linearly, $\Gamma \subset G$ is a subgroup, $S \subset G$ satisfies $\Gamma S \Gamma \subset S$, suppose that

$$S = \sqcup_{i=1}^{m} \Gamma \alpha_i.$$

If $v \in V$ is fixed by Γ, then

$$v\alpha_1 + \cdots + v\alpha_m$$

is also fixed by Γ.

Lemma 9.4 The matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $ad = n$ $d > 0$, $gcd(a, N) = 1$ and $0 \leq b \leq d - 1$ are a complete set of representatives for the right cosets of $\Gamma_0(N)$ on $M(n, N)$.

A key step in the proof is that for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(n, N)$, there exsits $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \Gamma_0(N)$ such that

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is upper triangular.

From Lemma 9.4, one can derive a formula for the coefficient of q-expansion of $T_k(n)f$ in terms of the coefficients of q-expansion of f, see Proposition 9.15.

Theorem 9.17. On the space $M_k(\Gamma_0(N))$, the Hecke operators satisfy

(a) For m and n with $gcd(m,n) = 1$, we have

$$T_k(m)T_k(n) = T_k(mn)$$

(b) For a prime power p^r, $r \geq 1$ such that $p \nmid N$,

$$T_k(p^r)T_k(p) = T_k(p^{r+1}) + p^{k-1}T_k(p^{r-1})$$

Hence $T_k(p^r)$ is a polynomial of $T_k(p)$ with integer coefficients.

(c) For a prime power p^r, $r \geq 1$ such that $p|N$,

$$T_k(p^r) = T_k(p)^r.$$

Petersson Inner Product.

For $N \in \mathbb{Z}_{>0}$, let V_N be the space of $\mathbb{C}$-valued continuous functions $f(x)$ on $\mathbb{R}$ such that

$$f(x+N) = f(x)$$

we define an inner product on V_N by

$$(f, g)_N = \int_0^N f(x)\overline{g(x)}dx$$

Using periodicity, we have

$$(f, g)_N = \int_b^{b+N} f(x)\overline{g(x)}dx$$

If $f, g \in V_N$, then $f, g \in V_{kN}$ $(k \in \mathbb{Z}_{\geq 1})$, the inner products

$$(f, g)_{kN} = k(f, g)_N.$$

On the vector space $V = \bigcup_{N=1}^{\infty} V_N$, we define an inner product as follows for $f, g \in V$, then there exists N such that $f, g \in V_N$, we define

$$(f, g) = \frac{1}{N} \int_b^{b+N} f(x)\overline{g(x)}dx$$

(f, g) is independent of the choice of N.

We want to define an inner product (f, g) on

$$M = \bigcup_{\Gamma:\text{congruence subgroups}} S_k(\Gamma)$$

so that

$$(f \circ [h]_k, g \circ [h]_k) = (f, g)$$

for every $h \in GL(2, \mathbb{Q})_+$.

First wet take the measure $\mu = \frac{1}{y^2} dxdy$ on $\mathcal{H}$. This measure is $SL(2, \mathbb{R})$-invariant. More generally, we have for a domain $D \subset \mathcal{H}$, any "good function" f,

$$\int_D f(g\tau) \frac{1}{y^2} dxdy = \int_{gD} f(\tau) \frac{1}{y^2} dxdy.$$

For any fundamental domain R_Γ of Γ, $\mu(R_\Gamma) < \infty$.

And if $\Gamma \subset \Gamma'$, then

$$\mu(R_\Gamma) = [\Gamma' : \Gamma]\mu(R'_\Gamma),$$

For

$$f, g \in M = \bigcup_{\Gamma} S_k(\Gamma),$$

we can find Γ so that $f, g \in S_k(\Gamma)$, we define

$$(f, g) = [SL(2, \mathbb{Z}) : \Gamma]^{-1} \int_{R_\Gamma} f(\tau)\overline{g(\tau)} y^k \frac{1}{y^2} dxdy$$

This is called the Petersson inner product.
(this is equal to the inner product on a single $S_k(\Gamma)$ in the text book up to a scalar).

It is easy to prove that

$$(f, g) = (f \circ [h]_k, g \circ [h]_k)$$

for any $h \in GL(2, \mathbb{Q})_+$.

Theorem 9.18. The Hecke operators $T_k(n)$ with $gcd(n, N) = 1$, on the space of cusp forms $S_k(\Gamma_0(N))$, are self adjoint relative to the Petersson inner product.

Theorem 9.19. The involution ω_N of $S_0(\Gamma_0(N))$ is self-adjoint and commutes with all $T_k(n)$ such that $gcd(n, N) = 1$.

Because operators $T_k(n)$ with $gcd(n, N) = 1$ are self-adjoint and commutes each other, the space $S_k(\Gamma_0(N))$ is an orthogonal direct sum of simultaneous eigenspaces for $T_k(n)$ with $gcd(n, N) = 1$. Two forms in the same simultaneous eigenspace are called to be equivalent.

Proposition 9.20. Theorem 9.21. Suppose $f \in S_k(\Gamma_0(N))$ is an eigenvector of all $T_k(n)$: $T_k(n) = \lambda(n)f$. If the q-expansion of f is

$$f(\tau) = \sum_{n=1}^{\infty} c_n q^n$$

, then

$$c_n = \lambda(n)c_1.$$

So $f \neq 0$ implies $c_1 \neq 0$.

Suppose $c_1 = 1$, we have

$$L(s, f) = \prod_{p:\text{prime}, p|N} \left(\frac{1}{1 - c_p p^{-s}} \right) \prod_{p:\text{prime}, p\nmid N} \left(\frac{1}{1 - c_p p^{-s} + p^{k-1-2s}} \right)$$

End