SUMMARY OF CHAPTER I OF HARTSHORNE'S "ALGEBRAIC GEOMETRY"

Notation. Throughout of this Chapter, k denotes an algebraic closed field.

Section I.1. Affine Varieties.

k: algebraically closed field. $\mathbf{A}^n$: set of all $(a_1, a_2, \ldots, a_n)$.

Algebraic sets, Zariski topology. **Affine algebraic variety** is an irreducible closed subsets in $\mathbf{A}^n$. An open set of an affine variety is called a **quasi-affine variety**.

For an ideal $\mathfrak{a} \subset k[x_1, \ldots, x_n]$, we define its zero set

$$Z(\mathfrak{a})\{a = (a_1, \ldots, a_n) \in \mathbf{A}^n \mid f(a) = 0 \text{ for all } f \in \mathfrak{a}\}.$$

For every subset $Y \subset \mathbf{A}^n$, we define

$$I(Y) = \{f \in k[x_1, \ldots, x_n] \mid f \text{ vanishes on } Y\}.$$

Then Z and I are order reversing and

$$ZI(Y) = \bar{Y}, \quad IZ(\mathfrak{a}) = \sqrt{\mathfrak{a}}.$$

So Z and I gives an one-to-one correspondence between the set of algebraic sets in $\mathbf{A}^n$ and the set of radical ideals in $k[x_1, \ldots, x_n]$. Under the above correspondence, the irreducible closed subsets are in one-to-one correspondence of prime in $k[x_1, \ldots, x_n]$.

If $Y \subset \mathbf{A}^n$ is an algebraic set, the **affine coordinate ring** of Y is defined to be $A(Y) = k[x_1, \ldots, x_n]/I(Y)$. It is a finitely generated k-algebra with no non-zero nilpotent elements.

Proposition. 1.5. *In a noetherian topological space X, every non-empty closed subset Y can be expressed as a finite union of $Y = Y_1 \cup \cdots \cup Y_r$ of irreducible closed subsets Y_i. If we require Y_i does not contain Y_j for $i \neq j$, then the decomposition is unique.*

Proof. The prove the existence of the decomposition, let $\mathcal{F}$ be the family of non-empty closed subsets that can't be decomposed. We want to prove $\mathcal{F}$ is the empty family. Suppose $\mathcal{F}$ is not empty, then $\mathcal{F}$ has a minimal element C using the noetherian assumption. This leads easily a contraction. To prove the uniqueness, we may assume $Y = X$ (otherwise, we consider Y instead of X). Suppose $X = Y_1 \cup \cdots \cup Y_r$ is a decomposition of irreducible closed subsets with $Y_i \not\subset Y_j$ for $i \neq j$. We prove that $\{Y_1, \ldots, Y_r\}$ is the set of maximal irreducible closed subsets in X (*this part of the argument is easier than the book*).

Corollary. 1.6. *Every algebraic set in $\mathbf{A}^n$ can expressed uniquely as a union of varieties, no one containing another.*

Dimension of a topological space. Krull dimension of a ring. Height of a prime ideal.

Theorem 1.8A. *Let k be a field, and let B be an integral domain which is a finitely generated k-algebra. Then*
(a) the dimension of B is equal to the transcendence degree of the quotient field $K(B)$ of B over k.
(b) For every prime ideal $\mathfrak{p}$ in B, we have

$$\text{height}\,\mathfrak{p} + \dim B/\mathfrak{p} = \dim B.$$

Proposition 1.13. *A variety in $\mathbf{A}^n$ has dimension $n-1$ iff it is the zero set $Z(f)$ of a single non-constant irreducible polynomial in $k[x_1, \ldots, x_n]$.*

Section I.2. Projective Varieties.

k; algebraically closed field. $\mathbf{P}^n_k = \mathbf{P}^n = k^{n+1} - \{0\}/k^*$. $S = k[x_0, x_1, \ldots, x_n]$.

Definition. A subset Y of $\mathbf{P}^n$ is an **algebraic set** if the zero set of a set T of homogeneous polynomials in S.

Zariski topology. Projective varieties. Quasi-projective varieties.

Proposition 2.2. *Let U_i $(i = 0, \ldots, n)$ be the subset of of $\mathbf{P}^n$ consisting of the points with homogeneous coordinates $(a_0, a_1, \ldots, a_n)$ with $a_i \neq 0$. Then U_i is an open subset and the map $\phi_i : U_i \to \mathbf{A}^n$ given by*

$$\phi_i : (a_0, a_1, \ldots, a_n) \mapsto (\frac{a_0}{a_i}, \ldots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i} \ldots, \frac{a_n}{a_i})$$

is a homeomorphism.

Corollary 2.3. *If Y is a projective (respectively, quasi-projective) variety, then Y is covered by the open subsets $Y \cap U_i$, $i = 0, 1, \ldots, n$, which are homeomorhic to affine (respectively, quasi-affine) varieties via the mapping ϕ_i defined above.*

Section I.3. Morphisms.

Definition. Let Y be a quasi-affine variety in $\mathbf{A}^n$ over the ground field k. A function $f : Y \to k$ is **regular at a point** $P \in Y$ if there is an open neighborhood U with $p \in U \subset Y$, and polynomials $g, h \in k[x_1, \ldots, x_n]$, such that h is nowhere zeros on U, and $f = g/h$ on U. We say that f is **regular on** Y if it is regular at every point of Y.

Lemma 3.1. *A regular function is continuous, where k is given the Zariski topology.*

Definition. Let Y be a quasi-projective variety in $\mathbf{P}^n$ over the ground field k. A function $f : Y \to k$ is **regular at a point** $P \in Y$ if there is an open neighborhood U with $p \in U \subset Y$, and homogeneous polynomials $g, h \in k[x_1, \ldots, x_n]$ of the same degree, such that h is nowhere zeros on U, and $f = g/h$ on U. We say that f is **regular on** Y if it is regular at every point of Y.

Definition. Let k be a fixed algebraically closed field. A **variety** over k is any affine, quasi-affine, projective, or quasi-projective variety. If X, Y are two varieties, a **morphism** $\phi : X \to Y$ is a continuous map such that for every open set $V \subset Y$, and every regular function $f : V \to k$, the function $f \circ \phi : \phi^{-1}(V) \to k$ is regular.

For an open subset $U \subset Y$, the ring of regular functions on U is denoted by $\mathcal{O}(U)$.

Theorem 3.2. *Let $Y \subset \mathbf{A}^n$ be an affine variety with coordinate ring $A(Y)$. Then*
(a) $\mathcal{O}(Y)$ is isomorphic to $A(Y)$;
(b) for each point $P \in Y$, let $\mathfrak{m}_P \subset A(Y)$ be ideal of functions vanishing at P. Then $P \mapsto \mathfrak{m}_P$ gives a 1-1 correspondence between the points of Y and the maximal ideals of $A(Y)$;
(c) for each P, $\mathcal{O}_p$ is isomorphic to $A(Y)_{\mathfrak{m}_P}$;
(d) $K(Y)$ is isomorphic to the quotient field of $A(Y)$, and hence $K(Y)$ is a finitely generated extension field of k of transcendence degree equal to $\dim Y$.

Proof. The proof of (b) (c) (d) are straightforward. For one, we have an obvious injective k-algebra homomorphism $A(Y) \to \mathcal{O}(Y)$. And we have $A(Y) \subset \mathcal{O}(Y) \subset K(Y) = \operatorname{Frac} A(Y)$. To prove $A(Y) = \mathcal{O}(Y)$, suppose $f \in \mathcal{O}(Y)$, for each point $P \in Y$, there is open neighborhood U_P containing P such that $f|_U = a_P/b_P$ for $a_P, b_P \in A(Y)$ and b_P never vanishes on U_P. Then $\{U_P\}_{P\in Y}$ is an open cover of Y, so it has a finite cover $Y = U_{P_1} \cup \cdots \cup U_{P_n}$. And $f|_{U_{P_i}} = a_i/b_i$, b_i never vanish on U_{P_i}. Since $Y = \cup_{i=1}^n U_{P_i}$, $(b_1, \ldots, b_n) = A(Y)$, so $b_1h_1 + \cdots + b_nh_n = 1$ for some $h_1, \ldots, h_n \in A(Y)$. Assume $h_1, \ldots, h_n$ are all non-zero (it should be clear how to proceed if this assumption is not satisfied), $f = \frac{a_1}{b_1} = \cdots = \frac{a_n}{b_n}$ as an element in $K(Y)$. It follows that $f = \frac{a_1h_1}{b_1h_1} = \cdots = \frac{a_nh_n}{b_nh_n}$. So

$$f = \frac{a_1h_1 + \cdots + a_nh_n}{b_1h_1 + \cdots + b_nh_n} = a_1h_1 + \cdots + a_nh_n \in A(Y).$$

□

Proposition 3.3. *Let $U_i \subset \mathbf{P}^n$ be the open set defined by the equation $x_i \neq 0$. Then the mapping $\phi_i : U_i \to \mathbf{A}^n$,*

$$[x_0, x_1, \ldots, x_n] \mapsto (x_0/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_n/x_i)$$

is an isomorphism of varieties.

Theorem 3.4. *Let $Y \subset \mathbf{P}^n$ be a projective variety with homogeneous coordinate ring $S(Y)$. Then:*
(a) $\mathcal{O}(Y) = k$;
(b) for any point $P \in Y$, let $\mathfrak{m}_P \subset S(Y)$ be the ideal generated by the set of homogeneous $f \in S(Y)$ such that $f(P) = 0$. Then $\mathcal{O}_P = S(Y)_{(\mathfrak{m}_P)}$;
(c) $K(Y) = S(Y)_{(0)}$.

Proposition 3.5. *Let X be any variety and Y be an affine variety. Then there is a natural bijective mapping of sets*

$$\alpha : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(A(Y), \mathcal{O}(X))$$

α is the obvious map. The inverse of α is the following: let $Y \subset k^n$ with $A(Y) = k[x_1, \ldots, x_n]/I$, we write x_i for $x_i + I$. If $h \in \mathrm{Hom}(A(Y), \mathcal{O}(X))$, then $h(x_1), \ldots, h(x_n) \in \mathcal{O}(X)$. We define a map $\psi : X \to k^n$ as follows, for $p \in X$,

$$\psi(p) = (h(x_1)(p), \ldots, h(x_n)(p)).$$

It is easy to $\mathrm{Im}\, \psi \in Y$ and by the following lemma, ψ is a morphism of varieties.

Lemma 3.6. *Let X be any variety, $Y \subset k^n$ be an affine variety, a map of sets $\psi : X \to Y$ is a morphism of variety iff $x_i \circ \psi$ is a regular function on X for $i = 1, \ldots, n$.*

Corollary 3.7. *Two affine varieties X and Y are isomorphic iff $A(X)$ and $A(Y)$ are isomorphic as k-algebras.*

Corollary 3.7. *The category of affine varieties and the category of finitely generated integral domains over k are anti-equivalent. The equivalence is given as $Y \mapsto A(Y)$.*

Section I.4. Rational Maps.

Corollary 3.7. *Let X and Y be varieties, and let $\phi, \psi : X \to Y$ be morphisms, suppose $\phi|_U = \psi|_U$ in some non-empty open set $U \subset X$, then $\phi = \psi$.*

The proof uses the variety structure on $P^n \times P^n$.

Let X, Y be varieties, a **rational map** $\phi : X \to Y$ is an equivalence class of pairs (U, ϕ), where $U \subset X$ is an non-empty subset, ϕ is a morphism of U to Y. The pairs (U_1, ϕ_1) and (U_2, ϕ_2) are equivalent iff $\phi_1|_{U_1 \cap U_2} = \phi_2|_{U_1 \cap U_2}$. The rational map represented by (U, ϕ) is **dominant** if $\operatorname{Im} \phi$ is dense in Y.

We need to justify the definition by proving that for any non-empty open subset $V \subset U$, $\overline{\phi(V)} = Y$. This can be proved as follows. If $\overline{\phi(V)} \neq Y$, then $\phi^{-1}(Y - \overline{\phi(V)})$ and V are disjoint non-empty open subsets of U, which contradicts the irreducibility of U.

Let X, Y be varieties, a **birational map** is a rational map $\phi : X \to Y$ that has an inverse.

Lemma 4.2. *Let Y be a hypersurface in A^n given by the equation $f(x_1, \ldots, x_n) = 0$, Then $A^n - Y$ is isomorphic to the hypersurface H in A^{n+1} given by the equation $x_{n+1} f = 1$. In particular, $A^n - Y$ is affine with its affine ring isomorphic to $k[x_1, \ldots, x_n]_f$.*

Proposition 4.3. *On any variety, there is a base for the topology consisting of open affine subsets.*

Section I.5. Non-Singular Varieties.

Definition. Let $Y \subset A^n$ be an affine variety, and let $f_1, \ldots, f_k \in k[x_1, \ldots, x_n]$ be a set of generators for the ideal of Y. Y is **nonsingular at** $P \in Y$ if the rank of the matrix $\{\frac{\partial f_i}{\partial x_j}(P)\}$ is $n - r$, where $r = \dim Y$. Y is **non-singular** if it is non-singular at every point.

Definition. Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $A/\mathfrak{m} = k$, A is a **regular local ring** if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim A$.

Theorem 5.1. *Let $Y \subset A^n$ be an affine variety. Let $P \in Y$ be a point. Then Y is nonsingular at P iff the local ring $\mathcal{O}_{P,Y}$ is a regular local ring.*

Sketch of Proof. Let $P = (a_1, \ldots, a_n) \in Y \subset A^n$. Let $\mathfrak{a}_P = \{f(x) \in k[x_1, \ldots, x_n] \mid f(P) = 0\}$. Then $\mathfrak{a}_P = (x_1 - a_1, \ldots, x_n - a_n)$. We define a linear map $\theta : \mathfrak{a}_P/\mathfrak{a}_P^2 \to k^n$ by

$$\theta(f + \mathfrak{a}_P^2) = (\frac{\partial f}{\partial x_1}(P), \ldots, \frac{\partial f}{\partial x_n}(P)).$$

It is easy to see that θ is an isomorphism. Let $\mathfrak{b}$ be the ideal of y, $A = k[x_1, \ldots, x_n]/\mathfrak{b}$. The maximal ideal of P in A is $\mathfrak{m} = \mathfrak{a}_P/\mathfrak{b}$. We have the exact sequence of vector spaces over k:

$$0 \to \mathfrak{b}/\mathfrak{b} \cap \mathfrak{a}_P^2 \to \mathfrak{a}_P/\mathfrak{a}_P^2 \to \mathfrak{m}/\mathfrak{m}^2 \to 0.$$

Under the isomorphism of θ,

$$\dim_k \theta(\mathfrak{b}/\mathfrak{b} \cap \mathfrak{a}_P^2) = \operatorname{rank}(\frac{\partial f_i}{\partial x_j}(P)).$$

So

$$\operatorname{rank}(\frac{\partial f_i}{\partial x_j}(P)) = n - r$$

iff $\dim_k \mathfrak{m}/\mathfrak{m}^2 = r$ iff the local ring $A_{\mathfrak{m}}$ is regular. □

Definition. Let Y be any variety, $P \in Y$ is called a **nonsingular point** if the local ring $\mathcal{O}_{P,Y}$ is regular. Y is **nonsingular** if every point in Y is nonsingular. Y is **singular** if it is not nonsingular.

Proposition 5.2A. *If A is a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k, then $\dim_k \mathfrak{m}/\mathfrak{m}^2 \geq \dim A$.*

Theorem 5.3. *Let Y be a variety, then the set $\operatorname{Sing} Y$ of singular points of Y is a proper closed subset of Y.*

Section I.6. Non-Singular Curves.

Let C be a nonsingular projective curve over k. The function field $K(C)$ is a finitely generated field extension over k with transcendence degree 1.

For example. If $C \subset \mathbf{P}^2$ is given by $y^2z - (x^3 + axz^2 + bz^3) = 0$. Suppose the equation $x^3+ax+b = 0$ has no repeated roots, then C is non-singular. The function field $K(C)$ is isomorphic to the fraction field of the integral domain

$$k[x, y]/(y^2 - (x^3 + ax + b)).$$

$K(C)$ is the extension of k by the generators x, y and x, y satisfies the relation

$$y^2 = x^3 + ax + b.$$

$k(x) \subset K(C)$. $K(C) = k(x)(y)$,y is algebraic over $k(x)$. So the transcendence degree of $K(C)$ over k is 1.

One of the main results (Theorem 6.9) of this section is the converse of the above. For a finitely generated extension K over k with transcendence degree 1, then there exists a nonsingular projective curve C over k such that the function field $K(C)$ is isomorphic to K. This C is unique up to isomorphism.

For example, for $K = k(x)$, the corresponding curve is $\mathbf{P}^1$.

If $\phi : C_1 \to C_2$ is a morphism of curves that is not a constant map, then ϕ induces a morphism of fields $\phi^* : K(C_2) \to K(C_1)$. ϕ^* a finite field extension over k. Conversely every finite field extension $f : K(C_2) \to K(C_1)$ is ϕ^* for a unique morphism $\phi : C_1 \to C_2$.

Section I.7. Intersections in Projective Spaces.

Proposition 7.1(Affine Dimension Theorem). *Let Y, Z be varieties of dimensions s, r in $\mathbf{A}^n$. Then each irreducible components of $Y \cap Z$ has dimensional $\geq s + r - n$.*

Proof. We may assume Y, Z are affine varieties in $\mathbf{A}^n$. Step 1. Prove the case that Z is a hypersurface in $\mathbf{A}^n$, i.e., $Z = Z(f)$ for some $f \in k[x_1, \ldots, x_n]$. Let $A(Y)$ be the affine coordinate ring of Y, then the irreducible components of $Y \cap Z$ corresponds to the minimal primes ideals in $A(Y)/(f)$. By Theorem 1.11A, each such minimal prime ideal has height one. The apply Theorem 1.8A. Second step. Let $Y \times Z$ embedded to $\mathbf{A}^{2n}$. We notice that $Y \cap Z$ is isomorphic to $Y \times Z \cap \Delta$, where $\Delta \overset{\text{def}}{=} \{((P, P) \in \mathbf{A}^{2n} \mid P \in \mathbf{A}^n\}$. Δ is given by the equation $x_1 - y_1 = 0, \ldots, x_n - y_n = 0$. □

Proposition 7.2(Projective Dimension Theorem). *Let Y, Z be varieties of dimensions s, r in $\mathbf{P}^n$. Then each irreducible components of Y, Z has dimensional $\geq s+r-n$. Furthermore, if $s+r-n \geq 0$, then $Y \cap Z$ is non-empty.*

Proof. The dimension inequality follows from the affine case. If $s+r-n \geq 0$. let $C(Y), C(Z) \subset \mathbf{A}^{n+1}$ be the cone of Y, Z respectively. Then $\dim C(Y) = \dim Y + 1 = s+1$, $\dim C(Z) \geq \dim Z + 1 = r+1$ (this can be proved using the chain of irreducible subsets, for any chain $Y_0 \subset \cdots \subset Y_s = Y$, we have the chain $\{0\} \subset C(Y_0) \subset \cdots \subset C(Y_s) = C(Y)$. So each irreducible components of $C(Y) \cap C(Z)$ has dimension $\geq s + 1 + r + 1 - (n+1) \geq 1$. But $C(Y) \cap C(Z)$ is not empty as contains 0. □

Definition. A **numerical polynomial** is a polynomial in $P(z) \in \mathbb{Q}[x]$ such that $P(n) \in \mathbb{Z}$ for n sufficiently large.

Proposition 7.3. *(a). If $P(x)$ is a numerical polynomial, then $P(x)$ can be written as*

$$P(z) = c_r \binom{z}{r} + \cdots + c_1 \binom{z}{r} + c_0$$

for some r and integers $c_0, c_1, \ldots, c_r \in \mathbb{Z}$. In particular $P(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$.
(b). If $f : \mathbb{Z} \to \mathbb{Z}$ be a map such that $\Delta f(n) = f(n+1) - f(n) = Q(n)$ for $n \gg 0$ for some numerical polynomial Q, then there is a numerical polynomial P such that $f(n) = P(n)$ for $n \gg 0$.

Let S be a graded ring, M be a graded S-module, for every integer l we denote $M(l)$ the same M-module but with gradation $M(l)_d = M_{d+l}$.

Proposition 7.4. *Let M be a finitely generated graded module over a noetherian graded ring S. Then there exists a a filtration of $0 = M^0 \subset M^1 \subset \cdots \subset M^r = M$ by graded submodules, such that for each i, $M^i/M^{i-1} \simeq (S/\mathfrak{p}_i)(l_i)$, where $\mathfrak{p}_i$ is a homogeneous ideal of S and $l_i \in \mathbb{Z}$. The filtration is not unique, but for any such filtration we have*
(a) if $\mathfrak{p}$ is homogeneous prime homogeneous ideal of S, then $\mathfrak{p} \supseteq \operatorname{Ann} M$ if and only $\mathfrak{p} \supseteq \mathfrak{p}_i$ for some i. In particular the minimal elements of the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ are just the minimal primes of M, i.e., the primes which minimal containing $\operatorname{Ann} M$.
(b) for each minimal prime ideal $\mathfrak{p}$ of M, the numner of times which $\mathfrak{p}$ occurs in the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ is equal to the length of $M_{\mathfrak{p}}$ over $\mathfrak{S}_{\mathfrak{p}}$ and hence is independent of the filtration.

Proof. The existence of the filtration follows from the book (using the assumption that S and M are noetherian.) (a) First we note that $\operatorname{Ann} M \subset \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$. And if $a_i \in \mathfrak{p}_i$, $i = 1, \ldots, r$, then $a_1 \cdots a_r \in \operatorname{Ann} M$. If $\mathfrak{p} \supseteq \operatorname{Ann} M$, we prove $\mathfrak{p} \supseteq \mathfrak{p}_i$ for some i. Suppose this is not true, we can find, for each i, $a_i \in \mathfrak{p}_i$, but $a_i \notin \mathfrak{p}$. then $a_1 \cdots a_r \in \operatorname{Ann} M$, $a_1 \cdots a_r \notin \mathfrak{p}$, contradicts to $\mathfrak{p} \supseteq \operatorname{Ann} M$. If $\mathfrak{p} \supseteq \mathfrak{p}_i \supset \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r \supset \operatorname{Ann} M$.

Theorem 7.5. (Hilbert-Serre) *(1) Let M be a finitely generated graded $S = k[x_0, x_1, \ldots, x_n]$-module. Let $\phi_M(l) = \dim_k M_l$, then there is a unique polynomial $P_M(z) \in \mathbb{Q}[z]$ such that $\phi_M(l) = P_M(l)$ for all $l \gg 0$. (2) Further more, $\deg P_M(z) = \dim Z(\operatorname{Ann} M)$, where $Z(\operatorname{Ann} M)$ is the zero set of the homogeneous ideal $\operatorname{Ann} M$ in $\mathbb{P}^n(k)$.*

Proof. First we note that for an exact sequence $0 \to M' \to M \to M'' \to 0$, we have

$$\operatorname{Ann}(M')\operatorname{Ann}(M'') \subset Ann(M) \subset \operatorname{Ann}(M') \cap \operatorname{Ann}(M'')$$

This implies that $Z(Ann(M)) = Z(Ann(M_1)) \cup Z(Ann(M_2))$, so

$$\dim Z(Ann(M)) = \max(\dim Z(Ann(M_1)), \dim Z(Ann(M_2))).$$

It is enough to prove the case $M = S/\mathfrak{p}(l)$ for some homogeneous prime ideal $\mathfrak{p}$. It is easy to prove the case $M = S/\mathfrak{p}$, where $\operatorname{Ann}(M) = \mathfrak{p}$. We may assume $\mathfrak{p} \neq (x_0, \ldots, x_n)$. Choose $x_i \notin \mathfrak{p}$. We have the exact sequence

$$0 \to M(1) \to M \to M'' = M/x_iM \to 0$$

where the first map is $a \mapsto x_1 a$.

$$dim(M/x_iM)_l = dim M_l - dim M_{l-1}$$

Using induction assumption, $dim(M/x_iM)_l = \phi_{M''}(l)$ for numerical polynomial $P_{M''}(l)$ for l large, so $\dim M_l = P_M(l)$ is also a numerical polynomial P_M for l large. We have $P_{M''}(l) = P_M(l) - P_M(l-1)$. So $\deg P_{M''} = \deg P_M - 1$. $Ann(M') = (x_i, \mathfrak{p})$, so $Z(\operatorname{Ann}(M')) = Z(\operatorname{Ann}(M)) \cap Z((x_i))$. □

Definition. The polynomial of the theorem is the **Hilbert polynomial** of M.

Definition. If $Y \subset \mathbf{P}^n(k)$ is an algebraic set of dimension r, let P_Y be the Hilbert polynomial of the homogeneous coordinate ring of Y, then $\deg P_Y = r$. We define the **degree** of Y to be $r!$ times the leading coefficient of P_Y.

Proposition 7.6.
(a) If $Y \subset \mathbf{P}^n$, Y is not empty, then the degree Y is a positive integer.
(b) If $Y = Y_1 \cup Y_2$, where Y_1 and Y_2 have the same dimension r, and $\dim Y_1 \cap Y_2 < r$, Then $\deg Y = \deg Y_1 + \deg Y_2$.
(c) $\deg \mathbf{P}^n = 1$
(d) If $H \subset \mathbf{P}^n$ is a hypersurface whose ideal is generated by a homogeneous polynomial of degree g, then $\deg Y = d$.

Proof. (a) (c) (d) are proved by direct computations. For (b), we use the exact sequence

$$0 \to S/I_1 \cap I_2 \to S/I_1 \oplus S/I_2 \to S/(I_1 + I_2) \to 0.$$

□

Let $Y \subset \mathbf{P}^n$ be a projective variety, and let $H \subset \mathbf{P}^n$ be a hypersurface not containing Y, $I(H) = (f)$, $\deg f = \deg H$. Then $Z(I_Y + I_H) = Y \cap H$. Notice that $I_Y + I_H \subset I_{Y \cap H}$. $Y \cap H = Z_1 \cup Z_2 \cup \cdots \cup Z_s$, Z_i's are the set of irreducible components of $Y \cap H$. Let $\mathfrak{p}_j$ be the prime ideal of Z_j. The **intersection multiplicity** is defined as

$$i(Y, H; Z_j) = \mu_{\mathfrak{p}_j}(S/(I_Y + I_H)).$$

Theorem 7.7. *Let Y be a variety of dimension ≥ 1 in $\mathbf{P}^n$, and let H be a hypersurface not containing Y. Let $Z_1, \ldots, Z_s$ be the irreducible components of $Y \cap H$. Then*

$$(\deg Y)(\deg H) = \sum_{j=1}^{s} i(Y, H; Z_j) \cdot \deg Z_j$$

Proof. Let $(f) = I_H$, consider the following exact sequence of graded S-modules

$$0 \longrightarrow S/I_Y(-d) \xrightarrow{f} S/I_Y \longrightarrow M \stackrel{\text{def}}{=} S/(I_Y + I_H) \longrightarrow 0$$

Let P_Y be the Hilbert polynomial for S/I_Y, similar meaning for P_M. Then we have

$$P_M(z) = P_Y(z) - P_Y(z - d).$$

$\deg P_Y = \dim Y = r$. Compare the coefficient of t^{r-1} in the above identity, we prove the identity in the theorem.

Corollary 7.8. (Bezout Theorem) *Let Y, Z be distinct curves in $\mathbf{P}^2$, having degree d, e. Let $Y \cap Z = \{P_1, \ldots, P_s\}$, then*

$$\sum_{j=1}^{s} i(Y, Z; P_j) = de.$$

Summary. $S = k[x_0, \ldots, x_n]$.
(1) Any finitely graded S-module M has a filtration of graded submodules

$$0 = M^0 \subset M^1 \subset \cdots \subset M^r = M$$

such that $M^i/M^{i-1} \cong S/\mathfrak{p}_i[l_i]$ for some homogeneous prime ideal $\mathfrak{p}_i$. The minimal members in the list $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ is the same as the minimal prime ideals containing $\operatorname{Ann}(M)$. The multiplicity of a minimal prime containing $\operatorname{Ann}(M)$ in the list $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ is independent of the filtration.
(2). For finitely graded S-module M, there is a polynomial $P_M(z)$ such that $P_M(l) = \dim M_l$ for $l \gg 0$.
(3). When $M = S/I_Y$, where Y is an algebraic set in $\mathbf{P}^n$ (not necessarily irreducible), we write $P_Y = P_{S/I_Y}$. then $\dim Y = \deg P_Y$. Suppose $\dim Y = r$. The $\deg Y$ is defined to be the $r!$ times the leading coefficient of $P_Y(z)$.
(4) For a homogeneous ideal $I \subset S$ (not necessarily prime), The minimal primes containing $I = \operatorname{Ann}(S/I)$ corresponds to the irreducible components of $Z(I)$.

The remaining part of this section is a proof of Pascal Hexagon Theorem. We need to extend Bezout Theorem.

A **generalized curve** in $\mathbf{P}^2$ is the zero set C in $\mathbf{P}^2$ given by a homogeneous polynomial $f = f_1 \cdots f_m$ (product of distinct irreducble polynomials). So $C = C_1 \cup \cdots \cup C_m$; $C_1, \ldots, C_m$ are curves.

$$\deg C \stackrel{\text{def}}{=} \deg f = \deg C_1 + \ldots \deg C_m.$$

If we have another generalized curve $D = D_1 \cup \cdots \cup D_m$, $P \in C \cap D$, we define

$$i(C, D, P) = \sum_{k=1,\ldots,m; j=1,\ldots,n} i(C_k, D_j, P).$$

The Bezout Theorem is implies that

$$\sum_P i(C, D; P) = (\deg C)(\deg D).$$

Theorem. *Let C_1 and C_2 be wo g-curves of degree n in $\mathbf{P}^2$ which intersect at n^2 points. Assume exactly nl $(l < n)$ of them lie in an irreducible curve E of degree l. Then the remaining $n(n-l)$ of points lie on a g-curve of degree at most $n - l$.*

Proof. Let C_1, C_2, E has equations $f_1(x, y, z), f_2(x, y, z), g(x, y, z)$, choose $P = (a, b, c) \in E \in C_1 \cap C_2$. Let

$$S(x, y, z) \stackrel{\text{def}}{=} f_1(a, b, c) f_2(x, y, z) - f_2(a, b, c) f_1(x, y, z).$$

$S(x, y, z) = 0$ gives a generalized curve F. $E \cap F$ contains at least $nl + 1$ points $(C_1 \cap C_2 \cap E) \cup \{P\}$. So by Bezout Theorem E is a component of F. So $S(, x, y, z) = g(x, y, z)h(x, y, z)$. The generalized curve $h(x, y, z) = 0$ has degree $\leq n - l$ and contains the other $n(n - l)$-pints.

Theorem. (**Pascal's Mystic Hexagon**) *Consider a hexagon inscribed in an irreducible conic in $\mathbf{P}^2$, then the three pairs of opposite sides of it meet in three collinear points .*

References

[Ha] R. Hartshorne, Algebraic Geometry.