Homework No.1 for Math 6170

Deadline: March 8.

All the problems are computational, just write down your answer, **no reasons needed**.

Problem 1. Let $C \subset \mathbb{A}^2(\mathbb{C})$ be the affine variety of dimension one defined by the equation

$$C = \{(x, y) \mid y^2 = x^2(x-1)\}$$

$\bar{C} \subset \mathbb{P}^2(\mathbb{C})$ be its projective closure.
(1) Find all the sigular points in C.
(2) How many are in $\bar{C} - C$? are they smooth points?
(3) Let $P = (1, 0) \in C$, it is easy to see P is a smooth point. Find a uniformizer at P.
(4) Find $\mathrm{ord}_P(y)$ and $\mathrm{ord}_P(\frac{y}{x-1})$

Problem 2. Let $C_1 = \mathbb{P}^1(\mathbb{C}) = \mathbb{A}^1(\mathbb{C}) \cup \{\infty\}$ and $C_2 = \mathbb{P}^1(\mathbb{C}) = \mathbb{A}^1(\mathbb{C}) \cup \{\infty\}$. Let $\mathbb{C}(C_1) = \mathbb{C}(X)$ denote the function field of C_1 and $\mathbb{C}(C_2) = \mathbb{C}(Y)$ denote the function field of C_2. Given a morphism of fields over $\mathbb{C}$ $\phi^* : \mathbb{C}(Y) \to \mathbb{C}(X)$ with

$$\phi^*(Y) = \frac{X^2 - 1}{X + 3}.$$

(1) Find the corresponding morphism of curves $\phi : C_1 \to C_2$.
(2) Find $\deg \phi$.
(3) Find all the ramified points.

Problem 3. Let C be the affine curve over $\mathbb{C}$ given by

$$C = \{(x, y) \mid y^2 - x^3 + x + 2 = 0\}$$

So the function field $\mathbb{C}(C)$ is

$$\mathbb{C}(C) = \mathrm{Frac}\, \mathbb{C}[X, Y]/(Y^2 - X^3 + X + 2).$$

(1). Find $g \in \mathbb{C}(C)$ such that $dY = gdX$.
(2). Let $P = (2, 2) \in C$, find a uniformizer at P.
(3) Find $\mathrm{ord}_P dX$.