Homework No. 3 for Math 6170, Due Date: May 5.

Problem 1. Let $E/\mathbb{Q}$ be the elliptic curve with Weierstrass equation

$$y^2 - \frac{1}{3}y = x^3 - \frac{1}{27}. \tag{0.1}$$

(1) Prove that $E(\mathbb{Q})$ has only three points. Find all the three points. Hint: you may use Fermat's result that the cubic Fermat's equation

$$a^3 + b^3 = c^3 \tag{0.2}$$

has only three projective solutions over $\mathbb{Q}$: $(1, 0, 1), (1, -1, 0), (0, 1, 1)$; and apply the transformation $a = -3x, b = -3y + z, c = -3y$.
(2) What is the group structure of $E(\mathbb{Q})$?

Problem 2. Let $[a_1, a_2, \ldots, a_N, 1] \in \mathbb{P}^N(\bar{\mathbb{Q}})$ and let

$$S = \{[a_1^n, a_2^n, \ldots, a_N^n, 1] \mid n = 1, 2, 3, \ldots\}$$

Suppose that there exists a constant C such that $H(P) \leq C$ for all $P \in S$.
(1) Prove that S is a finite set.
(2) Prove that each of a_i is either 0 or a root of unity.
Hint: Use Theorem VIII 5.2.