1. Introduction

Let $\mathcal{D}$ be a finite subset of $\mathbb{Z}^d$. $\mathcal{D}$ tiles $\mathbb{Z}^d$ if and only if $\mathbb{Z}^d$ can be written as a disjoint union of translates of $\mathcal{D}$, i.e., there is a set $\mathcal{C} \in \mathbb{Z}^d$ such that every point $v \in \mathbb{Z}^d$ can be expressed uniquely as $x + y$ with $x \in \mathcal{D}$ and $y \in \mathcal{C}$. In symbols, $\mathcal{D} \oplus \mathcal{C} = \mathbb{Z}^d$. $\mathcal{D}$ is called a tile and $\mathcal{C}$ the translation set.

In this note we give a sufficient and necessary condition for a subset $\mathcal{D}$ of $\mathbb{Z}^2$ with cardinality 4 to tile $\mathbb{Z}^2$. We may assume that $\mathcal{D}$ is not contained in a straight line. If $\mathcal{D}$ is contained in a line, then $\mathcal{D}$ can tile $\mathbb{Z}^2$ if and only if $\mathcal{D}$ can tile the set of integral points on that line, and such a sufficient and necessary condition for $\mathcal{D}$ was first given by Newman in [1]. In this note we prove the following result:

**Theorem 1.1.** Let $\mathcal{D}$ be a subset of $\mathbb{Z}^2$ with cardinality 4. Assume that $\mathcal{D}$ is not contained in a line, and furthermore $0 \in \mathcal{D}$. Then a sufficient and necessary condition for $\mathcal{D}$ can not tile $\mathbb{Z}^2$ is that there exists a $2 \times 2$ matrix $G$ so that

$$GD = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ p/q \end{pmatrix} \right\}$$

with $p, q \in \mathbb{Z} \setminus \{0\}$ and $p + q \in 2\mathbb{Z} + 1$; In another words, $\mathcal{D}$ can not tile $\mathbb{Z}^2$ if and only if $\mathcal{D} = \{v_1, v_2, v_3, v_4\}$ such that $v_2 - v_1 = \frac{p}{q}(v_4 - v_3)$ for some $p \in 2\mathbb{Z} \setminus \{0\}$ and $q \in 2\mathbb{Z} + 1$.

2. Proof of Theorem 1.1

**Proposition 2.1.** Let $\mathcal{A}$ be a finite subset of $\mathbb{Z}^d$. Then the following statements are equivalent:

(i) There exists $\mathcal{B} \subseteq \mathbb{Z}^d$ such that $\mathcal{A} \oplus \mathcal{B} = \mathbb{Z}^d$.  

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There exist a non-singular $d \times d$ matrix $G$ with rational entries and $D \subset \mathbb{Q}^d$ such that $(GA) \oplus D$ is a lattice in $\mathbb{R}^d$.

**Proof.** The direction (i) $\implies$ (ii) is clear. Now we show the opposite direction. Assume that $(GA) \oplus D$ is a lattice in $\mathbb{R}^d$, i.e., $(GA) \oplus D = H\mathbb{Z}^2$ for some $d \times d$ matrix $H$. Clearly $H$ is rational. We may assume that $H$ is non-singular (otherwise there exists $C \subset \mathbb{Q}^d$ so that $HZ^d \oplus C = \tilde{H}Z^2$). Define $p := pH^{-1}G$ is an integral matrix. Then $(E) \oplus \Lambda \ominus V = \mathbb{Z}^d$. Choose an integer $p$ so that $E := pH^{-1}G$ is an integral matrix. Note that $(E) \oplus (p\mathbb{Z}^d) = \mathbb{Z}^d$ and $E \subset \mathbb{Z}^d$. It follows that $\Lambda := (p\mathbb{Z}^d) \ominus (\Lambda \ominus V)$ is a lattice in $\mathbb{Z}^d$. Since $p$ is an integer, there exists a finite set $U \subset \mathbb{Z}^d$ so that $(p\mathbb{Z}^d) \oplus U = \mathbb{Z}^d$. We have $(E) \oplus \Lambda \ominus V = \mathbb{Z}^d$. Therefore $(E) \oplus \Lambda \ominus V = \mathbb{Z}^d$. Note that $\mathbb{Z}^d = (EZ^d) \ominus U$ for some finite set $U \subset \mathbb{Z}^d$ with $0 \in U$. We have $(E) \oplus \Lambda \ominus V = (EZ^d) \ominus U$. Letting $\Lambda = (EZ^d) \ominus (\Lambda \ominus V)$, we obtain $(E) \oplus \Lambda = EZ^d$. This implies $A \oplus (E^{-1}\Lambda) = \mathbb{Z}^d$.

**Corollary 2.2.** Let $A$ and $B$ be two finite subsets of $\mathbb{Z}^d$. If $A = GB$ for some non-singular $d \times d$ rational matrix $G$, then $A$ can tile $\mathbb{Z}^d$ if and only if $B$ can tile $\mathbb{Z}^d$.

**Lemma 2.3.** Let $C = \{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} p_1 \\ \frac{1}{2} + p_2 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} + p_3 \\ t + p_4 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} + p_5 \\ t + p_6 \\ 0 \end{pmatrix} \}$ for some $t \in \mathbb{Q}$ and $p_j \in \mathbb{Z}$ ($j = 1, \ldots, 6$). Then there exists $D \subset \mathbb{Q}^2$ such that $C \oplus D$ is a lattice in $\mathbb{R}^2$.

**Proof.** Let $t = \frac{m}{n}$, where $m \in \mathbb{Z}$, $n \in \mathbb{N}$ and $\gcd(m,n) = 1$. Define

$$D = \left\{ \begin{pmatrix} u \\ v + \frac{j}{2n} \end{pmatrix} : u,v \in \mathbb{Z}, j = 0,1,\ldots,n-1 \right\}.$$ 

Then one can check that

$$C \oplus D = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2n} \end{pmatrix} \mathbb{Z}^2.$$ 

This finishes the proof.

**Corollary 2.4.** Let $A$ be a subset of $\mathbb{Z}^2$ of cardinality 4. Assume $0 \in A$. Then $A$ can tile $\mathbb{Z}^2$ if $A = GC$ for some $2 \times 2$ non-singular rational matrix $G$ and some $C \subset \mathbb{R}^2$ which has the form as in Lemma 2.3.

**Remark 2.5.** I doubt that the above “if” can be replaced by “iff”.

**Lemma 2.6.** For any $u,v \in \mathbb{Q}$, one of the following three equations has a solution $(x,y,z) \in \mathbb{Z}^3$:

\begin{align*}
(1) & \quad \begin{pmatrix} u \\ v \\ z \end{pmatrix} \\
(2) & \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
(3) & \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\end{align*}
TILING $\mathbb{Z}^2$ BY A SET OF FOUR ELEMENTS

(i) $(\frac{1}{2} + x)u + (\frac{1}{2} + y)v = z$.
(ii) $xu + (\frac{1}{2} + y)v = \frac{1}{2} + z$.
(iii) $(\frac{1}{2} + x)u + yv = \frac{1}{2} + z$.

Proof. It is easily check that if one of the above three equations has an integral solution, then that equation also has an integral solution when we change $u, v$ to $\tilde{u}$ and $\tilde{v}$ so that $\tilde{u}/u, \tilde{v}/v \in \frac{2z+1}{2z+1}$. Thus to prove the lemma, we can assume without loss of generality that $u = 2^m$ and $v = 2^n$ for $m, n \in \mathbb{Z}$. Then it is a routine to check one of the above three equations must have an integral solution. □

Proposition 2.7. Let

$$D = \left\{ \begin{pmatrix} 0 \\ 0 \\ \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \end{pmatrix}, \begin{pmatrix} u \\ v \\ \end{pmatrix} \right\}$$

where $u, v \in \mathbb{Q}$. Then there exists a non-singular $2 \times 2$ rational matrix $G$ such that $GD$ has the same form as $C$ in Lemma 2.3 if $u, v$ do not satisfy anyone of the following conditions:

(i) $u = 1$ and $v \notin \frac{2z+1}{2z+1}$.
(ii) $v = 1$ and $u \notin \frac{2z+1}{2z+1}$.
(iii) $u = -v$ and $u \notin \frac{2z+1}{2z+1}$.

Proof. We will prove the existence of $G$ in each of the following scenarios:

(1) $u = 1$ and $v \in \frac{2z+1}{2z+1}$.
(2) $v = 1$ and $u \in \frac{2z+1}{2z+1}$.
(3) $u = -v$ and $u \in \frac{2z+1}{2z+1}$.
(4) $u \neq 1, v \neq 1$ and $u \neq -v$.

For scenario (1), let $v = \frac{2q+1}{2p+1}$, where $p, q \in \mathbb{Z}$. We may take $G = \begin{pmatrix} 0 & p + \frac{1}{2} \\ \frac{1}{2} & 2p + 1 \end{pmatrix}$. Then

$$GD = \left\{ \begin{pmatrix} 0 \\ 0 \\ \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \end{pmatrix}, \begin{pmatrix} \frac{1}{2} + p \\ 2p + 1 \\ \end{pmatrix}, \begin{pmatrix} \frac{1}{2} + q \\ 2q + 1 \\ \end{pmatrix} \right\}.$$

For scenario (2), let $u = \frac{2q+1}{2p+1}$, where $p, q \in \mathbb{Z}$. We may take $G = \begin{pmatrix} p + \frac{1}{2} & 0 \\ 2p + 1 & \frac{1}{2} \end{pmatrix}$. Then $GD$ has the same expression as that in scenario (1).

For scenario (3), let $u = \frac{2q+1}{2p+1}$, where $p, q \in \mathbb{Z}$. We may take $G = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 2p + 1 & p + \frac{1}{2} \end{pmatrix}$. Then $GD$ has the same expression as that in scenario (1).
Now let us turn to the scenario (4). By Lemma 2.6, one of the following equations has an integral solution \((x, y, z)\):

\[
\begin{align*}
(e1) \quad & (1/2 + x)u + (1/2 + y)v = z. \\
(e2) \quad & xu + (1/2 + y)v = 1/2 + z. \\
(e3) \quad & (1/2 + x)u + yv = 1/2 + z.
\end{align*}
\]

Assume at first that \((e1)\) has an integral solution \((x, y, z)\). Since \(u \neq -v\), there exists \(t \in \mathbb{Q}\) such that

\[
(t + x)u + (1/2 + t + y)v = 1/2 + z.
\]

Take \(G = \left(\begin{array}{ccc}
1/2 + x & 1/2 + y \\
t + x & t + y
\end{array}\right)\). Then \(GD = \left\{\left(\begin{array}{c}0 \\
0\end{array}\right), \left(\begin{array}{c}1/2 + x \\
t + x\end{array}\right), \left(\begin{array}{c}1/2 + y \\
t + y\end{array}\right), \left(\begin{array}{c}1/2 + z \\
z\end{array}\right)\right\}\).

Now we assume \((e2)\) has an integral solution \((x, y, z)\). Since \(v \neq 1\), there exists \(t \in \mathbb{Q}\) so that

\[
1/2 u + t v = 1/2 + t.
\]

Take \(G = \left(\begin{array}{ccc}
x & 1/2 + y \\
t & t
\end{array}\right)\). Then \(GD = \left\{\left(\begin{array}{c}0 \\
0\end{array}\right), \left(\begin{array}{c}x \\
t\end{array}\right), \left(\begin{array}{c}1/2 + y \\
t\end{array}\right), \left(\begin{array}{c}1/2 + z \\
t + t\end{array}\right)\right\}\).

If \((e3)\) has an integral solution, we may construct \(G\) in a similar way as above.

**Proposition 2.8.** Let

\[
\mathcal{D} = \left\{\left(\begin{array}{c}0 \\
0\end{array}\right), \left(\begin{array}{c}1 \\
0\end{array}\right), \left(\begin{array}{c}0 \\
1\end{array}\right), \left(\begin{array}{c}u \\
v\end{array}\right)\right\}
\]

where \(u, v \in \mathbb{Q}\). Then there exists no \(C\) such that \(\mathcal{D} \oplus C\) is a lattice if \(u, v\) does satisfy one of the following conditions:

\begin{itemize}
  \item[(i)] \(u = 1\) and \(v \notin \frac{2u+1}{2v+1}\).
  \item[(ii)] \(v = 1\) and \(u \notin \frac{2u+1}{2v+1}\).
  \item[(iii)] \(u = -v\) and \(u \notin \frac{2u+1}{2v+1}\).
\end{itemize}

**Proof.** Without loss of generality we may only consider case (ii), since the sets \(\mathcal{D}\) in cases (i) and (iii) differ from that in (ii) only by an affine map.

Assume \(u = \frac{p}{q}\) with \(p \in \mathbb{Z}, q \in \mathbb{N}\) and \(p + q \in 2\mathbb{Z} + 1\). Take \(G = \left(\begin{array}{cc}q & 0 \\
0 & 1\end{array}\right)\). Then

\[
GD = \left\{\left(\begin{array}{c}0 \\
0\end{array}\right), \left(\begin{array}{c}q \\
0\end{array}\right), \left(\begin{array}{c}0 \\
1\end{array}\right), \left(\begin{array}{c}p \\
1\end{array}\right)\right\}
\]

By Proposition 2.1, we only need to prove that \(GD\) can not tile \(\mathbb{Z}^2\).

Assume on the contrary that \(GD\) can tile \(\mathbb{Z}^2\), i.e., \((GD) \oplus \Lambda = \mathbb{Z}^2\). Then any \(x \in \mathbb{Z}^2\) can be uniquely written as \(x = x_1 + x_2\) with \(x_1 \in GD\) and \(x_2 \in \Lambda\). Define \(\phi: \mathbb{Z}^2 \to GD\) by
Let \( \{a_n\}_{n \in \mathbb{Z}} \) be the sequence defined by
\[
a_n = \begin{cases} 
1 & \text{if } \phi(n, 0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
2 & \text{if } \phi(n, 0) = \begin{pmatrix} 0 \\ q \end{pmatrix} \\
3 & \text{if } \phi(n, 0) = \begin{pmatrix} q \\ 0 \end{pmatrix} \\
4 & \text{if } \phi(n, 0) = \begin{pmatrix} p \\ 1 \end{pmatrix}
\end{cases}
\]

We have the following observations:

(a) For any \( n \in \mathbb{Z} \), \( a_{n+p} \neq a_n \) and \( a_{n+q} \neq a_n \).

(b) If \( a_n = 1 \) then \( a_{n+q} = 2 \). If \( a_n = 2 \) then \( a_{n-q} = 1 \). If \( a_n = 3 \) then \( a_{n+p} = 4 \). If \( a_n = 4 \) then \( a_{n-p} = 3 \).

Let us first prove (a). From \((GD) \oplus \Lambda = \mathbb{Z}^2\) we obtain \((GD-GD) \cap (\Lambda - \Lambda) = \{0\}\). Since \(\begin{pmatrix} p \\ 0 \end{pmatrix}, \begin{pmatrix} q \\ 0 \end{pmatrix} \in GD-GD\), we have \(\begin{pmatrix} p \\ 0 \end{pmatrix}, \begin{pmatrix} q \\ 0 \end{pmatrix} \not\in \Lambda - \Lambda\). Now assume (a) is not true. Without loss of generality we assume \(a_{n+p} = a_n\) for some \(n\). Then
\[
\begin{pmatrix} n \\ 0 \end{pmatrix} = y + \lambda_1, \quad \begin{pmatrix} n+p \\ 0 \end{pmatrix} = y + \lambda_2
\]
for some \(y \in GD\) and \(\lambda_1, \lambda_2 \in \Lambda\). It implies that \(\begin{pmatrix} p \\ 0 \end{pmatrix} = \lambda_2 - \lambda_1 \in \Lambda - \Lambda\), which leads to a contradiction. This proves (a). To prove (b) without loss of generality we prove that \(a_{n+q} = 2\) when \(a_n = 1\). Since \(a_n = 1\), we have \(\begin{pmatrix} n \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda\) for some \(\lambda \in \Lambda\). Therefore \(\begin{pmatrix} n+q \\ 0 \end{pmatrix} = \begin{pmatrix} q \\ 0 \end{pmatrix} + \lambda\), which implies \(a_{n+q} = 2\). This finishes the proof of (b).

According to (a) and (b), we have the following claims:

(c1) Assume \(p > 0\). If \(a_n \in \{1, 3\}\), then \(a_{n+p+q} \in \{1, 3\}\).

(c2) Assume \(p < 0\). If \(a_n \in \{1, 4\}\), then \(a_{n-p+q} \in \{2, 3\}\).

Without loss of generality we only prove (c1). First assume \(a_n = 1\). Then by (b) we have \(a_{n+q} = 2\). Thus by (a) we have \(a_{n+p+q} \neq 2\). In the same time by (b) we have \(a_{n+p+q} \neq 4\) since otherwise \(a_{n+q} = 3\). Therefore we always have \(a_n \in \{1, 3\}\) when \(a_n = 1\). Using an essentially identical argument, we can obtain that \(a_n \in \{1, 3\}\) when \(a_n = 3\). This finishes the proof of (c1).
Now assume \( p > 0 \). Then (c1) implies that the set \( \{0, 1, \ldots, p + q\} \) can be partitioned into two sets \( A \) and \( B \) such that there exists a large \( N \in \mathbb{N} \) so that for \( n > N \), \( a_n \in \{1, 3\} \) if \( n \mod (p + q) \in A \), and \( a_n \in \{2, 4\} \) if \( n \mod (p + q) \in B \). That means the density of those \( n \) with \( a_n \in \{1, 3\} \) in \( \mathbb{Z} \cap [N, \infty) \) is \( \#A/(p + q) \), and the density of the rest is \( \#B/(p + q) \). Since \( p + q \in 2\mathbb{Z} + 1 \), these two densities are different. However from (b), these two densities must be the same. This leads to a contradiction.

A contradiction can be derived on the same line for the case \( p < 0 \). We omit the details.

**Proof of Theorem 1.1** Since \( D \) is not contained in a line, there exists a non-singular rational \( 2 \times 2 \) matrix \( A \) so that \( AD = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \), \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), \( \begin{pmatrix} u & v \\ 0 & 1 \end{pmatrix} \) with \( u, v \in \mathbb{Q} \). Assume \( D \) can not tile \( \mathbb{Z}^2 \). Then by Proposition 2.1, There is no non-singular rational matrix \( G \) and \( C \subset \mathbb{Q}^2 \) such that \( GD \oplus C \) be a lattice. Therefore by Proposition 2.7 and Lemma 2.3, \( u, v \) do satisfy one of the following conditions:

(i) \( u = 1 \) and \( v \notin \frac{2z+1}{2z+1} \).

(ii) \( v = 1 \) and \( u \notin \frac{2z+1}{2z+1} \).

(iii) \( u = -v \) and \( u \notin \frac{2z+1}{2z+1} \).

Thus there exists a non-singular rational matrix \( B \) such that

\[
BAD = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & p/q \end{pmatrix} \right\}
\]

with \( p, q \in \mathbb{Z} \setminus \{0\} \) and \( p + q \in 2\mathbb{Z} + 1 \). This proves the necessity. The sufficiency is implied by Proposition 2.8.

**References**