

Bounded Semigroups of Matrices

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ABSTRACT

In this note are proved two conjectures of Daubechies and Lagarias. The first asserts that if Σ is a bounded set of matrices such that all left infinite products converge, then Σ generates a bounded semigroup. The second asserts the equality of two differently defined joint spectral radii for a bounded set of matrices. One definition involves the conventional spectral radius, and one involves the operator norm.

The occurrence of convergent infinite products of matrices pervades many current areas of mathematics. See, for example, the various articles in [6]. Recently, in studying curve and surface generation, several authors [1, 2, 5] have been led to sets of matrices all (or almost all) infinite products of which converge. Although the contexts vary, this infinite product convergence seems to be a fundamental underlying phenomenon. Thus in [5] Micchelli and Prautzsch are motivated by subdivision methods, in [2] Daubechies and Lagarias are motivated by wavelets and dilation equations, and in [1] Berger is motivated by iterated function systems and random algorithms for curve and surface generation.

In [2] Daubechies and Lagarias explore sets of matrices all infinite products of which converge. This note presents some additional results in that direction, and studies the general structure of bounded semigroups of

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matrices. It also proves two conjectures made in [2], called by them the *boundedness conjecture* and the *generalized spectral radius conjecture*.

Let $\mathcal{M} = \mathcal{M}_n$ denote the algebra of all real $n \times n$ matrices, and let $\|\cdot\|$ be a norm on $\mathbb{R}^{(n)}$. This norm induces a corresponding operator norm $\|\cdot\|$ on \mathcal{M} . Let $\Sigma \subseteq \mathcal{M}$ be a nonempty *bounded* set of matrices, and denote by $\mathcal{S} = \mathcal{S}(\Sigma)$ the semigroup generated by Σ augmented with the $n \times n$ identity matrix $I = I_n$, so that $\mathcal{S} = \bigcup_{m=0}^{\infty} \Sigma^m$, where $\Sigma^m = \prod_{i=1}^m M_i$; $M_i \in \Sigma$, $1 \leq i \leq m$. We say that Σ is LCP (left convergent products) if every infinite product from Σ left converges, i.e., if $\lim_{m \rightarrow \infty} M_m \cdots M_1$ exists for any sequence $(M_i)_{i=1}^{\infty}$ in Σ . In this case denote by Σ^{∞} the set of all such limits. Define

$$\|\Sigma\| = \sup\{\|M\| : M \in \Sigma\}, \quad \hat{\rho} = \hat{\rho}(\Sigma) = \limsup_{m \rightarrow \infty} \|\Sigma^m\|^{1/m}.$$

The quantity $\hat{\rho}(\Sigma)$ is a special case of the *joint spectral radius* of a bounded subset of a normed algebra defined in [7]. Observe that $\hat{\rho}$ does not depend on the particular choice for the norm on \mathcal{M} .

THEOREM I

(a) Product boundedness. *If Σ is LCP then \mathcal{S} is bounded. (That is, Σ is product bounded, using the terminology from [2].) In particular $\hat{\rho} \leq 1$.*

(b) Σ is LCP with $\Sigma^{\infty} = 0$ if and only if $\hat{\rho} < 1$.

Proof. (a): Let X be the subspace

$$X = \left\{ \mathbf{x} \in \mathbb{R}^{(n)} : \sup_{S \in \mathcal{S}} \|S\mathbf{x}\| < \infty \right\}.$$

Then X is invariant under each $M \in \Sigma$, and by the uniform boundedness principle $\|\mathcal{S}|_X\| < \infty$. Suppose X is *not* all of $\mathbb{R}^{(n)}$.

CLAIM. $\forall \mathbf{x} \notin X, C > 0$ there is $S \in \mathcal{S}$ such that

$$S\mathbf{x} \notin X \quad \text{and} \quad \|S\mathbf{x}\| > C.$$

Indeed, since $\mathbf{x} \notin X$, there exists

$$S' = M_m \cdots M_1 \quad \text{with} \quad M_i \in \Sigma, \quad 1 \leq i \leq m,$$

such that

$$\|S'x\| > \max(1, \|\mathcal{S}|_X\| \cdot \|\Sigma\|)C. \tag{1}$$

If $S'x \notin X$ then simply take $S = S'$. Otherwise choose $k < m$ such that

$$M_k \cdots M_1x \notin X \quad \text{but} \quad M_{k+1} \cdots M_1x \in X.$$

Since

$$\|S'x\| \leq \|\mathcal{S}|_X\| \cdot \|M_{k+1} \cdots M_1x\| \leq \|\mathcal{S}|_X\| \cdot \|\Sigma\| \cdot \|M_k \cdots M_1x\|,$$

we get from (1) that $\|M_k \cdots M_1x\| > C$. Thus $S = M_k \cdots M_1$ satisfies the Claim.

It follows now from the Claim that for each $x \notin X$ we can find a sequence $(S_i)_{i=1}^\infty$ in \mathcal{S} such that

$$\|S_m \cdots S_1x\| > m \quad \forall m,$$

which contradicts the LCP property.

(b): The “if” part is immediate, since $\|\Sigma^m\| < 1$ for some m . Suppose then that Σ is LCP with $\Sigma^\infty = 0$. Since we have established in part (a) above that \mathcal{S} is bounded, it follows from a “diagonal sequence argument,” as in Daubechies and Lagarias [2, Lemma 5.2], that $\hat{\rho} < 1$. ■

Let $\rho(M)$ denote the spectral radius of $M \in \mathcal{M}$, and define

$$\rho(\Sigma) = \sup\{\rho(M) : M \in \Sigma\}, \quad \rho_* = \rho_*(\Sigma) = \limsup_{m \rightarrow \infty} [\rho(\Sigma^m)]^{1/m}.$$

LEMMA II.

(a) \mathcal{S} is bounded if and only if there exists an operator norm ν on \mathcal{M} such that $\nu(\Sigma) \leq 1$.

(b) For any $\delta > \hat{\rho}$ there exists an operator norm $\nu = \nu_\delta$ on \mathcal{M} such that $\nu(\Sigma) \leq \delta$.

(c) Assume that the matrices $M \in \Sigma$ are all of the block upper-triangular form

$$M = \begin{pmatrix} M^{(1)} & & * \\ & \ddots & \\ 0 & & M^{(l)} \end{pmatrix},$$

where the $M^{(j)}$'s are square submatrices. Set $\Sigma^{(j)} = \{M^{(j)} : M \in \Sigma\}$. Then

$$\rho_*(\Sigma) = \max_{1 \leq j \leq l} \rho_*(\Sigma^{(j)}), \quad (2)$$

$$\hat{\rho}(\Sigma) = \max_{1 \leq j \leq l} \hat{\rho}(\Sigma^{(j)}). \quad (3)$$

Proof. (a): The “if” part is immediate, since $\nu(M_1 M_2) \leq \nu(M_1) \nu(M_2) \forall M_1, M_2 \in \mathcal{M}$. Suppose then that \mathcal{S} is bounded. For $\mathbf{x} \in \mathbb{R}^{(n)}$ define $\nu(\mathbf{x}) = \sup\{\|S\mathbf{x}\| : S \in \mathcal{S}\}$. (Here $\|\cdot\|$ is an arbitrary norm.)

(b): Since $\hat{\rho}(\Sigma/\delta) < 1$, it follows from Theorem I that $\mathcal{S}(\Sigma/\delta)$ is bounded. Thus, from part (a) of this lemma, there exists an operator norm ν on \mathcal{M} such that $\nu(\Sigma/\delta) \leq 1$.

(c): Equation (2) follows from block expansion and factorization of the determinant. To prove (3) write $M = D + N$, where $D = D_M$ is the block diagonal part of M ,

$$D = \begin{pmatrix} M^{(1)} & & 0 \\ & \ddots & \\ 0 & & M^{(l)} \end{pmatrix},$$

and $N = N_M$ is the nilpotent part of M , $N = M - D$. Let $\|\cdot\|$ be the l_1 -norm. (Actually almost any “standard” norm will do here.) This norm satisfies the properties: (i) the norm of any matrix with zeros in it can only go up (or stay the same) if nonzeros are substituted for some of the zero entries; (ii) $\|D\| = \max_{1 \leq j \leq l} \|M^{(j)}\|$. Then it is easy to see, using this norm, that

$$\hat{\rho}(\Sigma) \geq \hat{\rho}(\Sigma_D) = \max_{1 \leq j \leq l} \hat{\rho}(\Sigma^{(j)}).$$

On the other hand, given any $\delta > \max_{1 \leq j \leq l} \hat{\rho}(\Sigma^{(j)})$, there exists, by part (b)

of this lemma, an operator norm $\nu = \nu_\delta$ on \mathscr{A} such that $\nu(\Sigma_D) \leq \delta$. Observe that each term in the expansion of the product

$$M_m \cdots M_1 = (D_m + N_m) \cdots (D_1 + N_1) = D_m \cdots D_1 + \cdots + N_m \cdots N_1$$

which contains at least n nilpotent factors must be zero. Thus

$$\nu(\Sigma^m) \leq \sum_{k=1}^{n-1} \binom{m}{k} \nu^k(\Sigma_N) \nu^{m-k}(\Sigma_D) \leq C m^{n-1} \nu^m(\Sigma_D),$$

where C is a constant independent of m . From this it follows that $\hat{\rho}(\Sigma) \leq \nu(\Sigma_D) \leq \delta$. \blacksquare

Let $\mathscr{A} = \mathscr{A}(\Sigma)$ denote the algebra generated by Σ . Observe that \mathscr{A} is spanned by $\bigcup_{m=1}^{\infty} \Sigma^m$, so that we can choose a basis $B_1, \dots, B_N \in \bigcup_{m=1}^L \Sigma^m$ for \mathscr{A} , for some $L > 0$.

PROPOSITION III. *Suppose $\hat{\rho} = 1$. If \mathscr{A} is semisimple then \mathscr{S} is bounded.*

Proof. Let $a_m \downarrow 1$, and let $\nu_m = \nu_{a_m}$ be an operator norm on \mathscr{A} such that $\nu_m(\Sigma) \leq a_m$. The existence of ν_m is guaranteed by Lemma II(b). For each basis element B_i

$$\nu_m(B_i) \leq a_m^L \leq a_1^L \quad \forall m, \quad (4)$$

where L was defined above.

CLAIM. *The (ν_m) are bounded and equicontinuous on the (compact) sphere $\mathscr{C} = \{A \in \mathscr{A} : \|A\| = 1\}$. (Here $\|\cdot\|$ is an arbitrary norm.)*

Indeed, each $A \in \mathscr{A}$ has a unique representation $A = \sum \alpha_i B_i$, and so $\sum |\alpha_i|$ is a norm on \mathscr{A} . Since all norms on \mathscr{A} are equivalent, there exists $K > 0$ such that $\sum |\alpha_i| \leq K \|A\|$. From this it follows, using (4), that if $A \in \mathscr{C}$ then $\nu_m(A) \leq K \alpha_1^L \forall m$. Thus (ν_m) is bounded on \mathscr{C} . Similarly, if $\|A_1 - A_2\| < \epsilon$, $A_1, A_2 \in \mathscr{A}$, then

$$|\nu_m(A_1) - \nu_m(A_2)| \leq \nu_m(A_1 - A_2) \leq K \alpha_1^L \epsilon,$$

and so (ν_m) is equicontinuous.

Now that we have established our Claim, we can extract a convergent subsequence $\nu_{m_k} \rightarrow \nu$ on \mathcal{A} . Clearly ν is a seminorm on \mathcal{A} satisfying

$$\nu(A_1 A_2) \leq \nu(A_1) \nu(A_2) \quad \forall A_1, A_2 \in \mathcal{A},$$

$$\rho(A) \leq \nu(A) \quad \forall A \in \mathcal{A},$$

$$\nu(\Sigma) \leq 1.$$

Thus $\mathcal{I} = \{A \in \mathcal{A} : \nu(A) = 0\}$ is a nilpotent ideal in \mathcal{A} [3, p. 39]. Since \mathcal{A} is semisimple we must have $\mathcal{I} = 0$, and thus ν is a bona fide norm. By Lemma II(a) \mathcal{I} is bounded. ■

THEOREM IV (Spectral radius). $\rho_*(\Sigma) = \hat{\rho}(\Sigma)$ for all bounded sets Σ .

Proof. Since $\rho(M) \leq \|M\|$ for any operator norm $\|\cdot\|$, it is clear that $\rho_* \leq \rho$. By scaling if necessary, we may assume that $\hat{\rho} = 1$. Let $\mathcal{A} = \mathcal{A}(\Sigma)$ be the algebra generated by Σ . Suppose first that \mathcal{A} is semisimple. Then according to Proposition III, \mathcal{I} is bounded. Let $\Omega = \Omega(\Sigma)$ denote the set of all limit points of sequences $(S_m)_{m=1}^\infty$ where $S_m \in \Sigma^m \forall m$. Since \mathcal{A} is closed, $\Omega \subseteq \mathcal{A}$; and since \mathcal{I} is bounded, $\Omega \neq \emptyset$. Also $\Omega \neq \{0\}$, since Ω contains Σ^∞ and $\Sigma^\infty \neq \{0\}$ by Theorem I(b). Observe that Ω is a semigroup satisfying $\Omega \mathcal{I} = \Omega$. Thus the enveloping algebra $\mathcal{A}(\Omega)$ is a right ideal of \mathcal{A} .

If $\rho_* < 1$, then every element in Ω is nilpotent, and so it follows from Levitzki's theorem [3, Theorem VIII.5.1; 4, Theorem II.3.5] that $\mathcal{A}(\Omega)$ is a nilpotent right ideal of \mathcal{A} . Thus $\mathcal{A}(\Omega)$ must be contained in the radical of \mathcal{A} [3, p. 9], and since \mathcal{A} is semisimple, it follows that $\Omega = \mathcal{A}(\Omega) = 0$, a contradiction.

If \mathcal{A} is not semisimple, it can be simultaneously reduced via similarity so that each $A \in \mathcal{A}$ has the block upper-triangular form

$$A = \begin{pmatrix} A^{(1)} & & * \\ & \ddots & \\ 0 & & A^{(l)} \end{pmatrix},$$

where $A^{(j)}$'s are square submatrices, and each algebra $\mathcal{A}^{(j)} = \{A^{(j)} : A \in \mathcal{A}\}$ is semisimple. (Indeed, we can get each $\mathcal{A}^{(j)}$ to be in fact irreducible.)

Moreover each $\mathcal{A}^{(j)}$ is generated by $\Sigma^{(j)} = \{M^{(j)} : M \in \Sigma\}$, after similarity reduction; and thus our result follows now from Lemma II(c). ■

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