Linear Algebra

Min Yan

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Chapter 1

Vector Space

Linear structure is one of the most basic structures in mathematics. The key object for linear structure is vector space, characterised by the operations of addition and scalar multiplication. The key relation between vector space is linear transformation, characterised by preserving the two operations. The key example of vector space is the Euclidean space, which is the model for all finite dimensional vector spaces.

The theory of linear algebra can be developed over any field, which is a "number system" that allows the usual four arithmetic operations. In fact, a more general theory (of modules) can be developed over any ring, which is a system that allows addition, subtraction and multiplication (but not necessarily division). Since the linear algebra of real vector spaces already reflects most of the true spirit of linear algebra, we will concentrate on real vector spaces until Chapter 6.

1.1 Definition

1.1.1 Axioms of Vector Space

Definition 1.1.1. A *(real) vector space* is a set V, together with the operations of *addition* and *scalar multiplication*

$$\vec{u}+\vec{v}\colon V\times V\to V,\quad a\vec{u}\colon \mathbb{R}\times V\to V,$$

such that the following are satisfied.

1. Commutativity: $\vec{u}+\vec{v}=\vec{v}+\vec{u}$.

2. Additive associativity: $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$.

3. Zero: There is an element $\vec{0}\in V$ satisfying $\vec{u}+\vec{0}=\vec{u}=\vec{0}+\vec{u}$.

4. Negative: For any $\vec{u}$, there is $\vec{v}$ (to be denoted $-\vec{u}$), such that $\vec{u}+\vec{v}=\vec{0}=\vec{v}+\vec{u}$.

5. One: $1\vec{u}=\vec{u}$.

6. Multiplicative associativity: $(ab)\vec{u} = a(b\vec{u})$.

7. Scalar distributivity: $(a+b)\vec{u} = a\vec{u} + b\vec{u}$.

8. Vector distributivity: $a(\vec{u}+\vec{v}) = a\vec{u} + a\vec{v}$.

Due to the additive associativity, we may write $\vec{u}+\vec{v}+\vec{w}$ and even longer expressions without ambiguity.

Example 1.1.1. The zero vector space $\{\vec{0}\}$ consists of a single element $\vec{0}$. This leaves no choice for the two operations: $\vec{0}+\vec{0}=\vec{0}$, $a\vec{0}=\vec{0}$. It can be easily verified that all eight axioms are satisfied.

Example 1.1.2. The *Euclidean space* $\mathbb{R}^n$ is the set of n-tuples

$$\vec{x} = (x_1, x_2, \ldots, x_n), \quad x_i \in \mathbb{R}.$$

The i-th number x_i is the i-th *coordinate* of the vector. The Euclidean space is a vector space with coordinate wise addition and scalar multiplication

$$\begin{aligned}(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) &= (x_1+y_1, x_2+y_2, \ldots, x_n+y_n),\\ a(x_1, x_2, \ldots, x_n) &= (ax_1, ax_2, \ldots, ax_n).\end{aligned}$$

Geometrically, we may express a vector in the Euclidean space as a dot or an arrow from the *origin* $\vec{0} = (0,0,\ldots,0)$ to the dot. Figure 1.1.1 shows that the addition is described by parallelogram, and the scalar multiplication is described by stretching and shrinking.

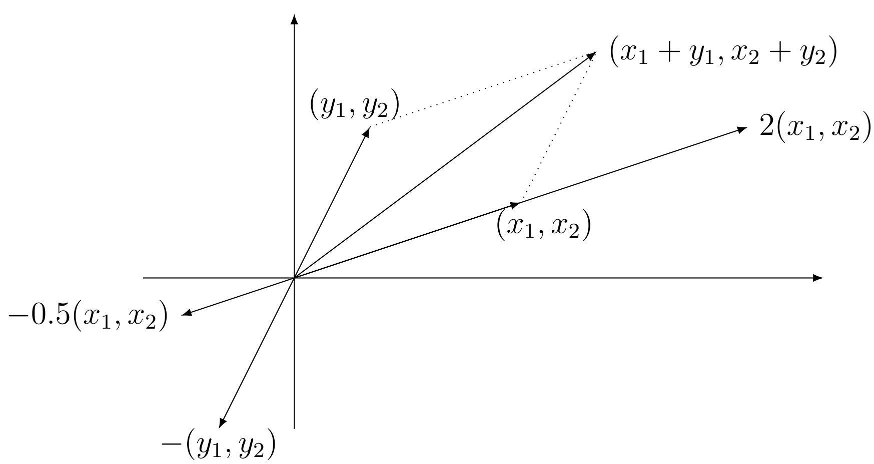

Figure 1.1.1: Euclidean space $\mathbb{R}^2$.

For the purpose of calculation (especially when mixed with matrices), it is more convenient to write a vector as a vertical $n \times 1$ matrix, or the transpose (indicated by superscript T) of a horizontal $1 \times n$ matrix

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = (x_1 \ x_2 \ \cdots \ x_n)^T.$$

Then the addition and scalar multiplication are

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}, \quad a \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_n \end{pmatrix}.$$

Example 1.1.3. Consider all polynomials of degree $\le n$

$$P_n = \{a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n\}.$$

We know how to add two polynomials together

$$\begin{aligned} &(a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n) + (b_0 + b_1 t + b_2 t^2 + \cdots + b_n t^n) \\ =&(a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + \cdots + (a_n + b_n)t^n, \end{aligned}$$

and how to multiplying a number to a polynomial

$$c(a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n) = ca_0 + ca_1 t + ca_2 t^2 + \cdots + ca_n t^n.$$

We can then verify that all eight axioms are satisfied. Therefore P_n is a vector space.

The coefficients of a polynomial provide a one-to-one correspondence

$$a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n \in P_n \longleftrightarrow (a_0, a_1, a_2, \ldots, a_n) \in \mathbb{R}^{n+1}.$$

Since the one-to-one correspondence preserves the addition and scalar multiplication, it identifies the polynomial vector space P_n with the Euclidean vector space $\mathbb{R}^{n+1}$. Such identification is an *isomorphism*.

The rigorous treatment of isomorphism will appear in Section 2.2.2.

Example 1.1.4. An $m \times n$ *matrix* A is mn numbers arranged in m rows and n columns. The number a_{ij} in the i-th row and j-column of A is the (i,j)-*entry* of A. We also denote the matrix by $A = (a_{ij})$.

All $m \times n$ matrices form a vector space $M_{m \times n}$ with the obvious addition and scalar multiplication. For example, in $M_{3 \times 2}$ we have

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix} + \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \\ y_{31} & y_{32} \end{pmatrix} = \begin{pmatrix} x_{11} + y_{11} & x_{12} + y_{12} \\ x_{21} + y_{21} & x_{22} + y_{22} \\ x_{31} + y_{31} & x_{32} + y_{32} \end{pmatrix},$$

$$a \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix} = \begin{pmatrix} ax_{11} & ax_{12} \\ ax_{21} & ax_{22} \\ ax_{31} & ax_{32} \end{pmatrix}.$$

We also have an isomorphism that identifies matrices with Euclidean vectors

$$\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{pmatrix} \in M_{3\times 2} \longleftrightarrow (x_1, x_2, x_3, y_1, y_2, y_3) \in \mathbb{R}^6.$$

Moreover, we have the general *transpose* isomorphism that identifies $m \times n$ matrices with $n \times m$ matrices (see Example 2.2.12 for the general formula)

$$A = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{pmatrix} \in M_{3\times 2} \longleftrightarrow A^T = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \in M_{2\times 3}.$$

A special case is the isomorphism in Example 1.1.2

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in M_{n\times 1} \longleftrightarrow \vec{x}^T = (x_1 \ x_2 \ \cdots \ x_n) \in M_{1\times n}.$$

The addition, scalar multiplication, and transpose of matrices are defined in the most "obvious" way. However, even simple definitions need to be justified. We can directly verify the expected properties by using the given formulae. In Sections 2.1.4 and 4.3.1, we give conceptual justifications for addition, scalar multiplication, and transpose.

Example 1.1.5. All infinite sequences $(x_n)_{n=1}^{\infty}$ of real numbers form a vector space, with the addition and scalar multiplications given by

$$(x_n) + (y_n) = (x_n + y_n), \quad a(x_n) = (ax_n).$$

Example 1.1.6. All smooth functions form a vector space C^∞, with the usual addition and scalar multiplication of functions. The vector space is not isomorphic to the usual Euclidean space because it is "infinite dimensional".

Exercise 1.1. Prove that $(a+b)(\vec{x}+\vec{y}) = a\vec{x} + b\vec{y} + b\vec{x} + a\vec{y}$ in any vector space.

Exercise 1.2. Introduce the following addition and scalar multiplication in $\mathbb{R}^2$

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_2, x_2 + y_1), \quad a(x_1, x_2) = (ax_1, ax_2).$$

Check which axioms of vector spaces are satisfied, and which are not satisfied.

Exercise 1.3. Introduce the following addition and scalar multiplication in $\mathbb{R}^2$

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, 0), \quad a(x_1, x_2) = (ax_1, 0).$$

Check which axioms of vector spaces are satisfied, and which are not satisfied.

Exercise 1.4. Introduce the following addition and scalar multiplication in $\mathbb{R}^2$

$$(x_1, x_2) + (y_1, y_2) = (x_1 + ky_1, x_2 + ly_2), \quad a(x_1, x_2) = (ax_1, ax_2).$$

Show that this makes $\mathbb{R}^2$ into a vector space if and only if $k = l = 1$.

Exercise 1.5. Show that all convergent sequences form a vector space.

Exercise 1.6. Show that all even smooth functions form a vector space.

Exercise 1.7. Explain that the transpose of matrix satisfies

$$(A + B)^T = A^T + B^T, \quad (cA)^T = cA^T, \quad (A^T)^T = A.$$

Section 2.4.2 gives conceptual explanation of the equalities.

1.1.2 Proof by Axiom

We establish some basic properties of vector spaces. You can directly verify these properties in the Euclidean space. For general vector spaces, however, we should derive these properties from the axioms.

Proposition 1.1.2. *The zero vector is unique.*

Proof. Suppose $\vec{0}_1$ and $\vec{0}_2$ are two zero vectors. By applying the first equality in Axiom 3 to $\vec{u} = \vec{0}_1$ and $\vec{0} = \vec{0}_2$, we get $\vec{0}_1 + \vec{0}_2 = \vec{0}_1$. By applying the second equality in Axiom 3 to $\vec{0} = \vec{0}_1$ and $\vec{u} = \vec{0}_2$, we get $\vec{0}_2 = \vec{0}_1 + \vec{0}_2$. Combining the two equalities, we get $\vec{0}_2 = \vec{0}_1 + \vec{0}_2 = \vec{0}_1$. □

Proposition 1.1.3. *If $\vec{u} + \vec{v} = \vec{u}$, then $\vec{v} = \vec{0}$.*

By Axioms 2, we also have $\vec{v} + \vec{u} = \vec{u}$, then $\vec{v} = \vec{0}$. Both properties are the *cancelation law*.

Proof. Suppose $\vec{u} + \vec{v} = \vec{u}$. By Axiom 3, there is $\vec{w}$, such that $\vec{w} + \vec{u} = \vec{0}$. We use $\vec{w}$ instead of $\vec{v}$ in the axiom, because $\vec{v}$ is already used in the proposition. Then

$$\begin{aligned}
\vec{v} &= \vec{0} + \vec{v} && \text{(Axiom 3)} \\
&= (\vec{w} + \vec{u}) + \vec{v} && \text{(choice of } \vec{w}\text{)} \\
&= \vec{w} + (\vec{u} + \vec{v}) && \text{(Axiom 2)} \\
&= \vec{w} + \vec{u} && \text{(assumption)} \\
&= \vec{0}. && \text{(choice of } \vec{w}\text{)}
\end{aligned}$$

□

Proposition 1.1.4. *$a\vec{u}=\vec{0}$ if and only if $a=0$ or $\vec{u}=\vec{0}$.*

Proof. By Axioms 3, 7, 8, we have

$$0\vec{u}+0\vec{u}=(0+0)\vec{u}=0\vec{u},\quad a\vec{0}+a\vec{0}=a(\vec{0}+\vec{0})=a\vec{0}.$$

Then by Proposition 1.1.3, we get $0\vec{u}=\vec{0}$ and $a\vec{0}=\vec{0}$. This proves the if part of the proposition.

The only if part means $a\vec{u}=\vec{0}$ implies $a=0$ or $\vec{u}=\vec{0}$. This is the same as $a\vec{u}=\vec{0}$ and $a\neq 0$ implying $\vec{u}=\vec{0}$. So we assume $a\vec{u}=\vec{0}$ and $a\neq 0$, and then apply Axioms 5, 6 and $a\vec{0}=\vec{0}$ (just proved) to get

$$\vec{u}=1\vec{u}=(a^{-1}a)\vec{u}=a^{-1}(a\vec{u})=a^{-1}\vec{0}=\vec{0}.$$ □

Exercise 1.8. Directly verify Propositions 1.1.2, 1.1.3, 1.1.4 in $\mathbb{R}^n$.

Exercise 1.9. Prove that the vector $\vec{v}$ in Axiom 4 is unique, and is $(-1)\vec{u}$. This justifies the notation $-\vec{u}$. Moreover, prove $-(-\vec{u})=\vec{u}$.

Exercise 1.10. Prove that $a\vec{v}=b\vec{v}$ if and only if $a=b$ or $\vec{v}=\vec{0}$.

Exercise 1.11. Prove the more general version of the cancelation law: $\vec{u}+\vec{v}_1=\vec{u}+\vec{v}_2$ implies $\vec{v}_1=\vec{v}_2$.

Exercise 1.12. We use Exercise 1.9 to define $\vec{u}-\vec{v}=\vec{u}+(-\vec{v})$. Prove the following properties

$$-(\vec{u}-\vec{v})=-\vec{u}+\vec{v},\quad -(\vec{u}+\vec{v})=-\vec{u}-\vec{v}.$$

1.2 Linear Combination

1.2.1 Linear Combination Expression

Combining addition and scalar multiplication gives *linear combination*

$$a_1\vec{v}_1+a_2\vec{v}_2+\cdots+a_n\vec{v}_n.$$

If we start with a nonzero seed vector $\vec{u}$, then all its linear combinations $a\vec{u}$ form a straight line passing through the origin $\vec{0}$. If we start with two non-parallel vectors $\vec{u}$ and $\vec{v}$, then all their linear combinations $a\vec{u}+b\vec{v}$ form a plane passing through the origin $\vec{0}$. See Figure 1.2.1.

Exercise 1.13. What are all the linear combinations of two parallel vectors $\vec{u}$ and $\vec{v}$?

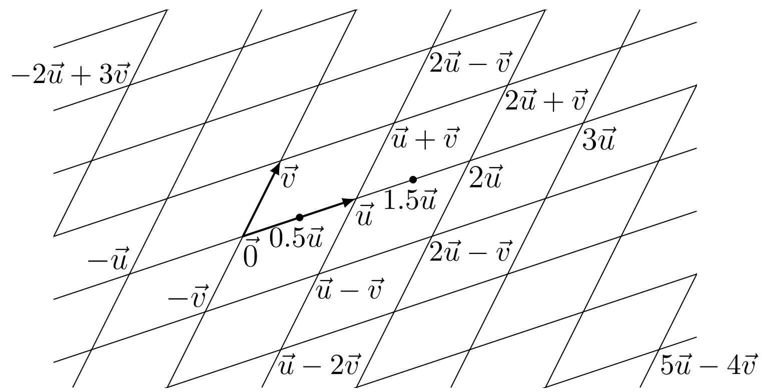

Figure 1.2.1: Linear combination.

By the axioms of vector spaces, we can easily verify the following

$$\begin{aligned}
&c_1(a_1\vec{v}_1+\cdots+a_n\vec{v}_n)+c_2(b_1\vec{v}_1+\cdots+b_n\vec{v}_n)\\
=&(c_1a_1\vec{v}_1+\cdots+c_1a_n\vec{v}_n)+(c_2b_1\vec{v}_1+\cdots+c_2b_n\vec{v}_n)\\
=&(c_1a_1+c_2b_1)\vec{v}_1+\cdots+(c_1a_n+c_2b_n)\vec{v}_n.
\end{aligned}$$

This show the linear combination of two linear combinations is still a linear combination. The fact can be easily extended to more linear combinations.

Proposition 1.2.1. *A linear combination of linear combinations is still a linear combination.*

Example 1.2.1. In $\mathbb{R}^3$, we try to express $\vec{v}=(10,11,12)$ as a linear combination of the following vectors

$$\vec{v}_1=(1,2,3),\quad \vec{v}_2=(4,5,6),\quad \vec{v}_3=(7,8,9).$$

This means finding suitable coefficients x_1,x_2,x_3, such that

$$\begin{aligned}
\vec{v}=\begin{pmatrix}10\\11\\12\end{pmatrix}&=x_1\vec{v}_1+x_2\vec{v}_2+x_3\vec{v}_3\\
&=x_1\begin{pmatrix}1\\2\\3\end{pmatrix}+x_2\begin{pmatrix}4\\5\\6\end{pmatrix}+x_3\begin{pmatrix}7\\8\\9\end{pmatrix}=\begin{pmatrix}x_1+4x_2+7x_3\\2x_1+5x_2+8x_3\\3x_1+6x_2+9x_3\end{pmatrix}.
\end{aligned}$$

In other words, we try to solve the system of linear equations

$$\begin{aligned}
x_1+4x_2+7x_3&=10,\\
2x_1+5x_2+8x_3&=11,\\
3x_1+6x_2+9x_3&=12.
\end{aligned}$$

By the way, we see the advantage of expressing Euclidean vectors in the vertical way in calculations.

To solve the system, we may eliminate x_1 in the second and third equations, by using $E_2 - 2E_1$ (multiply the first equation by -2 and add to the second equation) and $E_3 - 3E_1$ (multiply the first equation by -3 and add to the third equation). The result of the two operations is

$$\begin{aligned} x_1 + 4x_2 + 7x_3 &= 10, \\ - 3x_2 - 6x_3 &= -9, \\ - 6x_2 - 12x_3 &= -18. \end{aligned}$$

Then we use $E_3 - 2E_2$ to get

$$\begin{aligned} x_1 + 4x_2 + 7x_3 &= 10, \\ - 3x_2 - 6x_3 &= -9, \\ 0 &= 0. \end{aligned}$$

The last equation is trivial, and we only need to solve the first two equations. We may do $-\frac{1}{3}E_2$ (multiplying $-\frac{1}{3}$ to the second equation) to get

$$\begin{aligned} x_1 + 4x_2 + 7x_3 &= 10, \\ x_2 + 2x_3 &= 3, \\ 0 &= 0. \end{aligned}$$

From the second equation, we get $x_2 = 3 - 2x_3$. Substituting into the first equation, we get $x_1 = 10 - 4(3 - 2x_3) - 7x_3 = -2 + x_3$. The solution of the system is

$$x_1 = -2 + x_3, \quad x_2 = 3 - 2x_3, \quad x_3 \text{ arbitrary}.$$

We conclude that $\vec{v}$ is a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$, and there are many linear combination expressions, i.e., the expression is not unique.

Example 1.2.2. In P_2, we look for a, such that $p(t) = 10 + 11t + at^2$ is a linear combination of the following polynomials,

$$p_1(t) = 1 + 2t + 3t^2, \quad p_2(t) = 4 + 5t + 6t^2, \quad p_3(t) = 7 + 8t + 9t^2,$$

This means finding suitable coefficients x_1, x_2, x_3, such that

$$\begin{aligned} 10 + 11t + at^2 &= x_1(1 + 2t + 3t^2) + x_2(4 + 5t + 6t^2) + x_3(7 + 8t + 9t^2) \\ &= (x_1 + 4x_2 + 7x_3) + (2x_1 + 5x_2 + 8x_3)t + (3x_1 + 6x_2 + 9x_3)t^2. \end{aligned}$$

Comparing the coefficients of $1, t, t^2$, we get a system of linear equations

$$\begin{aligned} x_1 + 4x_2 + 7x_3 &= 10, \\ 2x_1 + 5x_2 + 8x_3 &= 11, \\ 3x_1 + 6x_2 + 9x_3 &= a. \end{aligned}$$

We use the same simplification process in Example 1.2.1 to simplify the system. First we get

$$\begin{aligned} x_1 + 4x_2 + \ \ 7x_3 &= 10, \\ - 3x_2 - \ \ 6x_3 &= -9, \\ - 6x_2 - 12x_3 &= a - 30. \end{aligned}$$

Then we get

$$\begin{aligned} x_1 + 4x_2 + 7x_3 &= 10, \\ - 3x_2 - 6x_3 &= -9, \\ 0 &= a - 12. \end{aligned}$$

If $a \neq 12$, then the last equation is a contradiction, and the system has no solution. If $a = 12$, then we are back to Example 1.2.1, and the system has (non-unique) solution.

We conclude $p(t)$ is a linear combination of $p_1(t), p_2(t), p_3(t)$ if and only if $a = 12$.

Exercise 1.14. Find the condition on a, such that the last vector can be expressed as a linear combination of the previous ones.

1. $(1, 2, 3), (4, 5, 6), (7, a, 9), (10, 11, 12)$.
2. $(1, 2, 3), (7, a, 9), (10, 11, 12)$.
3. $1 + 2t + 3t^2, 7 + at + 9t^2, 10 + 11t + 12t^2$.
4. $t^2 + 2t + 3, 7t^2 + at + 9, 10t^2 + 11t + 12$.
5. $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 5 \\ 5 & 6 \end{pmatrix}, \begin{pmatrix} 7 & a \\ a & 9 \end{pmatrix}, \begin{pmatrix} 10 & 11 \\ 11 & 12 \end{pmatrix}$.
6. $\begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix}, \begin{pmatrix} 7 & a \\ 9 & 9 \end{pmatrix}, \begin{pmatrix} 10 & 11 \\ 12 & 12 \end{pmatrix}$.

1.2.2 Row Operation

Examples 1.2.1 and 1.2.2 show that the problem of expressing a vector as a linear combination is equivalent to solving a system of linear equations. The shape of the vector (Euclidean, or polynomial, or some other form) is not important for the calculation. What is important is the coefficients in the vectors.

In general, to express a vector $\vec{b} \in \mathbb{R}^m$ as a linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \in \mathbb{R}^m$, we use $\vec{v}_i$ to form the columns of a matrix

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n), \quad \vec{v}_i = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix}. \tag{1.2.1}$$

We denote the linear combination by $A\vec{x}$

$$
\begin{aligned}
A\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n &= x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \\
&= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}. \qquad (1.2.2)
\end{aligned}
$$

Then the linear combination expression

$$A\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{b}$$

means a *system of linear equations*

$$
\begin{aligned}
a_{11}x_1 + \ a_{12}x_2 + \cdots + \ a_{1n}x_n &= b_1, \\
a_{21}x_1 + \ a_{22}x_2 + \cdots + \ a_{2n}x_n &= b_2, \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m.
\end{aligned}
$$

We call A the *coefficient matrix* of the system, and call $\vec{b} = (b_1, b_2, \dots, b_n)$ the right side. The *augmented matrix* of the system is

$$(A\ \vec{b}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}.$$

We have the correspondences

$$
\begin{aligned}
\text{equations} &\Longleftrightarrow \text{rows of } (A\ \vec{b}), \\
\text{variables} &\Longleftrightarrow \text{columns of } A.
\end{aligned}
$$

A system of linear equations can be solved by the process of *Gaussian elimination*, as illustrated in Examples 1.2.1 and 1.2.2. The idea is to eliminate variables, and thereby simplify equations. This is equivalent to the similar simplifications of the augmented matrix $(A\ \vec{b})$. For example, the Gaussian elimination process in Example 1.2.1 corresponds to the following operations on rows

$$\begin{pmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{pmatrix} \xrightarrow[R_3-3R_1]{R_2-2R_1} \begin{pmatrix} 1 & 4 & 7 & 10 \\ 0 & -3 & -6 & -9 \\ 0 & -6 & -12 & -18 \end{pmatrix} \xrightarrow{\substack{R_3-2R_2 \\ -\frac{1}{3}R_2}} \begin{pmatrix} 1 & 4 & 7 & 10 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (1.2.3)$$

In general, we may use three types of *row operations*, which do not change solutions of a system of linear equations.

- $R_i \leftrightarrow R_j$: exchange the i-th and j-th rows.
- cR_i: multiply a number $c \neq 0$ to the i-th row.
- $R_i + cR_j$: add the c multiple of the j-th row to the i-th row.

In Example 1.2.1, we use the third operation to create zero coefficients (and therefore simpler matrix). We use the second operation to simplify the coefficients (say, -3 is changed to 1). We may use the first operation to rearrange the equations from the most complicated (i.e., longest) to the simplest (i.e., shortest). We did not use this operation in the example because the arrangement is already from the most complicated to the simplest.

Example 1.2.3. We rearrange the order of vectors as $\vec{v}_2, \vec{v}_3, \vec{v}_1$ in Example 1.2.1. The corresponding row operations tell us how to express $\vec{v}$ as a linear combination of $\vec{v}_2, \vec{v}_3, \vec{v}_1$. We remark that the first row operation is also used.

$$\begin{pmatrix} 4 & 7 & 1 & 10 \\ 5 & 8 & 2 & 11 \\ 6 & 9 & 3 & 12 \end{pmatrix} \xrightarrow[R_3-R_1]{R_2-R_1} \begin{pmatrix} 4 & 7 & 1 & 10 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{pmatrix} \xrightarrow[R_3-2R_2]{R_1-4R_2} \begin{pmatrix} 0 & 3 & -3 & 6 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[R_1\leftrightarrow R_2]{-\frac{1}{3}R_1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1-R_2} \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The system is simplified to $x_2 + 2x_1 = -1$ and $x_3 - x_1 = 2$. We get the general solution

$$x_2 = -1 - 2x_1, \quad x_3 = 2 + x_1, \quad x_1 \text{ arbitrary}.$$

Exercise 1.15. Explain that row operations can always be reversed:

- The reverse of $R_i \leftrightarrow R_j$ is $R_i \leftrightarrow R_j$.
- The reverse of cR_i is $c^{-1}R_i$.
- The reverse of $R_i + cR_j$ is $R_i - cR_j$.

Exercise 1.16. Explain that row operations do not change the solutions of the corresponding system.

1.2.3 Row Echelon Form

We use three row operations to simplify a matrix. The simplest *shape* we can achieve is called the *row echelon form*. For the matrices in Examples 1.2.1 and 1.2.3, the row echelon form is

$$\begin{pmatrix} \bullet & * & * & * \\ 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \bullet \neq 0, \quad * \text{ can be any number.} \tag{1.2.4}$$

The entries indicated by $\bullet$ are called the *pivots*. The rows and columns containing the pivots are *pivot rows* and *pivot columns*. In the row echelon form (1.2.4), the pivot rows are the first and second, and the pivot columns are the first and second. The following are all the 2×3 row echelon forms

$$\begin{pmatrix} \bullet & * & * \\ 0 & \bullet & * \end{pmatrix}, \begin{pmatrix} \bullet & * & * \\ 0 & 0 & \bullet \end{pmatrix}, \begin{pmatrix} \bullet & * & * \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \bullet & * \\ 0 & 0 & \bullet \end{pmatrix},$$

$$\begin{pmatrix} 0 & \bullet & * \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \bullet \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In general, a row echelon form has the shape of upside down staircase (indicating simpler and simpler linear equations), and the shape is characterised by the locations of the pivots. The pivots are the leading nonzero entries in the rows. They appear in the first several rows, and in later and later positions. The subsequent non-pivot rows are completely zero. We note that each row has at most one pivot and each column has at most one pivot. Therfore

$$\text{number of pivot rows} = \text{number of pivots} = \text{number of pivot columns}.$$

For an $m \times n$ matrix, the number above is no more than the number of rows and columns

$$\text{number of pivots} \le \min\{m, n\}. \tag{1.2.5}$$

Exercise 1.17. How can row operations improve the following shapes to become upside down staircase?

1. $\begin{pmatrix} 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 \\ \bullet & * & * & * \end{pmatrix}$. 2. $\begin{pmatrix} \bullet & * & * & * \\ \bullet & * & * & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$. 3. $\begin{pmatrix} 0 & \bullet & * & * \\ 0 & \bullet & * & * \\ \bullet & * & * & * \end{pmatrix}$. 4. $\begin{pmatrix} 0 & \bullet & * & * \\ 0 & 0 & \bullet & * \\ \bullet & * & * & * \end{pmatrix}$.

Then explain why the shape (1.2.4) cannot be further improved?

Exercise 1.18. Display all the 2×2 row echelon forms. How about 3×2 row echelon forms?

Exercise 1.19. How many $m \times n$ row echelon forms are there?

If $a \neq 12$ in Example 1.2.2, then the augmented matrix of the system has the following row echelon form

$$\begin{pmatrix} \bullet & * & * & * \\ 0 & \bullet & * & * \\ 0 & 0 & 0 & \bullet \end{pmatrix}$$

The row $(0\ 0\ 0\ \bullet)$ represents the equation $0 = \bullet$, a contradiction. Therefore the system has no solution. We remark that the row $(0\ 0\ 0\ \bullet)$ means the last column is pivot.

If $a = 12$, then we do not have the contradiction, and the system has solution. Section 1.2.4 shows the existence of solution even more explicitly.

The discussion leads to the first part of the following.

Theorem 1.2.2. *A system of linear equations $A\vec{x} = \vec{b}$ has solution if and only if $\vec{b}$ is not a pivot column of the augmented matrix $(A\ \vec{b})$. The solution is unique if and only if all columns of A are pivot.*

1.2.4 Reduced Row Echelon Form

Although a row echelon form has the simplest shape, we may still simplify individual entries in a row echelon form. First we may divide rows (using the second row operation) by the nonzero numbers at the pivots, so that the pivots are occupied by 1. Then we use these 1 to eliminate (using the third row operation) all terms above the pivots. The result is the simplest *matrix* one can get by row operations, called the *reduced row echelon form*.

For the row echelon form (1.2.4), this means

$$\begin{pmatrix} \bullet & * & * & * \\ 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here is one more example

$$\begin{pmatrix} \bullet & * & * & * & * \\ 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & \bullet & * \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & * & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * & * \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & * & 0 & 0 & * & * \\ 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * \end{pmatrix}.$$

The corresponding systems of linear equations are also the simplest

$$\begin{aligned} x_1 \quad + c_{13}x_3 &= d_1, \\ x_2 + c_{23}x_3 &= d_2, \\ 0 &= 0. \end{aligned} \qquad\qquad \begin{aligned} x_1 + c_{12}x_2 \qquad + c_{15}x_5 &= d_1, \\ x_3 \quad + c_{25}x_5 &= d_2, \\ x_4 + c_{35}x_5 &= d_3. \end{aligned}$$

Then we can literally read off the solutions of the two systems. The general solution of the first system is

$$x_1 = d_1 - c_{13}x_3, \quad x_2 = d_2 - c_{23}x_3, \quad x_3 \text{ arbitrary}.$$

For the obvious reason, we call x_3 a *free variable*, and x_1, x_2 *non-free variables*. The general solution of the second system is

$$x_1 = d_1 - c_{12}x_2 - c_{15}x_5, \quad x_3 = d_2 - c_{25}x_5, \quad x_4 = d_3 - c_{35}x_5, \quad x_2, x_5 \text{ arbitrary}.$$

Here x_2, x_5 are free, and x_1, x_3, x_4 are not free.

Exercise 1.20. Display all the 2×2, 2×3, 3×2 and 3×4 reduced row echelon forms.

Exercise 1.21. Given the reduced row echelon form of the augmented matrix of the system of linear equations, find the general solution.

1. $\begin{pmatrix} 1 & a_1 & 0 & b_1 \\ 0 & 0 & 1 & b_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

2. $\begin{pmatrix} 1 & a_1 & 0 & b_1 & 0 \\ 0 & 0 & 1 & b_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

3. $\begin{pmatrix} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 \end{pmatrix}$.

4. $\begin{pmatrix} 1 & a_1 & a_2 & b_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

5. $\begin{pmatrix} 1 & a_1 & a_2 & b_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

6. $\begin{pmatrix} 1 & a_1 & 0 & a_2 & b_1 \\ 0 & 0 & 1 & a_3 & b_2 \end{pmatrix}$.

7. $\begin{pmatrix} 1 & 0 & a_1 & a_2 & b_1 \\ 0 & 1 & a_3 & a_4 & b_2 \end{pmatrix}$.

8. $\begin{pmatrix} 1 & 0 & a_1 & b_1 \\ 0 & 1 & a_2 & b_2 \end{pmatrix}$.

9. $\begin{pmatrix} 0 & 1 & 0 & a_1 & b_1 \\ 0 & 0 & 1 & a_2 & b_2 \end{pmatrix}$.

10. $\begin{pmatrix} 1 & 0 & a_1 & b_1 \\ 0 & 1 & a_2 & b_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

11. $\begin{pmatrix} 1 & 0 & a_1 & 0 & a_2 & b_1 \\ 0 & 1 & a_3 & 0 & a_4 & b_2 \\ 0 & 0 & 0 & 1 & a_5 & b_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

12. $\begin{pmatrix} 1 & 0 & a_1 & 0 & a_2 & b_1 & 0 \\ 0 & 1 & a_3 & 0 & a_4 & b_2 & 0 \\ 0 & 0 & 0 & 1 & a_5 & b_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

Exercise 1.22. Given the general solution of the system of linear equations, find the reduced row echelon form of the augmented matrix.

1. $x_1 = -x_3$, $x_2 = 1 + x_3$; x_3 arbitrary.
2. $x_1 = -x_3$, $x_2 = 1 + x_3$; x_3, x_4 arbitrary.
3. $x_2 = -x_4$, $x_3 = 1 + x_4$; x_1, x_4 arbitrary.
4. $x_2 = -x_4$, $x_3 = x_4 - x_5$; x_1, x_4, x_5 arbitrary.
5. $x_1 = 1 - x_2 + 2x_5$, $x_3 = 1 + 2x_5$, $x_4 = -3 + x_5$; x_2, x_5 arbitrary.
6. $x_1 = 1 + 2x_2 + 3x_4$, $x_3 = 4 + 5x_4 + 6x_5$; x_2, x_4, x_5 arbitrary.
7. $x_1 = 2x_2 + 3x_4 - x_6$, $x_3 = 5x_4 + 6x_5 - 4x_6$; x_2, x_4, x_5, x_6 arbitrary.

We see that, if the system has solution (i.e., $\vec{b}$ is not pivot column of $(A\ \vec{b})$), then the reduced row echelon form is equivalent to the general solution. Since the solution is independent of the choice of the row operations, we know the reduced row echelon form of a matrix is unique.

We also see the following correspondence

variables in $A\vec{x} = \vec{b}$	free	non-free
columns in A	non-pivot	pivot

In particular, the uniqueness of solution means no freedom for the variables. By the correspondence above, this means all columns of A are pivot. This explains the second part of Theorem 1.2.2.

1.3 Basis

1.3.1 Basis and Coordinate

In Example 1.2.2, we see the linear combination problem for polynomials is equivalent to the linear combination problem for Euclidean vectors. In Example 1.1.3, the equivalence is given by expressing polynomials as linear combinations of $1, t, t^2, \dots, t^n$ with unique coefficients. In general, we ask for the following.

Definition 1.3.1. A ordered set $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ of vectors in a vector space V is a *basis* of V, if for any $\vec{x} \in V$, there exist unique $x_1, x_2, \dots, x_n$, such that

$$\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n.$$

The coefficients $x_1, x_2, \dots, x_n$ are the *coordinates* of $\vec{x}$ with respect to the (ordered) basis. The unique expression means that the *α-coordinate map*

$$\vec{x} \in V \mapsto [\vec{x}]_\alpha = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$$

is well defined. The reverse map is given by the linear combination

$$(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mapsto x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n \in V.$$

The two way maps identify (called *isomorphism*) the general vector space V with the Euclidean space $\mathbb{R}^n$. Moreover, the following result shows that the coordinate preserves linear combinations. Therefore we may use the coordinate to translate linear algebra problems (such as linear combination expression) in a general vector space to corresponding problems in an Euclidean space.

Proposition 1.3.2. $[\vec{x} + \vec{y}]_\alpha = [\vec{x}]_\alpha + [\vec{y}]_\alpha$, $[a\vec{x}]_\alpha = a[\vec{x}]_\alpha$.

Proof. Let $[\vec{x}]_\alpha = (x_1, x_2, \dots, x_n)$ and $[\vec{y}]_\alpha = (y_1, y_2, \dots, y_n)$. Then by the definition of coordinates, we have

$$\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n, \quad \vec{y} = y_1\vec{v}_1 + y_2\vec{v}_2 + \cdots + y_n\vec{v}_n.$$

Adding the two together, we have

$$\vec{x}+\vec{y}=(x_1+y_1)\vec{v}_1+(x_2+y_2)\vec{v}_2+\cdots+(x_n+y_n)\vec{v}_n.$$

By the definition of coordinates, this means

$$\begin{aligned}[\vec{x}+\vec{y}]_\alpha &= (x_1+y_1, x_2+y_2, \dots, x_n+y_n)\\ &= (x_1,x_2,\dots,x_n)+(y_1,y_2,\dots,y_n)=[\vec{x}]_\alpha+[\vec{y}]_\alpha.\end{aligned}$$

The proof of $[a\vec{x}]_\alpha = a[\vec{x}]_\alpha$ is similar. □

Example 1.3.1. The *standard basis* vector $\vec{e}_i$ in $\mathbb{R}^n$ has the i-th coordinate 1 and all other coordinates 0. For example, the standard basis vectors of $\mathbb{R}^3$ are

$$\vec{e}_1=(1,0,0),\quad \vec{e}_2=(0,1,0),\quad \vec{e}_3=(0,0,1).$$

By the equality

$$(x_1,x_2,\dots,x_n)=x_1\vec{e}_1+x_2\vec{e}_2+\cdots+x_n\vec{e}_n,$$

any vector is a linear combination of the standard basis vectors. Moreover, the equality shows that, if two expressions on the right are equal

$$x_1\vec{e}_1+x_2\vec{e}_2+\cdots+x_n\vec{e}_n=y_1\vec{e}_1+y_2\vec{e}_2+\cdots+y_n\vec{e}_n,$$

then the two vectors are also equal

$$(x_1,x_2,\dots,x_n)=(y_1,y_2,\dots,y_n).$$

Of course this means exactly $x_1=y_1, x_2=y_2, \dots, x_n=y_n$, i.e., the uniqueness of the coefficients. Therefore the standard basis vectors for the standard basis $\epsilon = \{\vec{e}_1,\vec{e}_2,\dots,\vec{e}_n\}$ of $\mathbb{R}^n$. The equality can be interpreted as

$$[\vec{x}]_\epsilon=\vec{x}.$$

If we change the order in the standard basis, then we should also change the order of coordinates

$$\begin{aligned}[(x_1,x_2,x_3)]_{\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}}&=(x_1,x_2,x_3),\\ [(x_1,x_2,x_3)]_{\{\vec{e}_2,\vec{e}_1,\vec{e}_3\}}&=(x_2,x_1,x_3),\\ [(x_1,x_2,x_3)]_{\{\vec{e}_3,\vec{e}_2,\vec{e}_1\}}&=(x_3,x_2,x_1).\end{aligned}$$

Example 1.3.2. Any polynomial of degree $\le n$ is of the form

$$p(t)=a_0+a_1t+a_2t^2+\cdots+a_nt^n.$$

The formula can be interpreted as that $p(t)$ is a linear combination of *monomials* $1,t,t^2,\dots,t^n$. For the uniqueness of the linear combination, we consider the equality

$$a_0+a_1t+a_2t^2+\cdots+a_nt^n=b_0+b_1t+b_2t^2+\cdots+b_nt^n.$$

The equality usually mean equal functions. In other words, if we substitute any real number in place of t, the two sides have the same value. Taking $t = 0$, we get $a_0 = b_0$. Dividing the remaining equality by $t \neq 0$, we get

$$a_1 + a_2 t + \cdots + a_n t^{n-1} = b_1 + b_2 t + \cdots + b_n t^{n-1} \quad \text{for } t \neq 0.$$

Taking $\lim_{t\to 0}$ on both sides, we get $a_1 = b_1$. Inductively, we find that $a_i = b_i$ for all i. This shows the uniqueness of the linear combination. Therefore $1, t, t^2, \dots, t^n$ form a basis of P_n. We have

$$[a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n]_{\{1,t,t^2,\dots,t^n\}} = (a_0, a_1, a_2, \dots, a_n).$$

Example 1.3.3. Consider the monomials $1, t-1, (t-1)^2$ at $t_0 = 1$. Any quadratic polynomial is a linear combination of $1, t-1, (t-1)^2$

$$\begin{aligned} a_0 + a_1 t + a_2 t^2 &= a_0 + a_1[1 + (t-1)] + a_2[1 + (t-1)]^2 \\ &= (a_0 + a_1 + a_2) + (a_1 + 2a_2)(t-1) + a_2(t-1)^2. \end{aligned}$$

Moreover, if two linear combinations are equal

$$a_0 + a_1(t-1) + a_2(t-1)^2 = b_0 + b_1(t-1) + b_2(t-1)^2,$$

then substituting t by $t+1$ gives the equality

$$a_0 + a_1 t + a_2 t^2 = b_0 + b_1 t + b_2 t^2.$$

This means $a_0 = b_0$, $a_1 = b_1$, $a_2 = b_2$, or the uniqueness of the linear combination expression. Therefore $1, t-1, (t-1)^2$ form a basis of P_2. In general, $1, t-t_0, (t-t_0)^2, \dots, (t-t_0)^n$ form a basis of P_n.

Exercise 1.23. Show that the following 3×2 matrices form a basis of the vector space $M_{3\times 2}$.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

In general, how many matrices are in a basis of the vector space $M_{m\times n}$ of $m \times n$ matrices?

Exercise 1.24. For an ordered basis $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is of V, explain that $[\vec{v}_i]_\alpha = \vec{e}_i$.

Exercise 1.25. A *permutation* of $\{1, 2, \dots, n\}$ is a one-to-one correspondence $\pi\colon \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$. Let $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be a basis of V.

1. Show that $\pi(\alpha) = \{\vec{v}_{\pi(1)}, \vec{v}_{\pi(2)}, \dots, \vec{v}_{\pi(n)}\}$ is still a basis.
2. What is the relation between $[\vec{x}]_\alpha$ and $[\vec{x}]_{\pi(\alpha)}$?

1.3.2 Spanning Set

The definition of basis consists of two parts, the *existence* of linear combination expression for all vectors, and the *uniqueness* of the expression. We study the two properties separately. The existence is the following property.

Definition 1.3.3. A set of vectors $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ in a vector space V *spans* V if any vector in V can be expressed as a linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$.

For $V = \mathbb{R}^m$, we form the $m \times n$ matrix $A = (\vec{v}_1\ \vec{v}_2\ \cdots \vec{v}_n)$. Then the vectors spanning V means the system of linear equations

$$A\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{b}$$

has solution for *all* right side $\vec{b}$.

Example 1.3.4. For $\vec{u} = (a, b)$ and $\vec{v} = (c, d)$ to span $\mathbb{R}^2$, we need the system of two linear equations to have solution for all p, q

$$\begin{aligned} ax + cy &= p, \\ bx + dy &= q. \end{aligned}$$

We multiply the first equation by b and the second equation by a, and then subtract the two. We get

$$(bc - ad)y = bp - aq.$$

If $ad \neq bc$, then we can solve for y. We may further substitute y into the original equations to get x. Therefore the system has solution for all p, q.

We conclude that (a, b) and (c, d) span $\mathbb{R}^2$ in case $ad \neq bc$. For example, by $1 \cdot 4 \neq 3 \cdot 2$, we know $(1, 2)$ and $(3, 4)$ span $\mathbb{R}^2$.

Exercise 1.26. Show that a linear combination of $(1, 2)$ and $(2, 4)$ is always of the form $(a, 2a)$. Then explain that the two vectors do not span $\mathbb{R}^2$.

Exercise 1.27. Explain that, if $ad = bc$, then (a, b) and (c, d) do not span $\mathbb{R}^2$.

Example 1.3.5. The following vectors span $\mathbb{R}^3$

$$\vec{v}_1 = (1, 2, 3), \quad \vec{v}_2 = (4, 5, 6), \quad \vec{v}_3 = (7, 8, 9),$$

if and only if $(\vec{v}_1\ \vec{v}_2\ \vec{v}_3)\vec{x} = \vec{b}$ has solution for all $\vec{b}$. We apply the same row operations in (1.2.3) to the augmented matrix

$$(\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{b}) = \begin{pmatrix} 1 & 4 & 7 & b_1 \\ 2 & 5 & 8 & b_2 \\ 3 & 6 & 9 & b_3 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 7 & b_1' \\ 0 & 1 & 2 & b_2' \\ 0 & 0 & 0 & b_3' \end{pmatrix}.$$

Although we may calculate the explicit formulae for b_i', which are linear combinations of b_1, b_2, b_3, we do not need these. All we need to know is that, since b_1, b_2, b_3 are arbitrary, and the row operations can be reversed (see Exercise 1.15), the right side b_1', b_2', b_3' in the row echelon form are also arbitrary. In particular, it is possible to have $b_3' \neq 0$, and the system has no solution. Therefore the three vectors do not span $\mathbb{R}^3$.

If we change the third vector to $\vec{v}_3 = (7, 8, a)$, then the same row operations give

$$(\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{b}) = \begin{pmatrix} 1 & 4 & 7 & b_1 \\ 2 & 5 & 8 & b_2 \\ 3 & 6 & a & b_3 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 7 & b_1' \\ 0 & 1 & 2 & b_2' \\ 0 & 0 & a-9 & b_3' \end{pmatrix}.$$

If $a \neq 9$, then the last column is not pivot. By Theorem 1.2.2, the system always has solution. Therefore $(1,2,3), (4,5,6), (7,8,a)$ span $\mathbb{R}^3$ if and only if $a \neq 9$.

Example 1.3.5 can be summarized as the following criterion for a set of vectors to span the Euclidean space.

Proposition 1.3.4. *Let $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \subset \mathbb{R}^m$ and $A = (\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n)$. The following are equivalent.*

1. *α spans $\mathbb{R}^m$.*
2. *$A\vec{x} = \vec{b}$ has solution for all $\vec{b} \in \mathbb{R}^m$.*
3. *All rows of A are pivot. In other words, the row echelon form of A has no zero row $(0\ 0\ \cdots\ 0)$.*

Moreover, we have $m \leq n$ in the above cases.

For the last property $n \geq m$, we note that all rows pivot implies the number of pivots is m. Then by (1.2.5), we get $m \leq \min\{m, n\} \leq n$.

Example 1.3.6. To find out whether $(1,2,3), (4,5,6), (7,8,a), (10,11,b)$ span $\mathbb{R}^3$, we apply the row operations in (1.2.3)

$$\begin{pmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & a & b \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 7 & 10 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & a-9 & b-12 \end{pmatrix}.$$

The row echelon form depends on the values of a and b. If $a \neq 9$, then the result is already a row echelon form

$$\begin{pmatrix} \bullet & * & * & * \\ 0 & \bullet & * & * \\ 0 & 0 & \bullet & * \end{pmatrix}.$$

If $a=9$, then the result is

$$\begin{pmatrix} \bullet & * & * & * \\ 0 & \bullet & * & * \\ 0 & 0 & 0 & b-12 \end{pmatrix}.$$

Then we have two possible row echelon forms

$$\begin{pmatrix} \bullet & * & * & * \\ 0 & \bullet & * & * \\ 0 & 0 & 0 & \bullet \end{pmatrix} \text{ for } b\neq 12; \quad \begin{pmatrix} \bullet & * & * & * \\ 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ for } b=12.$$

By Proposition 1.3.4, the vectors $(1,2,3),(4,5,6),(7,8,a),(10,11,b)$ span $\mathbb{R}^3$ if and only if $a\neq 9$, or $a=9$ and $b\neq 12$.

If we restrict the row operations to the first three columns, then we find that all rows are pivot if and only if $a\neq 9$. Therefore the first three vectors span $\mathbb{R}^3$ if and only if $a\neq 9$.

We may also restrict the row operations to the first, second and fourth columns, and find that all rows are pivot if and only if $b\neq 12$. This is the condition for $(1,2,3),(4,5,6),(10,11,b)$ to span $\mathbb{R}^3$.

Exercise 1.28. Find row echelon form and determine whether the column vectors span the Euclidean space.

1. $\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 1 & 2 \\ 3 & 1 & 2 & 3 \end{pmatrix}$.
2. $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$.
3. $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$.
4. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & a \end{pmatrix}$.
5. $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & a \end{pmatrix}$.
6. $\begin{pmatrix} 0 & 2 & -1 & 4 \\ -1 & 3 & 0 & 1 \\ 2 & -4 & -1 & 2 \\ 1 & 1 & -2 & 7 \end{pmatrix}$.
7. $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$.
8. $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & a \\ 0 & 1 & a & b \end{pmatrix}$.

Exercise 1.29. If $\vec{v}_1,\vec{v}_2,\dots,\vec{v}_n$ span V, prove that $\vec{v}_1,\vec{v}_2,\dots,\vec{v}_n,\vec{w}$ span V.

The property means that any set bigger than a spanning set is also a spanning set.

Exercise 1.30. Suppose $\vec{w}$ is a linear combination of $\vec{v}_1,\vec{v}_2,\dots,\vec{v}_n$. Prove that $\vec{v}_1,\vec{v}_2,\dots,\vec{v}_n,\vec{w}$ span V if and only if $\vec{v}_1,\vec{v}_2,\dots,\vec{v}_n$ span V.

The property means that, if one vector is a linear combination of the others, then we may remove the vector without changing the spanning set property.

Exercise 1.31. Suppose $\vec{v}_1,\vec{v}_2,\dots,\vec{v}_n\in V$. If each of $\vec{w}_1,\vec{w}_2,\dots,\vec{w}_m$ is a linear combination of $\vec{v}_1,\vec{v}_2,\dots,\vec{v}_n$, and $\vec{w}_1,\vec{w}_2,\dots,\vec{w}_m$ span V, prove that $\vec{v}_1,\vec{v}_2,\dots,\vec{v}_n$ also span V.

Exercise 1.32. Prove that the following are equivalent for a set of vectors in V.

1. $\vec{v}_1,\dots,\vec{v}_i,\dots,\vec{v}_j,\dots,\vec{v}_n$ span V.

2. $\vec{v}_1, \dots, \vec{v}_j, \dots, \vec{v}_i, \dots, \vec{v}_n$ span V.
3. $\vec{v}_1, \dots, c\vec{v}_i, \dots, \vec{v}_n$, $c \neq 0$, span V.
4. $\vec{v}_1, \dots, \vec{v}_i + c\vec{v}_j, \dots, \vec{v}_j, \dots, \vec{v}_n$ span V.

Example 1.3.7. By the last part of Proposition 1.3.4, three vectors $(1, 0, \sqrt{2},\ \pi)$, $(\log 2, e, 100, -0.5)$, $(\sqrt{3}, e^{-1}, \sin 1, 2.3)$ cannot span $\mathbb{R}^4$.

Exercise 1.33. If $m > n$ in Proposition 1.3.4, what can you conclude?

Exercise 1.34. Explain that the vectors do not span the Euclidean space. Then interpret the result in terms of systems of linear equations.

1. $(10, -2, 3, 7, 2), (0, 8, -2, 5, -4), (8, -9, 3, 6, 5)$.
2. $(10, -2, 3, 7, 2), (0, 8, -2, 5, -4), (8, -9, 3, 6, 5), (7, -9, 3, -5, 6)$.
3. $(0, -2, 3, 7, 2), (0, 8, -2, 5, -4), (0, -9, 3, 6, 5), (0, -5, 4, 2, -7), (0, 4, -1, 3, -6)$.
4. $(6, -2, 3, 7, 2), (-4, 8, -2, 5, -4), (6, -9, 3, 6, 5), (8, -5, 4, 2, -7), (-2, 4, -1, 3, -6)$.

1.3.3 Linear Independence

Definition 1.3.5. A set of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are *linearly independent* if the coefficients in linear combination are unique

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = y_1\vec{v}_1 + y_2\vec{v}_2 + \cdots + y_n\vec{v}_n \implies x_1 = y_1,\ x_2 = y_2,\ \dots,\ x_n = y_n.$$

The vectors are *linearly dependent* if they are not linearly independent.

For $V = \mathbb{R}^m$, we form the matrix $A = (\vec{v}_1\ \vec{v}_2\ \cdots \vec{v}_n)$. Then the linear independence of the column vectors means the solution of the system of linear equations

$$A\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{b}$$

is unique. By Theorem 1.2.2, we have the following criterion for linear independence.

Proposition 1.3.6. *Let $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \subset \mathbb{R}^m$ and $A = (\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n)$. The following are equivalent.*

1. *α is linearly independent.*
2. *The solution of $A\vec{x} = \vec{b}$ is unique.*
3. *All columns of A are pivot.*

Moreover, we have $m \geq n$ in the above cases.

For the last property $m \geq n$, we note that all columns pivot implies the number of pivots is n. Then by (1.2.5), we get $n \leq \min\{m, n\} \leq m$.

Example 1.3.8. We try to find the condition on a, such that

$$\vec{v}_1 = (1,2,3,4), \quad \vec{v}_2 = (5,6,7,8), \quad \vec{v}_3 = (9,10,11,a)$$

are linearly independent. We carry out the row operations

$$(\vec{v}_1\ \vec{v}_2\ \vec{v}_3) = \begin{pmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & a \end{pmatrix} \xrightarrow[R_2-R_1]{\substack{R_4-R_3 \\ R_3-R_2}} \begin{pmatrix} 1 & 5 & 9 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & a-11 \end{pmatrix}$$
$$\xrightarrow[R_4-R_2]{\substack{R_1-R_2 \\ R_3-R_2}} \begin{pmatrix} 0 & 4 & 8 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & a-12 \end{pmatrix} \xrightarrow[R_3\leftrightarrow R_4]{R_1\leftrightarrow R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 8 \\ 0 & 0 & a-12 \\ 0 & 0 & 0 \end{pmatrix}.$$

We find all three columns are pivot if and only if $a \neq 12$. This is the condition for the three vectors to be linearly independent.

Exercise 1.35. Determine the linear independence of the column vectors in Exercises 1.28.

Exercise 1.36. Prove that the following are equivalent.

1. $\vec{v}_1, \dots, \vec{v}_i, \dots, \vec{v}_j, \dots, \vec{v}_n$ are linearly independent.
2. $\vec{v}_1, \dots, \vec{v}_j, \dots, \vec{v}_i, \dots, \vec{v}_n$ are linearly independent.
3. $\vec{v}_1, \dots, c\vec{v}_i, \dots, \vec{v}_n$, $c \neq 0$, are linearly independent.
4. $\vec{v}_1, \dots, \vec{v}_i + c\vec{v}_j, \dots, \vec{v}_j, \dots, \vec{v}_n$ are linearly independent.

Example 1.3.9. By the last part of Proposition 1.3.6, four vectors $(1, \log 2, \sqrt{3})$, $(0, e, e^{-1})$, $(\sqrt{2}, 100, \sin 1)$, $(\pi, -0.5, 2.3)$ in $\mathbb{R}^3$ are linearly dependent.

Exercise 1.37. If $m < n$ in Proposition 1.3.6, what can you conclude?

Exercise 1.38. Explain that the vectors are linearly dependent. Then interpret the result in terms of systems of linear equations.

1. $(1,2,3), (2,3,1), (3,1,2), (1,3,2), (3,2,1), (2,1,3)$.
2. $(1,3,2,-4), (10,-2,3,7), (0,8,-2,5), (8,-9,3,6), (7,-9,3,-5)$.
3. $(1,3,2,-4), (10,-2,3,7), (0,8,-2,5), (\pi, 3\pi, 2\pi, -4\pi)$.
4. $(1,3,2,-4,0), (10,-2,3,7,0), (0,8,-2,5,0), (8,-9,3,6,0), (7,-9,3,-5,0)$.

The criterion for linear independence in Proposition 1.3.6 does not depend on the right side. This means we only need to verify the uniqueness for the case $\vec{b} = \vec{0}$. We call the corresponding system $A\vec{x} = \vec{0}$ *homogeneous*. The homogeneous system always has the zero solution $\vec{x} = \vec{0}$. Therefore we only need to ask the uniqueness of the solution of $A\vec{x} = \vec{0}$.

The relation between the uniqueness for $A\vec{x} = \vec{b}$ and the uniqueness for $A\vec{x} = \vec{0}$ holds in general vector space.

Proposition 1.3.7. *A set of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly independent if and only if*

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0} \implies x_1 = x_2 = \cdots = x_n = 0.$$

Proof. The property in the proposition is the special case of $y_1 = \cdots = y_n = 0$ in the definition of linear independence. Conversely, if the special case holds, then

$$\begin{aligned} & x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = y_1\vec{v}_1 + y_2\vec{v}_2 + \cdots + y_n\vec{v}_n \\ \implies & (x_1 - y_1)\vec{v}_1 + (x_2 - y_2)\vec{v}_2 + \cdots + (x_n - y_n)\vec{v}_n = \vec{0} \\ \implies & x_1 - y_1 = x_2 - y_2 = \cdots = x_n - y_n = 0. \end{aligned}$$ □

Example 1.3.10. By Proposition 1.3.7, a single vector $\vec{v}$ is linearly independent if and only if $a\vec{v} = \vec{0}$ implies $a = 0$. By Proposition 1.1.4, the property means exactly $\vec{v} \neq \vec{0}$.

Example 1.3.11. To show that $\cos t, \sin t, e^t$ are linearly independent, we only need to verify that the equality $x_1 \cos t + x_2 \sin t + x_3 e^t = 0$ implies $x_1 = x_2 = x_3 = 0$.

If the equality holds, then by evaluating at $t = 0, \frac{\pi}{2}, \pi$, we get

$$x_1 + x_3 = 0, \quad x_2 + x_3 e^{\frac{\pi}{2}} = 0, \quad -x_1 + x_3 e^{\pi} = 0.$$

Adding the first and third equations together, we get $x_3(1 + e^{\pi}) = 0$. This implies $x_3 = 0$. Substituting $x_3 = 0$ to the first and second equations, we get $x_1 = x_2 = 0$.

Example 1.3.12. To show $t(t-1), t(t-2), (t-1)(t-2)$ are linearly independent, we onsider

$$a_1 t(t-1) + a_2 t(t-2) + a_3 (t-1)(t-2) = 0.$$

Since the equality holds for all t, we may take $t = 0$ and get $a_3(-1)(-2) = 0$. Therefore $a_3 = 0$. Similarly, by taking $t = 1$ and $t = 2$, we get $a_2 = 0$ and $a_1 = 0$.

In general, suppose $t_0, t_1, \dots, t_n$ are distinct, and[1]

$$p_i(t) = \prod_{j \neq i} (t - t_j) = (t - t_0)(t - t_1) \cdots \widehat{(t - t_i)} \cdots (t - t_n)$$

[1]The notation $\hat{?}$ is the mathematical convention that the term ? is missing.

is the product of all $t - t_*$ except $t - t_i$. Then $p_0(t), p(1(t), \dots, p_n(t)$ are linearly independent.

Exercise 1.39. For $a \neq b$, show that e^{at} and e^{bt} are linearly independent. What about e^{at}, e^{bt} and e^{ct}?

Exercise 1.40. Prove that $\cos t, \sin t, e^t$ do not span the vector space of all smooth functions, by showing that the constant function 1 is not a linear combination of the three functions.

Hint: Take several values of t in $x_1 \cos t + x_2 \sin t + x_3 e^t = 1$ and then derive contradiction.

Exercise 1.41. Determine whether the given functions are linearly independent, and whether $f(x), g(t)$ can be expressed as linear combinations of given functions.

1. $\cos^2 t, \sin^2 t$. $f(t) = 1$, $g(t) = t$.
2. $\cos^2 t, \sin^2 t, 1$. $f(t) = \cos 2t$, $g(t) = t$.
3. $1, t, e^t, te^t$. $f(t) = (1+t)e^t$, $g(t) = f'(t)$.
4. $\cos^2 t, \cos 2t$. $f(t) = a$, $g(t) = a + \sin^2 t$.

The following result says that linear dependence means some vector is a "waste". The proof makes use of division by nonzero number.

Proposition 1.3.8. *A set of vectors are linearly dependent if and only if one vector is a linear combination of the other vectors.*

Proof. If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly dependent, then by Proposition 1.3.7, we have $x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}$, with some $x_i \neq 0$. Then we get

$$\vec{v}_i = -\frac{x_1}{x_i}\vec{v}_1 - \cdots - \frac{x_{i-1}}{x_i}\vec{v}_{i-1} - \frac{x_{i+1}}{x_i}\vec{v}_{i+1} - \cdots - \frac{x_n}{x_i}\vec{v}_n.$$

This shows that the i-th vector is a linear combination of the other vectors.

Conversely, if

$$\vec{v}_i = x_1\vec{v}_1 + \cdots + x_{i-1}\vec{v}_{i-1} + x_{i+1}\vec{v}_{i+1} + \cdots + x_n\vec{v}_n,$$

then the left side is a linear combination with coefficients $(0, \dots, 0, 1_{(i)}, 0, \dots, 0)$, and the right side has coefficients $(x_1, \dots, x_{i-1}, 0_{(i)}, x_{i+1}, \dots, x_n)$. Since coefficients are different, by definition, the vectors are linearly dependent. □

Example 1.3.13. By Proposition 1.3.8, two vectors $\vec{u}$ and $\vec{v}$ are linearly dependent if and only if either $\vec{u}$ is a linear combination of $\vec{v}$, or $\vec{v}$ is a linear combination of $\vec{u}$. In other words, the two vectors are parallel.

Two vectors are linearly independent if and only if they are not parallel.

Exercise 1.42. Prove that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n, \vec{w}$ are linearly independent if and only if $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly independent, and $\vec{w}$ is not a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$.

The property implies that any subset of a linearly independent set is also linearly independent. What does this tell you about linear dependence?

Exercise 1.43. Prove that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly dependent if and only if some $\vec{v}_i$ is a linear combination of the *previous vectors* $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{i-1}$.

1.3.4 Minimal Spanning Set

Basis has two aspects, span and linear independence. If we already know that some vectors span a vector space, then we can achieve linear independence (and therefore basis) by deleting "unnecessary" vectors.

Definition 1.3.9. A vector space is *finite dimensional* if it is spanned by finitely many vectors.

Theorem 1.3.10. *In a finite dimensional vector space, a set of vectors is a basis if and only if it is a minimal spanning set. Moreover, any finite spanning set contains a minimal spanning set and therefore a basis.*

By a minimal spanning set α, we mean α spans V, and any subset strictly smaller than α does not span V.

Proof. Suppose $\alpha = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ spans V. The set α is either linearly independent, or linearly dependent.

If α is linearly independent, then it is a basis by definition. Moreover, by Proposition 1.3.8, we know $\vec{v}_i$ is not a linear combination of $\vec{v}_1, \cdots, \vec{v}_{i-1}, \vec{v}_{i+1} \ldots, \vec{v}_n$. Therefore after deleting $\vec{v}_i$, the remaining vectors $\vec{v}_1, \cdots, \vec{v}_{i-1}, \vec{v}_{i+1} \ldots, \vec{v}_n$ do not span V. This proves that α is a minimal spanning set.

If α is linearly dependent, then by Proposition 1.3.8, we may assume $\vec{v}_i$ is a linear combination of

$$\alpha' = \alpha - \{\vec{v}_i\} = \{\vec{v}_1, \cdots, \vec{v}_{i-1}, \vec{v}_{i+1} \ldots, \vec{v}_n\}.$$

By Proposition 1.2.1 (also see Exercise 1.30), linear combinations of α are also linear combinations of α'. Therefore we get a strictly smaller spanning set α'. Then we may ask whether α' is linearly dependent. If the answer is yes, then α' contains strictly smaller spanning set. The process continues and, since α is finite, will stop after finitely many steps. By the time we stop, we get a linearly independent spanning set. By definition, this is a basis.

We proved that "independence $\Longrightarrow$ minimal" and "dependence $\Longrightarrow$ not minimal". This implies that "independence $\Longleftrightarrow$ minimal". Since a spanning set is independent if and only if it is a basis, we get the first part of the theorem. Then the second part is contained in the proof above in case α is linearly dependent. $\square$

The intuition behind Theorem 1.3.10 is the following. Imagine that α is all the people in a company, and V is all the things the company wants to do. Then α spanning V means that the company can do all the things it wants to do. However, the company may not be efficient in the sense that if somebody's duty can be fulfilled by the others (the person is a linear combination of the others), then the company can fire the person and still do all the things. By firing unnecessary persons one after another, eventually everybody is indispensable (linearly independent). The result is that the company can do everything, and is also the most efficient.

Example 1.3.14. By taking $a = b = 10$ in Example 1.3.6, we get the row operations

$$\begin{pmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 10 & 10 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 7 & 10 \\ 0 & -3 & -6 & -9 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

Since all rows are pivot, the four vectors $(1,2,3),(4,5,6),(7,8,10),(11,12,10)$ span $\mathbb{R}^3$. By restricting the row operations to the first three columns, the row echelon form we get still has all rows being pivot. Therefore the spanning set can be reduced to a strictly smaller spanning set $(1,2,3),(4,5,6),(7,8,10)$. Alternatively, we may view the matrix as the augmented matrix of a system of linear equations. Then the row operation implies that the fourth vector $(10,11,10)$ is a linear combination of the first three vectors. This means that the fourth vector is a waste, and we can delete the fourth vector to get a strictly smaller spanning set.

Is the smaller spanning set $(1,2,3),(4,5,6),(7,8,10)$ minimal? If we further delete the third vector, then we are talking about the same row operation applied to the first two columns. The row echelon form we get has only two pivot rows, and third row is not pivot. Therefore $(1,2,3),(4,5,6)$ do not span $\mathbb{R}^3$. One may also delete the second or the first vector and do the similar investigation. In fact, by Proposition 1.3.4, two vectors can never span $\mathbb{R}^3$.

Exercise 1.44. Suppose we have row operations

$$(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_6) \to \begin{pmatrix} \bullet & * & * & * & * & * \\ 0 & 0 & \bullet & * & * & * \\ 0 & 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & 0 & \bullet \end{pmatrix}.$$

Explain that the six vectors span $\mathbb{R}^4$ and $\vec{v}_1, \vec{v}_3, \vec{v}_4, \vec{v}_6$ form a minimal spanning set (and therefore a basis).

Exercise 1.45. Show that the vectors span P_3 and then find a minimal spanning set.

1. $1+t, 1+t^2, 1+t^3, t+t^2, t+t^3, t^2+t^3$.
2. $t+2t^2+3t^3, -t-2t^2-3t^3, 1+2t^2+3t^3, 1-t, 1+t+3t^3, 1+t+2t^2$.

1.3.5 Maximal Independent Set

If we already know that some vectors are linearly independent, then we can achieve the span property (and therefore basis) by adding "independent" vectors. Therefore we have two ways of constructing basis

$$\text{span vector space} \xrightarrow{\text{delete vectors}} \text{basis} \xleftarrow{\text{add vectors}} \text{linearly independent}$$

Using the analogy of company, linear independence means there is no waste. What we need to achieve is to do all the things the company wants to do. If there is a job that the existing employees cannot do, then we need to hire somebody who can do the job. The new hire is linearly independent of the existing employees because the person can do something the others cannot do. We keep adding new necessary people until the company can do all the things, and therefore achieve the span.

Theorem 1.3.11. *In a finite dimensional vector space, a set of vectors is a basis if and only if it is a maximal linearly independent set. Moreover, any linearly independent set can be extended to a maximal linearly independent set and therefore a basis.*

By a maximal linearly independent set α, if α is linearly independent, and any set strictly bigger than α is linearly dependent.

Proof. Suppose $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a linearly independent set in V. The set α either spans V, or does not span V.

If α spans V, then it is a basis by definition. Moreover, any vector $\vec{v} \in V$ is a linear combination of α. By Proposition 1.3.8, adding $\vec{v}$ to α makes the set linearly dependent. Therefore α is a maximal linearly independent set.

If α does not span V, then there is a vector $\vec{v} \in V$ which is not a linear combination of α. If $\alpha' = \alpha \cup \{\vec{v}\} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n, \vec{v}\}$ is linearly dependent, then by Proposition 1.3.7, we have

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n + a\vec{v} = \vec{0},$$

and some coefficients are nonzero. If $a = 0$, then some of $a_1, a_2, \dots, a_n$ are nonzero. By Proposition 1.3.7, this contradicts the linear independence of α. If $a \neq 0$, then we get

$$\vec{v} = -\frac{a_1}{a}\vec{v}_1 - \frac{a_2}{a}\vec{v}_2 + \cdots - \frac{a_n}{a}\vec{v}_n,$$

contradicting the assumption that $\vec{v}$ is not a linear combination of α. This proves that α' is a linearly independent set strictly bigger than α.

Now we may ask whether α' spans V. If the answer is no, then we can enlarge α' further by adding another vector that is not a linear combination of α'. The result is again a bigger linearly independent set. The process continues, and must stop after finitely many steps due to the finite dimension assumption (full justification by

Proposition 1.3.13). By the time we stop, we get a linearly independent spanning set, which is a basis by definition.

We proved that "span $V \implies$ maximal" and "not span $V \implies$ not maximal". This implies that "span $V \iff$ maximal". Since a linearly independent set spans the vector space if and only if it is a basis, we get the first part of the theorem. Then the second part is contained in the proof above in case α does not span V. $\square$

Example 1.3.15. We take the transpose of the matrix in Example 1.3.14, and carry out row operations (this is the *column operations* on the earlier matrix)

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \\ 11 & 12 & 10 \end{pmatrix} \xrightarrow[R_2-R_1]{\substack{R_4-R_3 \\ R_3-R_2}} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 4 \\ 3 & 3 & 0 \end{pmatrix} \xrightarrow[R_4-R_2]{R_3-R_2} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{pmatrix} \xrightarrow[R_4+3R_3]{R_2-3R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

By Proposition 1.3.6, this shows that $(1,4,7,11), (2,5,8,12), (3,6,10,10)$ are linearly independent. However, since the last row is not pivot (or the last part of Proposition 1.3.4), the three vectors do not span $\mathbb{R}^4$.

To enlarge the linearly independent set of three vectors, we try to add a vector so that the same row operations produces $(0,0,0,1)$. The vector can be obtained by reversing the operations on $(0,0,0,1)$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \xleftarrow[R_4+R_3]{\substack{R_2+R_1 \\ R_3+R_2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \xleftarrow[R_3+R_2]{R_4+R_2} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \xleftarrow[R_2+3R_1]{R_4-3R_3} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then we have row operations

$$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 10 & 0 \\ 11 & 12 & 10 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This shows that $(1,4,7,11), (2,5,8,12), (3,6,10,10), (0,0,0,1)$ form a basis.

If we try a more interesting vector $(4,3,2,1)$

$$\begin{pmatrix} 4 \\ 19 \\ 36 \\ 46 \end{pmatrix} \xleftarrow[R_4+R_3]{\substack{R_2+R_1 \\ R_3+R_2}} \begin{pmatrix} 4 \\ 15 \\ 17 \\ 10 \end{pmatrix} \xleftarrow[R_3+R_2]{R_4+R_2} \begin{pmatrix} 4 \\ 15 \\ 2 \\ -5 \end{pmatrix} \xleftarrow[R_2+3R_1]{R_4-3R_3} \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix},$$

then we find that $(1,4,7,11), (2,5,8,12), (3,6,10,10), (4,19,36,46)$ form a basis.

Exercise 1.46. For the column vectors in Exercise 1.28, find a linearly independent subset, and then extend to a basis. Note that the linearly independent subset should be as big as possible, to avoid the lazy choice such as picking the first column only.

Exercise 1.47. Explain that $t^2(t-1), t(t^2-1), t^2-4$ are linearly independent. Then extend to a basis of P_3.

1.3.6 Dimension

Let V be a finite dimensional vector space. By Theorem 1.3.10, V has a basis $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$. Then the coordinate with respect to α translates the linear algebra of V to the Euclidean space $[\cdot]_\alpha \colon V \leftrightarrow \mathbb{R}^n$.

Let $\beta = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_k\}$ be another basis of V. Then the α-coordinate translates β into a basis $[\beta]_\alpha = \{[\vec{w}_1]_\alpha, [\vec{w}_2]_\alpha, \dots, [\vec{w}_k]_\alpha\}$ of $\mathbb{R}^n$. Therefore $[\beta]_\alpha$ (a set of k vectors) spans $\mathbb{R}^n$ and is also linearly independent. By the last parts of Propositions 1.3.4 and 1.3.6, we get $k = n$. This shows that the following concept is well defined.

Definition 1.3.12. The *dimension* of a (finite dimensional) vector space is the number of vectors in a basis.

We denote the dimension by $\dim V$. By Examples 1.3.1, 1.3.2, and Exercise 1.23, we have $\dim \mathbb{R}^n = n$, $\dim P_n = n+1$, $\dim M_{m\times n} = mn$.

If $\dim V = m$, then V can be identified with the Euclidean space $\mathbb{R}^m$, and the linear algebra in V is the same as the linear algebra in $\mathbb{R}^m$. For example, we may change $\mathbb{R}^m$ in Propositions 1.3.4 and 1.3.6 to any vector space V of dimension m, and get the following.

Proposition 1.3.13. *Suppose V is a finite dimensional vector space.*

1. *If n vectors span V, then* $\dim V \le n$.
2. *If n vectors in V are linearly independent, then* $\dim V \ge n$.

Continuation of the proof of Theorem 1.3.11. The proof of the theorem creates bigger and bigger linearly independent sets of vectors. By Proposition 1.3.13, however, the set is no longer linearly independent when the number of vectors is $> \dim V$. This means that, if the set α we start with has n vectors, then the construction in the proof stops after at most $\dim V - n$ steps.

We note that the argument uses Theorem 1.3.10, for the existence of basis and then the concept of dimension. What we want to prove is Theorem 1.3.11, which is not used in the argument. □

Exercise 1.48. Explain that the vectors do not span the vector space.

1. $3+\sqrt{2}t-\pi t^2-3t^3$, $e+100t+2\sqrt{3}t^2$, $4\pi t-15.2t^2+t^3$.
2. $\begin{pmatrix} 3 & 8 \\ 4 & 9 \end{pmatrix}, \begin{pmatrix} 2 & 8 \\ 6 & 5 \end{pmatrix}, \begin{pmatrix} 4 & 7 \\ 5 & 0 \end{pmatrix}$.

3. $\begin{pmatrix} \pi & \sqrt{3} \\ 1 & 2\pi \end{pmatrix}, \begin{pmatrix} \sqrt{2} & \pi \\ -10 & 2\sqrt{2} \end{pmatrix}, \begin{pmatrix} 3 & 100 \\ -77 & 6 \end{pmatrix}, \begin{pmatrix} \sin 2 & \pi \\ \sqrt{2}\pi & 2\sin 2 \end{pmatrix}$.

Exercise 1.49. Explain that the vectors are linearly dependent.

1. $3 + \sqrt{2}t - \pi t^2$, $e + 100t + 2\sqrt{3}t^2$, $4\pi t - 15.2t^2$, $\sqrt{\pi} + e^2 t^2$.
2. $\begin{pmatrix} 3 & 8 \\ 4 & 9 \end{pmatrix}, \begin{pmatrix} 2 & 8 \\ 6 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -2 & 4 \end{pmatrix}$.
3. $\begin{pmatrix} \pi & \sqrt{3} \\ 1 & 2\pi \end{pmatrix}, \begin{pmatrix} \sqrt{2} & \pi \\ -10 & 2\sqrt{2} \end{pmatrix}, \begin{pmatrix} 3 & 100 \\ -77 & 6 \end{pmatrix}, \begin{pmatrix} \sin 2 & \pi \\ \sqrt{2}\pi & 2\sin 2 \end{pmatrix}$.

Theorem 1.3.14. *Suppose α is a collection of vectors in a finite dimensional vector space V. Then any two of the following imply the third.*

1. *The number of vectors in α is $\dim V$.*
2. *α spans V.*
3. *α is linearly independent.*

To prove the theorem, we may translate into Euclidean space. Then by Propositions 1.3.4 and 1.3.6, we only need to prove the following properties about system of linear equations.

Theorem 1.3.15. *Suppose A is an $m \times n$ matrix. Then any two of the following imply the third.*

1. *$m = n$.*
2. *$A\vec{x} = \vec{b}$ has solution for all $\vec{b}$.*
3. *The solution of $A\vec{x} = \vec{b}$ is unique.*

Proof. If the second and third statement hold, then by Propositions 1.3.4 and 1.3.6, we have $m \le n$ and $m \ge n$. Therefore the first statement holds.

Now we assume the first statement, and prove that the second and third are equivalent. The first statement means A is an $n \times n$ matrix. By Proposition 1.3.4, the second statement means all rows are pivot, i.e., the number of pivots is n. By Proposition 1.3.6, the third statement means all columns are pivot, i.e., the number of pivots is n. Therefore the two statements are the same. □

Example 1.3.16. By Example 1.3.8, the three quadratic polynomials $t(t-1), t(t-2), (t-1)(t-2)$ are linearly independent. By Theorem 1.3.14 and $\dim P_2 = 3$, we know the three vectors form a basis of P_2.

For the general discussion, see Example 2.2.13.

Exercise 1.50. Use Theorem 1.3.10 to give another proof of the first part of Proposition 1.3.13. Use Theorem 1.3.11 to give another proof of the second part of Proposition 1.3.13.

Exercise 1.51. Suppose the number of vectors in α is $\dim V$. Explain the following are equivalent.

1. α spans V.
2. α is linearly independent.
3. α is a basis.

Exercise 1.52. Show that the vectors form a basis.

1. $(1,1,0),(1,0,1),(0,1,1)$ in $\mathbb{R}^3$.
2. $(1,1,-1),(1,-1,1),(-1,1,1)$ in $\mathbb{R}^3$.
3. $(1,1,1,0),(1,1,0,1),(1,0,1,1),(0,1,1,1)$ in $\mathbb{R}^4$.

Exercise 1.53. Show that the vectors form a basis.

1. $1+t, 1+t^2, t+t^2$ in P_2.
2. $\begin{pmatrix}1&1\\1&0\end{pmatrix}, \begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\1&1\end{pmatrix}, \begin{pmatrix}0&1\\1&1\end{pmatrix}$ in $M_{2\times 2}$.

Exercise 1.54. For which a do the vectors form a basis?

1. $(1,1,0),(1,0,1),(0,1,a)$ in $\mathbb{R}^3$.
2. $(1,-1,0),(1,0,-1),(0,1,a)$ in $\mathbb{R}^3$.
3. $(1,1,1,0),(1,1,0,1),(1,0,1,1),(0,1,1,a)$ in $\mathbb{R}^4$.

Exercise 1.55. For which a do the vectors form a basis?

1. $1+t, 1+t^2, t+at^2$ in P_2.
2. $\begin{pmatrix}1&1\\1&0\end{pmatrix}, \begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\1&1\end{pmatrix}, \begin{pmatrix}0&1\\1&a\end{pmatrix}$ in $M_{2\times 2}$.

Exercise 1.56. Show that $(a,b),(c,d)$ form a basis of $\mathbb{R}^2$ if and only if $ad \neq bc$. What is the condition for a to be a basis of $\mathbb{R}^1$?

Exercise 1.57. If the columns of a matrix form a basis of the Euclidean space, what is the reduced row echelon form of the matrix?

Exercise 1.58. Show that α is a basis if and only if β is a basis.

1. $\alpha = \{\vec{v}_1, \vec{v}_2\}$, $\beta = \{\vec{v}_1 + \vec{v}_2, \vec{v}_1 - \vec{v}_2\}$.
2. $\alpha = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, $\beta = \{\vec{v}_1 + \vec{v}_2, \vec{v}_1 + \vec{v}_3, \vec{v}_2 + \vec{v}_3\}$.
3. $\alpha = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, $\beta = \{\vec{v}_1, \vec{v}_1 + \vec{v}_2, \vec{v}_1 + \vec{v}_2 + \vec{v}_3\}$.
4. $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$, $\beta = \{\vec{v}_1, \vec{v}_1 + \vec{v}_2, \dots, \vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_n\}$.

Exercise 1.59. Use Exercises 1.32 and 1.37 to prove that the following are equivalent.

1. $\{\vec{v}_1, \dots, \vec{v}_i, \dots, \vec{v}_j, \dots, \vec{v}_n\}$ is a basis.
2. $\{\vec{v}_1, \dots, \vec{v}_j, \dots, \vec{v}_i, \dots, \vec{v}_n\}$ is a basis.
3. $\{\vec{v}_1, \dots, c\vec{v}_i, \dots, \vec{v}_n\}$, $c \neq 0$, is a basis.
4. $\{\vec{v}_1, \dots, \vec{v}_i + c\vec{v}_j, \dots, \vec{v}_j, \dots, \vec{v}_n\}$ is a basis.

1.3.7 Calculation of Coordinate

In Examples 1.3.1 and 1.3.1, we see the coordinates with respect to simple bases are quite easy to find. For the more complicated bases, it takes some serious calculation to find the coordinates.

Example 1.3.17. We have a basis of $\mathbb{R}^3$ (confirmed by later row operations)

$$\alpha = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \{(1, -1, 0), (1, 0, -1), (1, 1, 1)\}.$$

The coordinate of a general vector $\vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ with respect to α is the unique solution of a system of linear equation, whose augmented matrix is

$$(\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{b}) = \begin{pmatrix} 1 & 1 & 1 & x_1 \\ -1 & 0 & 1 & x_2 \\ 0 & -1 & 1 & x_3 \end{pmatrix}.$$

Of course we can find the coordinate $[\vec{x}]_\alpha$ by carrying out the row operations on this augmented matrix.

Alternatively, we may first try to find the α-coordinates of the standard basis vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$. Then the α-coordinate of a general vector is

$$[(x_1, x_2, x_3)]_\alpha = [x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3]_\alpha = x_1[\vec{e}_1]_\alpha + x_2[\vec{e}_2]_\alpha + x_3[\vec{e}_3]_\alpha.$$

The coordinate $[\vec{e}_i]_\alpha$ is calculated by row operations on the augmented matrix $(\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{e}_i)$. Note that the three augmented matrices have the same coefficient matrix part. Therefore we may combine the three calculations together by carry out the following row operations

$$(\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{e}_1\ \vec{e}_2\ \vec{e}_3) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_2 \leftrightarrow R_3]{\substack{R_3+R_1+R_2 \\ R_1 \leftrightarrow R_2}} \begin{pmatrix} -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 3 & 1 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{\substack{-R_1 \\ -R_2 \\ \frac{1}{3}R_3}} \begin{pmatrix} 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \xrightarrow{\substack{R_1+R_3 \\ R_2+R_3}} \begin{pmatrix} 1 & 0 & 0 & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

If we restrict the row operations to the first four columns $(\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{e}_1)$, then the reduced row echelon form is

$$(I\ [\vec{e}_1]_\alpha) = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \end{pmatrix}, \quad \text{where } I = (I\ [\vec{e}_1]_\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (\vec{e}_1\ \vec{e}_2\ \vec{e}_3).$$

The fourth column of the reduced echelon form is the solution $[\vec{e}_1]_\alpha = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Similarly, we get $[\vec{e}_2]_\alpha = (-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$ (fifth column) and $[\vec{e}_3]_\alpha = (\frac{1}{3}, -\frac{2}{3}, \frac{1}{3})$ (sixth column).

By Proposition 1.3.2, the α-coordinate of a general vector in $\mathbb{R}^3$ is

$$\begin{aligned} [\vec{b}]_\alpha &= x_1[\vec{e}_1]_\alpha + x_2[\vec{e}_2]_\alpha + x_3[\vec{e}_3]_\alpha \\ &= x_1 \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} + x_2 \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} + x_3 \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{pmatrix} \vec{x}. \end{aligned}$$

In general, if $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a basis of $\mathbb{R}^n$, then all rows and all columns of $A = (\vec{v}_1\ \vec{v}_2\ \dots\ \vec{v}_n)$ are pivot. This means that the reduced row echelon form of A is the *identity matrix*

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = (\vec{e}_1\ \vec{e}_2\ \cdots\ \vec{e}_n).$$

In other words, we have row operations changing A to I. Applying the same row operations to the $n \times 2n$ matrix $(A\ I)$, we get

$$(A\ I) \to (I\ B).$$

Then columns of B are the coordinates $[\vec{e}_i]_\alpha$ of $\vec{e}_i$ with respect to α, and the general α-coordinate is given by

$$[\vec{x}]_\alpha = B\vec{x}.$$

Exercise 1.60. Find the coordinates of a general vector in Euclidean space with respect to basis.

1. $(0,1),(1,0)$.
2. $(1,2),(3,4)$.
3. $(a,0),(0,b),\ a,b\neq 0$.
4. $(\cos\theta,\sin\theta),(-\sin\theta,\cos\theta)$.
5. $(1,1,0),(1,0,1),(0,1,1)$.
6. $(1,2,3),(0,1,2),(0,0,1)$.
7. $(0,1,2),(0,0,1),(1,2,3)$.
8. $(0,-1,2,1),(2,3,2,1),(-1,0,3,2),(4,1,2,3)$.

Exercise 1.61. Find the coordinates of a general vector with respect to the basis in Exercise 1.52.

Exercise 1.62. Determine whether vectors form a basis of $\mathbb{R}^n$. Moreover, find the coordinates with respect to the basis.

1. $\vec{e}_1-\vec{e}_2,\ \vec{e}_2-\vec{e}_3,\ \dots,\ \vec{e}_{n-1}-\vec{e}_n,\ \vec{e}_n-\vec{e}_1$.
2. $\vec{e}_1-\vec{e}_2,\ \vec{e}_2-\vec{e}_3,\ \dots,\ \vec{e}_{n-1}-\vec{e}_n,\ \vec{e}_1+\vec{e}_2+\cdots+\vec{e}_n$.
3. $\vec{e}_1+\vec{e}_2,\ \vec{e}_2+\vec{e}_3,\ \dots,\ \vec{e}_{n-1}+\vec{e}_n,\ \vec{e}_n+\vec{e}_1$.
4. $\vec{e}_1,\vec{e}_1+2\vec{e}_2,\vec{e}_1+2\vec{e}_2+3\vec{e}_3,\ \dots,\ \vec{e}_1+2\vec{e}_2+\cdots+n\vec{e}_n$.

Exercise 1.63. Determine whether polynomials form a basis of P_n. Moreover, find the coordinates with respect to the basis.

1. $1-t,\ t-t^2,\ \dots,\ t^{n-1}-t^n,\ t^n-1$.
2. $1+t,\ t+t^2,\ \dots,\ t^{n-1}+t^n,\ t^n+1$.
3. $1,\ 1+t,\ 1+t^2,\ \dots,\ 1+t^n$.
4. $1,\ t-1,\ (t-1)^2,\ \dots,\ (t-1)^n$.

Exercise 1.64. Suppose $ad\neq bc$. Find the coordinate of a vector in $\mathbb{R}^2$ with respect to the basis $(a,b),(c,d)$ (see Exercise 1.56).

Exercise 1.65. Suppose there is an identification of a vector space V with Euclidean space $\mathbb{R}^n$. In other words, there is a one-to-one correspondence $F\colon V\to\mathbb{R}^n$ preserving vector space operations

$$F(\vec{u}+\vec{v})=F(\vec{u})+F(\vec{v}),\quad F(a\vec{u})=aF(\vec{u}).$$

Let $\vec{v}_i=F^{-1}(\vec{e}_i)\in V$.

1. Prove that $\alpha=\{\vec{v}_1,\vec{v}_2,\dots,\vec{v}_n\}$ is a basis of V.
2. Prove that $F(\vec{x})=[\vec{x}]_\alpha$.

Chapter 2

Linear Transformation

Linear transformation is the relation between vector spaces. We discuss general concepts about maps such as onto, one-to-one, and invertibility, and then specialise these concepts to linear transformations. We also introduce matrices as the formula for linear transformations. Then operations of linear transformations become operations of matrices.

2.1 Linear Transformation and Matrix

Definition 2.1.1. A map $L\colon V \to W$ between vector spaces is a *linear transformation* if it preserves two operations in vector spaces

$$L(\vec{u}+\vec{v}) = L(\vec{u}) + L(\vec{v}), \quad L(c\vec{u}) = cL(\vec{u}).$$

If $V = W$, then we also call L a *linear operator*. If $W = \mathbb{R}$, then we also call L a *linear functional*.

Geometrically, the preservation of two operations means the preservation of parallelogram and scaling.

The collection of all linear transformations from V to W is denoted $\mathrm{Hom}(V, W)$. Here "Hom" refers to *homomorphism*, which means maps preserving algebraic structures.

Example 2.1.1. The identity map $I(\vec{v}) = \vec{v}\colon V \to V$ is a linear operator. We also denote by I_V to emphasise the vector space V.

The zero map $O(\vec{v}) = \vec{0}\colon V \to W$ is a linear transformation.

Example 2.1.2. Proposition 1.3.2 means that the α-coordinate map is a linear transformation.

Example 2.1.3. The rotation R_θ of the plane by angle θ and the reflection (flipping) F_θ with respect to the direction of angle ρ are linear, because they clearly preserve parallelogram and scaling.

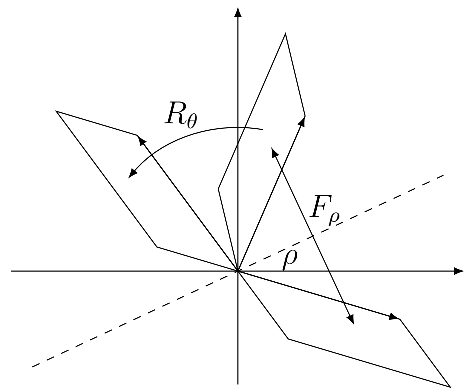

Figure 2.1.1: Rotation by angle θ and flipping with respect to angle ρ.

Example 2.1.4. The projection of $\mathbb{R}^3$ to a plane $\mathbb{R}^2$ passing through the origin is a linear transformation. More generally, any (orthogonal) projection of $\mathbb{R}^3$ to a plane inside $\mathbb{R}^3$ and passing through the origin is a linear operator.

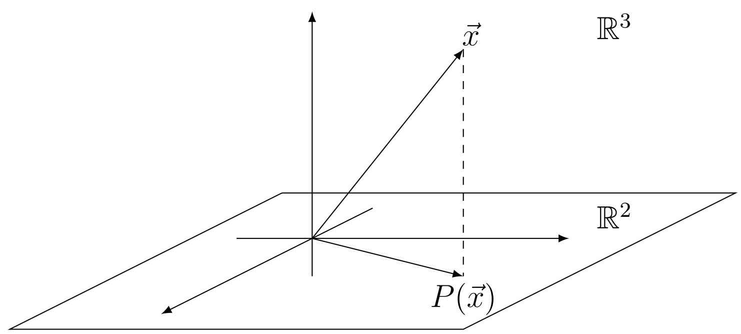

Figure 2.1.2: Projection from $\mathbb{R}^3$ to $\mathbb{R}^2$.

Example 2.1.5. The evaluation of functions at several places is a linear transformation

$$L(f) = (f(0), f(1), f(2)) \colon C^\infty \to \mathbb{R}^3.$$

In the reverse direction, the linear combination of several functions is a linear transformation

$$L(x_1, x_2, x_3) = x_1 \cos t + x_2 \sin t + x_3 e^t \colon \mathbb{R}^3 \to C^\infty.$$

The idea is extended in Exercise 2.19.

Example 2.1.6. In C^∞, taking the derivative is a linear operator

$$f \mapsto f' \colon C^\infty \to C^\infty.$$

The integration is a linear operator

$$f(t) \mapsto \int_0^t f(\tau)d\tau \colon C^\infty \to C^\infty.$$

Multiplying a fixed function $a(t)$ is also a linear operator

$$f(t) \mapsto a(t)f(t) \colon C^\infty \to C^\infty.$$

Exercise 2.1. Is the map a linear transformation?

1. $(x_1, x_2, x_3) \mapsto (x_1, x_2 + x_3) \colon \mathbb{R}^3 \to \mathbb{R}^2$.
2. $(x_1, x_2, x_3) \mapsto (x_1, x_2 x_3) \colon \mathbb{R}^3 \to \mathbb{R}^2$.
3. $(x_1, x_2, x_3) \mapsto (x_3, x_1, x_2) \colon \mathbb{R}^3 \to \mathbb{R}^3$.
4. $(x_1, x_2, x_3) \mapsto x_1 + 2x_2 + 3x_3 \colon \mathbb{R}^3 \to \mathbb{R}$.

Exercise 2.2. Is the map a linear transformation?

1. $f \mapsto f^2 \colon C^\infty \to C^\infty$.
2. $f(t) \mapsto f(t^2) \colon C^\infty \to C^\infty$.
3. $f \mapsto f'' \colon C^\infty \to C^\infty$.
4. $f(t) \mapsto f(t-2) \colon C^\infty \to C^\infty$.
5. $f(t) \mapsto f(2t) \colon C^\infty \to C^\infty$.
6. $f \mapsto f' + 2f \colon C^\infty \to C^\infty$.
7. $f \mapsto (f(0) + f(1), f(2)) \colon C^\infty \to \mathbb{R}^2$.
8. $f \mapsto f(0)f(1) \colon C^\infty \to \mathbb{R}$.
9. $f \mapsto \int_0^1 f(t)dt \colon C^\infty \to \mathbb{R}$.
10. $f \mapsto \int_0^t \tau f(\tau)d\tau \colon C^\infty \to C^\infty$.

2.1.1 Linear Transformation of Linear Combination

A linear transformation $L \colon V \to W$ preserves linear combination

$$\begin{aligned} L(x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n) &= x_1L(\vec{v}_1) + x_2L(\vec{v}_2) + \cdots + x_nL(\vec{v}_n) \\ &= x_1\vec{w}_1 + x_2\vec{w}_2 + \cdots + x_n\vec{w}_n, \qquad \vec{w}_i = L(\vec{v}_i). \end{aligned} \tag{2.1.1}$$

If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ span V, then any $\vec{x} \in V$ is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, and the formula implies that a linear transformation is determined by its values on a spanning set.

Proposition 2.1.2. *If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ span V, then two linear transformations L, K on V are equal if and only if $L(\vec{v}_i) = K(\vec{v}_i)$ for each i.*

Conversely, given assigned values $\vec{w}_i = L(\vec{v}_i)$ on a spanning set, the following says when the formula (2.1.1) gives a well defined linear transformation.

Proposition 2.1.3. *If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ span a vector space V, and $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n$ are vectors in W, then* (2.1.1) *gives a well defined linear transformation $L\colon V \to W$ if and only if*

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0} \implies x_1\vec{w}_1 + x_2\vec{w}_2 + \cdots + x_n\vec{w}_n = \vec{0}.$$

Proof. The formula gives a well defined map if and only if

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = y_1\vec{v}_1 + y_2\vec{v}_2 + \cdots + y_n\vec{v}_n$$

implies

$$x_1\vec{w}_1 + x_2\vec{w}_2 + \cdots + x_n\vec{w}_n = y_1\vec{w}_1 + y_2\vec{w}_2 + \cdots + y_n\vec{w}_n.$$

Let $z_i = x_i - y_i$. Then the condition becomes

$$z_1\vec{v}_1 + z_2\vec{v}_2 + \cdots + z_n\vec{v}_n = \vec{0}$$

implying

$$z_1\vec{w}_1 + z_2\vec{w}_2 + \cdots + z_n\vec{w}_n = \vec{0}.$$

This is the condition in the proposition.

After showing L is well defined, we still need to verify that L is a linear transformation. For

$$\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n, \quad \vec{y} = y_1\vec{v}_1 + y_2\vec{v}_2 + \cdots + y_n\vec{v}_n,$$

by (2.1.1), we have

$$\begin{aligned}
L(\vec{x}+\vec{y}) &= L((x_1+y_1)\vec{v}_1 + (x_2+y_2)\vec{v}_2 + \cdots + (x_n+y_n)\vec{v}_n) \\
&= (x_1+y_1)\vec{w}_1 + (x_2+y_2)\vec{w}_2 + \cdots + (x_n+y_n)\vec{w}_n \\
&= (x_1\vec{w}_1 + x_2\vec{w}_2 + \cdots + x_n\vec{w}_n) + (y_1\vec{w}_1 + y_2\vec{w}_2 + \cdots + y_n\vec{w}_n) \\
&= L(\vec{x}) + L(\vec{y}).
\end{aligned}$$

We can similarly show $L(c\vec{x}) = cL(\vec{x})$. □

If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ is a basis of V, then the condition of Proposition 2.1.3 is satisfied (the first $\implies$ is due to linear independence)

$$\begin{aligned}
x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0} &\implies x_1 = x_2 = \cdots = x_n = 0 \\
&\implies x_1\vec{w}_1 + x_2\vec{w}_2 + \cdots + x_n\vec{w}_n = \vec{0}.
\end{aligned}$$

This leads to the following result.

Proposition 2.1.4. *If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ is a basis of V, then* (2.1.1) *gives one-to-one correspondence between linear transformations $L\colon V \to W$ and the collections of n vectors $\vec{w}_1 = L(\vec{v}_1), \vec{w}_2 = L(\vec{v}_2), \dots, \vec{w}_n = L(\vec{v}_n)$ in W.*

Example 2.1.7. The rotation R_θ in Example 2.1.3 takes $\vec{e}_1 = (1,0)$ to the vector $(\cos\theta, \sin\theta)$ of radius 1 and angle θ. It also takes $\vec{e}_2 = (0,1)$ to the vector $(\cos\left(\theta+\frac{\pi}{2}\right), \sin\left(\theta+\frac{\pi}{2}\right)) = (-\sin\theta, \cos\theta)$ of radius 1 and angle $\theta + \frac{\pi}{2}$. We get

$$R_\theta(\vec{e}_1) = (\cos\theta, \sin\theta), \quad R_\theta(\vec{e}_2) = (-\sin\theta, \cos\theta).$$

Example 2.1.8. The derivative linear transformation $P_3 \to P_2$ is determined by the derivatives of the monomials $1' = 0$, $t' = 1$, $(t^2)' = 2t$, $(t^3)' = 3t^2$. It is also determined by the derivatives of another basis (see Example 1.3.3) of P_3: $1' = 0$, $(t-1)' = 1$, $((t-1)^2)' = 2(t-1)$, $((t-1)^3)' = 3(t-1)^2$.

Exercise 2.3. Suppose $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are vectors in V, and L is a linear transformation on V. Prove the following.

1. If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly dependent, then $L(\vec{v}_1), L(\vec{v}_2), \dots, L(\vec{v}_n)$ are linearly dependent.
2. If $L(\vec{v}_1), L(\vec{v}_2), \dots, L(\vec{v}_n)$ are linearly independent, then $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly independent.

2.1.2 Linear Transformation between Euclidean Spaces

By Proposition 2.1.4, linear transformations $L\colon \mathbb{R}^n \to W$ are in one-to-one correspondence with the collections of n vectors in W

$$\vec{w}_1 = L(\vec{e}_1),\ \vec{w}_2 = L(\vec{e}_2),\ \dots, \vec{w}_n = L(\vec{e}_n).$$

In case $W = \mathbb{R}^m$ is also a Euclidean space, a linear transformation $L\colon \mathbb{R}^n \to \mathbb{R}^m$ corresponds to the *matrix of linear transformation*

$$A = (\vec{w}_1\ \vec{w}_2\ \cdots\ \vec{w}_n) = (L(\vec{e}_1)\ L(\vec{e}_2)\ \cdots\ L(\vec{e}_n)).$$

The formula for L is the left (1.2.2) of a system of linear equations

$$L(\vec{x}) = x_1\vec{w}_1 + x_2\vec{w}_2 + \cdots + x_n\vec{w}_n = A\vec{x}. \tag{2.1.2}$$

Example 2.1.9. The matrix of the identity operator $I\colon \mathbb{R}^n \to \mathbb{R}^n$ is the *identity matrix*, still denoted by I (or I_n to indicate the size)

$$(I(\vec{e}_1)\ I(\vec{e}_2)\ \cdots\ I(\vec{e}_n)) = (\vec{e}_1\ \vec{e}_2\ \cdots\ \vec{e}_n) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = I_n.$$

The zero transformation $O = \mathbb{R}^n \to \mathbb{R}^m$ has the *zero matrix*

$$O = O_{m\times n} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Example 2.1.10. By Example 2.1.7, we have the matrix of the rotation R_θ in Example 2.1.3

$$(R_\theta(\vec{e}_1)\ R_\theta(\vec{e}_2)) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The reflection F_ρ of $\mathbb{R}^2$ with respect to the direction of angle ρ takes $\vec{e}_1$ to the vector of radius 1 and angle 2ρ, and also takes $\vec{e}_2$ to the vector of radius 1 and angle $2\rho - \frac{\pi}{2}$. Therefore the matrix of F_ρ is

$$\begin{pmatrix} \cos 2\rho & \cos\left(2\rho - \frac{\pi}{2}\right) \\ \sin 2\rho & \sin\left(2\rho - \frac{\pi}{2}\right) \end{pmatrix} = \begin{pmatrix} \cos 2\rho & \sin 2\rho \\ \sin 2\rho & -\cos 2\rho \end{pmatrix}.$$

Example 2.1.11. The projection in Example 2.1.4 takes the standard basis $\vec{e}_1 = (1,0,0)$, $\vec{e}_2 = (0,1,0)$, $\vec{e}_3 = (0,0,1)$ of $\mathbb{R}^3$ to $(1,0)$, $(0,1)$, $(0,0)$ in $\mathbb{R}^2$. The matrix of the projection is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Example 2.1.12. The linear transformation corresponding to the matrix

$$A = \begin{pmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{pmatrix}$$

is

$$L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + 4x_2 + 7x_3 + 10x_4 \\ 2x_1 + 5x_2 + 8x_3 + 11x_4 \\ 3x_1 + 6x_2 + 9x_3 + 12x_4 \end{pmatrix} : \mathbb{R}^4 \to \mathbb{R}^3.$$

We note that

$$L(\vec{e}_1) = \begin{pmatrix} 1 + 4\cdot 0 + 7\cdot 0 + 10\cdot 0 \\ 2\cdot 1 + 5\cdot 0 + 8\cdot 0 + 11\cdot 0 \\ 3\cdot 1 + 6\cdot 0 + 9\cdot 0 + 12\cdot 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

is the first column of A. Similarly, $L(\vec{e}_2), L(\vec{e}_3), L(\vec{e}_4)$ are the second, third and fourth columns of A.

Example 2.1.13. The orthogonal projection P of $\mathbb{R}^3$ to the plane $x + y + z = 0$ is a linear transformation. The columns of the matrix of P are the projections of the standard basis vectors to the plane. These projections are not easy to see directly. On the other hand, we can easily find the projections of some other vectors.

First, the vectors $\vec{v}_1 = (1,-1,0)$ and $\vec{v}_2 = (1,0,-1)$ lie in the plane because they satisfy $x+y+z = 0$. Since the projection clearly preserves the vectors on the plane, we get $P(\vec{v}_1) = \vec{v}_1$ and $P(\vec{v}_2) = \vec{v}_2$.

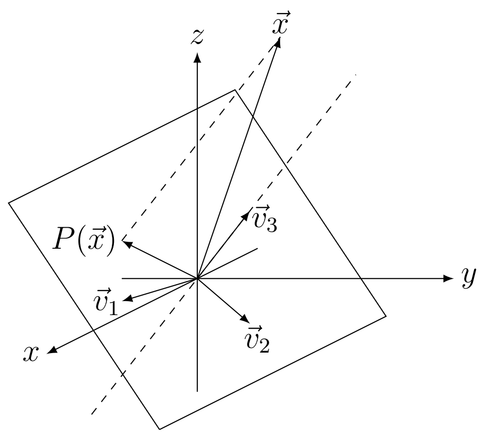

Figure 2.1.3: Projection to the plane $x + y + z = 0$.

Second, the vector $\vec{v}_3 = (1, 1, 1)$ is the coefficients of $x+y+z = 0$, and is therefore orthogonal to the plane. Since the projection kills vectors orthogonal to the plane, we get $P(\vec{v}_3) = \vec{0}$.

In Example 1.3.17, we found $[\vec{e}_1]_\alpha = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. This implies

$$P(\vec{e}_1) = \tfrac{1}{3}P(\vec{v}_1) + \tfrac{1}{3}P(\vec{v}_2) + \tfrac{1}{3}P(\vec{v}_3) = \tfrac{1}{3}\vec{v}_1 + \tfrac{1}{3}\vec{v}_2 + \tfrac{1}{3}\vec{0} = (\tfrac{2}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}).$$

By the similar idea, we get

$$\begin{aligned} P(\vec{e}_2) &= -\tfrac{2}{3}\vec{v}_1 + \tfrac{1}{3}\vec{v}_2 + \tfrac{1}{3}\vec{0} = (-\tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{1}{3}), \\ P(\vec{e}_3) &= \tfrac{1}{3}\vec{v}_1 - \tfrac{2}{3}\vec{v}_2 + \tfrac{1}{3}\vec{0} = (-\tfrac{1}{3}, -\tfrac{1}{3}, \tfrac{2}{3}). \end{aligned}$$

We conclude that the matrix of P is

$$(P(\vec{e}_1)\ P(\vec{e}_2)\ P(\vec{e}_3)) = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

Example 2.1.14. For the basis $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in Example 2.1.13, suppose we know a linear transformation $L\colon \mathbb{R}^3 \to \mathbb{R}^4$ satisfies

$$L(\vec{v}_1) = \vec{w}_1 = (1, 2, 3, 4), \quad L(\vec{v}_2) = \vec{w}_2 = (5, 6, 7, 8), \quad L(\vec{v}_3) = \vec{w}_3 = (9, 10, 11, 12).$$

To find the matrix of L, we note that a linear transformation preserves column operations. For example, we have $L(\vec{v}_1 + c\vec{v}_2) = L(\vec{v}_1) + cL(\vec{v}_2) = \vec{w}_1 + c\vec{w}_2$. Now suppose some column operations change $(\vec{v}_1\ \vec{v}_2\ \vec{v}_3)$ to $(\vec{e}_1\ \vec{e}_2\ \vec{e}_3)$. We apply the same column operations to $(\vec{w}_1\ \vec{w}_2\ \vec{w}_3)$ and obtain $(\vec{u}_1\ \vec{u}_2\ \vec{u}_3)$. Then we get $L(\vec{e}_i) = \vec{u}_i$, and the matrix of L. Therefore we carry out column operations

$$\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{pmatrix} \xrightarrow[C_3+C_2]{C_3+C_1} \begin{pmatrix} 1 & 1 & 3 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 5 & 15 \\ 2 & 6 & 18 \\ 3 & 7 & 21 \\ 4 & 8 & 24 \end{pmatrix}$$

$$\xrightarrow[\substack{C_2 \leftrightarrow C_3 \\ C_1 \leftrightarrow C_2}]{\frac{1}{3}C_3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 5 & 1 & 5 \\ 6 & 2 & 6 \\ 7 & 3 & 7 \\ 8 & 4 & 8 \end{pmatrix} \xrightarrow[C_3-C_1]{C_2-C_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 5 & -4 & 0 \\ 6 & -4 & 0 \\ 7 & -4 & 0 \\ 8 & -4 & 0 \end{pmatrix} \xrightarrow[-C_3]{-C_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 4 & 0 \\ 6 & 4 & 0 \\ 7 & 4 & 0 \\ 8 & 4 & 0 \end{pmatrix}.$$

We conclude that the matrix of L is

$$\begin{pmatrix} 5 & 4 & 0 \\ 6 & 4 & 0 \\ 7 & 4 & 0 \\ 8 & 4 & 0 \end{pmatrix}.$$

In general, if $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ is a basis of $\mathbb{R}^n$, and a linear transformation $L\colon \mathbb{R}^n \to \mathbb{R}^m$ satisfies $L(\vec{v}_i) = \vec{w}_i$, then the matrix A of L is given by column operations

$$\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_n \end{pmatrix} \xrightarrow{\text{col op}} \begin{pmatrix} I \\ A \end{pmatrix}.$$

Exercise 2.4. Find the matrix of the linear operator of $\mathbb{R}^2$ that sends $(1,2)$ to $(1,0)$ and sends $(3,4)$ to $(0,1)$.

Exercise 2.5. Find the matrix of the linear operator of $\mathbb{R}^3$ that reflects with respect to the plane $x+y+z=0$.

Exercise 2.6. Use the method in Example 2.1.14 to calculate Example 2.1.13 in another way.

Exercise 2.7. Find the matrix of the linear transformation $L\colon \mathbb{R}^3 \to \mathbb{R}^4$ determined by $L(1,-1,0) = (1,2,3,4)$, $L(1,0,-1) = (2,3,4,1)$, $L(1,1,1) = (3,4,1,2)$.

Exercise 2.8. Suppose a linear transformation $L\colon \mathbb{R}^3 \to P_3$ satisfies $L(1,-1,0) = 1 + 2t + 3t^2 + 4t^3$, $L(1,0,-1) = 2 + 3t + 4t^2 + t^3$, $L(1,1,1) = 3 + 4t + t^2 + 2t^3$. Find $L(x,y,z)$.

2.1.3 Operation of Linear Transformation

The *addition* of two linear transformations $L, K\colon V \to W$ is

$$(L+K)(\vec{v}) = L(\vec{v}) + K(\vec{v})\colon V \to W.$$

The following shows that $L+K$ preserves the addition

$$\begin{aligned}
(L+K)(\vec{u}+\vec{v}) &= L(\vec{u}+\vec{v}) + K(\vec{u}+\vec{v}) && \text{(definition of } L+K\text{)}\\
&= (L(\vec{u}) + L(\vec{v})) + (K(\vec{u}) + K(\vec{v})) && \text{(}L \text{ and } K \text{ preserve addition)}\\
&= L(\vec{u}) + K(\vec{u}) + L(\vec{v}) + K(\vec{v}) && \text{(Axioms 1 and 2)}\\
&= (L+K)(\vec{u}) + (L+K)(\vec{v}). && \text{(definition of } L+K\text{)}
\end{aligned}$$

We can similarly verify $(L+K)(c\vec{u}) = c(L+K)(\vec{u})$.

The *scalar multiplication* of a linear transformation is

$$(cL)(\vec{v}) = cL(\vec{v})\colon V \to W.$$

We can similarly show that cL is a linear transformation.

Proposition 2.1.5. $\mathrm{Hom}(V,W)$ *is a vector space.*

Proof. The following proves $L+K = K+L$

$$(L+K)(\vec{u}) = L(\vec{u}) + K(\vec{u}) = K(\vec{u}) + L(\vec{u}) = (K+L)(\vec{u}).$$

The first and third equalities are due to the definition of addition in $\mathrm{Hom}(V,W)$. The second equality is due to Axiom 1 of vector space.

We can similarly prove the associativity $(L+K)+M = L+(K+M)$. The zero vector in $\mathrm{Hom}(V,W)$ is the zero transformation $O(\vec{v}) = \vec{0}$ in Example 2.1.1. The negative of $L \in \mathrm{Hom}(V,W)$ is $K(\vec{v}) = -L(\vec{v})$. The other axioms can also be verified, and are left as exercises. $\square$

We can *compose* linear transformations $K\colon U \to V$ and $L\colon V \to W$ with matching domain and range

$$(L \circ K)(\vec{v}) = L(K(\vec{v}))\colon U \to W.$$

The composition preserves the addition

$$\begin{aligned}
(L\circ K)(\vec{u}+\vec{v}) &= L(K(\vec{u}+\vec{v})) = L(K(\vec{u}) + K(\vec{v}))\\
&= L(K(\vec{u})) + L(K(\vec{v})) = (L\circ K)(\vec{u}) + (L\circ K)(\vec{v}).
\end{aligned}$$

The first and fourth equalities are due to the definition of composition. The second and third equalities are due to the linearity of L and K. We can similarly verify that the composition preserves the scalar multiplication. Therefore the composition is a linear transformation.

Example 2.1.15. The composition of two rotations is still a rotation: $R_{\theta_1} \circ R_{\theta_2} = R_{\theta_1+\theta_2}$.

Example 2.1.16. Consider the differential equation

$$(1+t^2)f'' + (1+t)f' - f = t + 2t^3.$$

The left is the addition of three transformations $f \mapsto (1+t^2)f''$, $f \mapsto (1+t)f'$, $f \mapsto -f$.

Let $D(f) = f'$ be the derivative linear transformation in Example 2.1.6. Let $M_a(f) = af$ be the linear transformation of multiplying a function $a(t)$. Then $(1+t^2)f''$ is the composition $D^2 = M_{1+t^2} \circ D \circ D$, $(1+t)f'$ is the composition $M_{1+t} \circ D$, and $f \mapsto -f$ is the linear transformation M_{-1}. Therefore the left side of the differential equation is the linear transformation

$$L = M_{1+t^2} \circ D \circ D + M_{1+t} \circ D + M_{-1}.$$

The differential equation can be expressed as $L(f(t)) = b(t)$ with $b(t) = t+2t^3 \in C^\infty$.

In general, a *linear differential equation* of order n is

$$a_0(t)\frac{d^n f}{dt^n} + a_1(t)\frac{d^{n-1} f}{dt^{n-1}} + a_2(t)\frac{d^{n-2} f}{dt^{n-2}} + \cdots + a_{n-1}(t)\frac{df}{dt} + a_n(t)f = b(t).$$

If the coefficient functions $a_0(t), a_1(t), \dots, a_n(t)$ are smooth, then the left side is a linear transformation $C^\infty \to C^\infty$.

Exercise 2.9. Interpret the Newton-Leibniz formula $f(t) = f(0) + \int_0^t f'(\tau)d\tau$ as an equality of linear transformations.

Exercise 2.10. The *trace* of a square matrix $A = (a_{ij})$ is the sum of its diagonal entries

$$\mathrm{tr}A = a_{11} + a_{22} + \cdots + a_{nn}.$$

Explain that the trace is a linear functional on the vector space of $M_{n\times n}$ of $n \times n$ matrices, and $\mathrm{tr}A^T = \mathrm{tr}A$.

Exercise 2.11. Fix a vector $\vec{v} \in V$. Prove that the evaluation map $L \mapsto L(\vec{v})\colon \mathrm{Hom}(V, W) \to W$ is a linear transformation.

Exercise 2.12. Let $L\colon V \to W$ be a linear transformation.

1. Prove that $L \circ (K_1 + K_2) = L \circ K_1 + L \circ K_2$ and $L \circ (aK) = a(L \circ K)$.
2. Explain that the first part means that the map $L_* = L\circ\cdot\colon \mathrm{Hom}(U, V) \to \mathrm{Hom}(U, W)$ is a linear transformation.
3. Prove that $I_* = I$, $(L + K)_* = L_* + K_*$, $(aL)_* = aL_*$ and $(L \circ K)_* = L_* \circ K_*$.

4. Prove that $L \mapsto L_*\colon \mathrm{Hom}(V,W) \to \mathrm{Hom}(\mathrm{Hom}(U,V),\ \mathrm{Hom}(U,W))$ is a linear transformation.

Exercise 2.13. Let $L\colon V \to W$ be a linear transformation.

1. Prove that $(K_1 + K_2)\circ L = K_1 \circ L + K_2 \circ L$ and $(aK)\circ L = a(K\circ L)$.
2. Prove that $L^* = \cdot \circ L\colon \mathrm{Hom}(W,U) \to \mathrm{Hom}(V,U)$ is a linear transformation.
3. Prove that $I^* = I$, $(L+K)^* = L^* + K^*$, $(aL)^* = aL^*$ and $(L\circ K)^* = K^*\circ L^*$.
4. Prove that $L \mapsto L^*\colon \mathrm{Hom}(V,W) \to \mathrm{Hom}(\mathrm{Hom}(W,U),\ \mathrm{Hom}(V,U))$ is a linear transformation.

Exercise 2.14. Denote by $\mathrm{Map}(X,Y)$ all the maps from a set X to another set Y. For a map $f\colon X \to Y$ and any set Z, define

$$f_* = f\circ\colon \mathrm{Map}(Z,X) \to \mathrm{Map}(Z,Y), \quad f^* = \circ f\colon \mathrm{Map}(Y,Z) \to \mathrm{Map}(X,Z).$$

Prove that $(f\circ g)_* = f_*\circ g_*$ and $(f\circ g)^* = g^*\circ f^*$. This shows that $(L\circ K)_* = L_*\circ K_*$ in Exercise 2.12 and $(L\circ K)^* = K^*\circ L^*$ in Exercise 2.13 do not require linear transformation.

2.1.4 Matrix Operation

In Section 2.1.2, we have the equivalence

$$L \in \mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^m) \longleftrightarrow A \in M_{m\times n}, \quad L(\vec{x}) = A\vec{x}, \quad A = (L(\vec{e}_1)\ L(\vec{e}_2)\ \cdots\ L(\vec{e}_n)).$$

Any operation we can do on one side should be reflected on the other side. For example, we know $\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^m)$ is a vector space. We define the addition and scalar multiplication of matrices in $M_{m\times n}$ in such a way to make the invertible map $\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^m) \longleftrightarrow M_{m\times n}$ an isomorphism of vector spaces.

Let $L, K\colon \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations, with respective matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix}.$$

Then the i-th column of the matrix of $L+K\colon \mathbb{R}^n \to \mathbb{R}^m$ is

$$(L+K)(\vec{e}_i) = L(\vec{e}_i) + K(\vec{e}_i) = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix} + \begin{pmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{mi} \end{pmatrix} = \begin{pmatrix} a_{1i} + b_{1i} \\ a_{2i} + b_{2i} \\ \vdots \\ a_{mi} + b_{mi} \end{pmatrix}.$$

The *addition* of two matrices (of the same size) is the matrix of $L+K$

$$A+B=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{pmatrix}.$$

Similarly,the *scalar multiplication* cA of a matrix is the matrix of the linear transformation cL

$$cA=\begin{pmatrix} ca_{11} & ca_{12} & \dots & ca_{1n} \\ ca_{21} & ca_{22} & \dots & ca_{2n} \\ \vdots & \vdots & & \vdots \\ ca_{m1} & ca_{m2} & \dots & ca_{mn} \end{pmatrix}.$$

We emphasise that the formulae for $A+B$ and cA are not because they are the obvious thing to do, but because they reflect the concepts of $L+K$ and cL for linear transformations.

Let $L\colon \mathbb{R}^n \to \mathbb{R}^m$ and $K\colon \mathbb{R}^k \to \mathbb{R}^n$ be linear transformations, with respective matrices

$$A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad B=\begin{pmatrix} b_{11} & b_{12} & \dots & b_{1k} \\ b_{21} & b_{22} & \dots & b_{2k} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nk} \end{pmatrix}.$$

To get the matrix of the composition $L\circ K\colon \mathbb{R}^k \to \mathbb{R}^m$, we note that

$$B=(\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_k), \quad K(\vec{e}_i)=\vec{v}_i=\begin{pmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{ni} \end{pmatrix}.$$

Then the i-th column of the matrix of $L\circ K$ is (see (1.2.1) and (1.2.2))

$$\begin{aligned} (L\circ K)(\vec{e}_i) &= L(K(\vec{e}_i)) = L(\vec{v}_i) = A\vec{v}_i \\ &= \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{ni} \end{pmatrix} = \begin{pmatrix} a_{11}b_{1i}+a_{12}b_{2i}+\dots+a_{1n}b_{ni} \\ a_{21}b_{1i}+a_{22}b_{2i}+\dots+a_{2n}b_{ni} \\ \vdots \\ a_{m1}b_{1i}+a_{m2}b_{2i}+\dots+a_{mn}b_{ni} \end{pmatrix}. \end{aligned}$$

We define the *multiplication* of two matrices (of matching size) to be the matrix of $L\circ K$

$$AB=\begin{pmatrix} c_{11} & c_{12} & \dots & c_{1k} \\ c_{21} & c_{22} & \dots & c_{2k} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mk} \end{pmatrix}, \quad c_{ij}=a_{i1}b_{1j}+\dots+a_{in}b_{nj}.$$

The ij-entry of AB is obtained by multiplying the i-th row of A and the j-th column of B. For example, we have

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} \end{pmatrix}.$$

Example 2.1.17. The zero map O in Example 2.1.1 corresponds to the zero matrix O in Example 2.1.9. Since $O + L = L = L + O$, we get $O + A = A = A + O$.

The identity map I in Example 2.1.1 corresponds to the identity matrix I in Example 2.1.9. Since $I \circ L = L = L \circ I$, we get $IA = A = AI$.

Example 2.1.18. The composition of maps satisfies $(L \circ K) \circ M = L \circ (K \circ M)$. The equality is also satisfied by linear transformations. Correspondingly, we get the associativity $(AB)C = A(BC)$ of the matrix multiplication.

It is very complicated to verify $(AB)C = A(BC)$ by multiplying rows and columns. The conceptual explanation makes such computation unnecessary.

Example 2.1.19. In Example 2.1.10, we obtained the matrix of rotation R_θ. Then the equality $R_{\theta_1} \circ R_{\theta_2} = R_{\theta_1+\theta_2}$ in Example 2.1.15 corresponds to the multiplication of matrices

$$\begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix} = \begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{pmatrix}.$$

On the other hand, we calculate the left side by multiplying the rows of the first matrix with the columns of the second matrix, and we get

$$\begin{pmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & -\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2 \\ \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2 & \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \end{pmatrix}.$$

By comparing the two sides, we get the familiar trigonometric identities

$$\begin{aligned} \cos(\theta_1+\theta_2) &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2, \\ \sin(\theta_1+\theta_2) &= \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2. \end{aligned}$$

Example 2.1.20. Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We try to find matrices X satisfying $AX = I$. Let $X = \begin{pmatrix} x & z \\ y & w \end{pmatrix}$. Then the equality becomes

$$AX = \begin{pmatrix} x+2y & z+2w \\ 3x+4y & 3z+4w \end{pmatrix} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This means two systems of linear equations

$$\begin{aligned} x+2y&=1, & z+2w&=0,\\ 3x+4y&=0; & 3z+4w&=1. \end{aligned}$$

We can solve two systems simultaneously by the row operations

$$(A\ I)=\begin{pmatrix}1&2&1&0\\3&4&0&1\end{pmatrix}\to\begin{pmatrix}1&2&1&0\\0&-2&-3&1\end{pmatrix}\to\begin{pmatrix}1&0&-2&1\\0&1&\frac{3}{2}&-\frac{1}{2}\end{pmatrix}.$$

From the first three columns, we get $x=-2$ and $y=\frac{3}{2}$. from the first, second and fourth columns, we get $z=1$ and $w=-\frac{1}{2}$. Therefore

$$X=\begin{pmatrix}-2&1\\\frac{3}{2}&-\frac{1}{2}\end{pmatrix}.$$

In general, to solve $AX=B$, we may carry out the row operation on the matrix $(A\ B)$.

Exercise 2.15. Composing the reflection F_ρ in Examples 2.1.3 and 2.1.10 with itself is the identity. Explain that this means the trigonometric identity $\cos^2\theta+\sin^2\theta=1$.

Exercise 2.16. Geometrically, one can see the following compositions of rotations and refelctions.

1. $R_\theta\circ F_\rho$ is a reflection. What is the angle of reflection?
2. $F_\rho\circ R_\theta$ is a reflection. What is the angle of reflection?
3. $F_{\rho_1}\circ F_{\rho_2}$ is a rotation. What is the angle of rotation?

Interpret the geometrical observations as trigonometric identities.

Exercise 2.17. Use some examples (say rotations and reflections) to show that for two $n\times n$ matrices, AB may not be equal to BA.

Exercise 2.18. Use the formula for matrix addition to show the commutativity $A+B=B+A$ and the associativity $(A+B)+C=A+(B+C)$. Then give a conceptual explanation to the properties without using calculation.

Exercise 2.19. Explain that the addition and scalar multiplication of matrices make the set $M_{m\times n}$ of $m\times n$ matrices into a vector space. Moreover, the matrix of linear transformation gives an isomorphism (i.e., invertible linear transformation) $\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^m)\to M_{m\times n}$.

Exercise 2.20. Explain that Exercise 2.12 means that the matrix multiplication satisfies $A(B+C)=AB+AC$, $A(aB)=a(AB)$, and the left multiplication $X\mapsto AX$ is a linear transformation.

Exercise 2.21. Explain that Exercise 2.13 means that the matrix multiplication satisfies $(A+B)C = AC+BC$, $(aA)B = a(AB)$, and the right multiplication $X \mapsto XA$ is a linear transformation.

Exercise 2.22. Let A be an $m \times n$ matrix and let B be a $k \times m$ matrix. For the trace defined in Example 2.10, explain that $\mathrm{tr}AXB$ is a linear functional for $X \in M_{n\times k}$.

Exercise 2.23. Let A be an $m \times n$ matrix and let B be an $n \times m$ matrix. Prove that the trace defined in Example 2.10 satisfies $\mathrm{tr}AB = \mathrm{tr}BA$.

Exercise 2.24. Add or multiply matrices, whenever you can.

1. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.
2. $\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$.
3. $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.
4. $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.
5. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$.
6. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

Exercise 2.25. Find the n-th power matrix A^n (i.e., multiply the matrix to itself n times).

1. $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.
2. $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$.
3. $\begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix}$.
4. $\begin{pmatrix} 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix}$.
5. $\begin{pmatrix} a & b & 0 & 0 \\ 0 & a & b & 0 \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix}$.

Exercise 2.26. Solve the matrix equations.

1. $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} X = \begin{pmatrix} 7 & 8 \\ 9 & 10 \end{pmatrix}$.
2. $\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} X = \begin{pmatrix} 7 & 10 \\ 8 & 11 \\ a & b \end{pmatrix}$.
3. $\begin{pmatrix} 1 & 2 \\ 5 & 6 \\ 9 & 10 \end{pmatrix} X = \begin{pmatrix} 3 & 4 \\ 7 & 8 \\ 11 & b \end{pmatrix}$.

Exercise 2.27. For the transpose of 2×2 matrices, verify that $(AB)^T = B^T A^T$ (the conceptual reason will be given in Section 2.4.2).) Then use this to solve the matrix equations.

1. $X \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
2. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} X \begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Exercise 2.28. Let $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Find all the matrices X satisfying $AX = XA$. Generalise your result to *diagonal matrix*

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}.$$

2.1.5 Elementary Matrix and LU-Decomposition

The row operations can be interpreted as multiplying on the left by some special matrices, called *elementary matrices*. In fact, the special matrix is obtained by applying the same row operation to the identity matrix I. For example, by exchanging the 2nd and 4th rows, by multiplying c to the 2nd row, and by adding c multiples of the 3rd row to the 1st row, we get respectively

$$T_{24} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad D_2(c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad E_{13}(c) = \begin{pmatrix} 1 & 0 & c & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then

1. $T_{ij}A$ exchanges i-th and j-th rows of A.
2. $D_i(c)A$ multiplies c to the i-th row of A.
3. $E_{ij}(c)A$ adds the c multiple of the j-th row to the i-th row.

Note that the elementary matrices can also be obtained by applying similar *column operations* (already appeared in Exercises 1.32, 1.37, 1.59) to I. Then we have

1. AT_{ij} exchanges i-th and j-th columns of A.
2. $AD_i(a)$ multiplies a to the i-th column of A.
3. $AE_{ij}(a)$ adds the a multiple of the i-th column to the j-th column.

We know that any matrix can become (reduced) row echelon form after some row operations. This means that multiplying the left of the matrix by some elementary matrices gives a (reduced) row echelon form.

Example 2.1.21. We have

$$T_{24}D_2(c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} = D_4(c)T_{24}.$$

The equality can be obtained in four ways:

1. $T_{24}\times$: Exchange the 2nd and 4th rows of $D_2(c)$.
2. $\times D_2(c)$: Multiplying c to the 2nd column of T_{24}.
3. $\times T_{24}$: Exchange the 2nd and 4th columns of $D_4(c)$.
4. $D_4(c)\times$: Multiplying c to the 4th row of T_{24}.

In general, we get $T_{ij}D_i(c) = D_j(c)T_{ij}$. We also have $T_{ij}D_j(c) = D_i(c)T_{ij}$. Moreover, we have $T_{ij}D_k(c) = D_k(c)T_{ij}$ for distinct i, j, k.

Example 2.1.22. The row operations in Example 1.2.1 can be interpreted as

$$\begin{aligned}
&\begin{pmatrix}1 & 0 & 0\\ 0 & -\frac{1}{3} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -2 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -3 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 4 & 7 & 10\\ 2 & 5 & 8 & 11\\ 3 & 6 & 9 & 12\end{pmatrix}\\
=&\begin{pmatrix}1 & 4 & 7 & 10\\ 0 & 1 & 2 & 3\\ 0 & 0 & 0 & 0\end{pmatrix} = U.
\end{aligned}$$

Here U is an upper triangle matrix. On the other hand, we used row operations cR_i and $R_i + cR_j$ with $j < i$. This means that all the elementary matrices are lower triangular. The we move the elementary matrices to the right, and use $D_i(c)^{-1} = D_i(c^{-1})$ and $E_{ij}(c)^{-1} = E_{ij}(-c)$ to get

$$\begin{aligned}
&\begin{pmatrix}1 & 4 & 7 & 10\\ 2 & 5 & 8 & 11\\ 3 & 6 & 9 & 12\end{pmatrix}\\
=&\begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 3 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 2 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 4 & 7 & 10\\ 0 & 1 & 2 & 3\\ 0 & 0 & 0 & 0\end{pmatrix}\\
=&\begin{pmatrix}1 & 0 & 0\\ 2 & -3 & 0\\ 3 & -6 & 1\end{pmatrix}\begin{pmatrix}1 & 4 & 7 & 10\\ 0 & 1 & 2 & 3\\ 0 & 0 & 0 & 0\end{pmatrix} = LU.
\end{aligned}$$

We may also move -3 to U (i.e., remove $D_2(-\frac{1}{3})$) and get

$$\begin{pmatrix}1 & 4 & 7 & 10\\ 2 & 5 & 8 & 11\\ 3 & 6 & 9 & 12\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 2 & 1\end{pmatrix}\begin{pmatrix}1 & 4 & 7 & 10\\ 0 & -3 & -6 & -9\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

The example shows that, if we can use only $R_i + cR_j$ with $j < i$ to get a row echelon form U of A, then we can write $A = LU$, where L is the combination of the inverse of these $R_i + cR_j$, and is a lower triangular square matrix with nonzero diagonals. This is the *LU-decomposition* of the matrix A.

Not every matrix has LU-decomposition. For example, if $a_{11} = 0$ in A, then we need to first exchange rows to make the term nonzero

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 \end{pmatrix} \xrightarrow[R_3+R_2]{R_3-2R_1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = U.$$

This gives

$$PA = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = LU.$$

Here the left multiplication by P permutes rows. In general, every matrix has LU-decomposition after suitable permutation of rows.

The LU-decomposition is useful for solving $A\vec{x} = \vec{b}$. We may first solve $L\vec{y} = \vec{b}$ to get $\vec{y}$ and then solve $U\vec{x} = \vec{y}$. Since L is lower triangular, it is easy to get the unique solution $\vec{y}$ by forward substitution. Since U is upper triangular, we can use backward substituting to solve $U\vec{x} = \vec{y}$.

Exercise 2.29. Write down the 5×5 matrices: T_{24}, T_{42}, $D_4(c)$, $E_{35}(c)$, $E_{53}(c)$.

Exercise 2.30. What do you get by multiplying $T_{13}E_{13}(-2)D_2(3)$ to the left of

$$\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}.$$

What about multiplying on the right?

Exercise 2.31. Explain the following equalities

$$\begin{aligned} T_{ij}^2 &= I, \\ D_i(a)D_i(b) &= D_i(ab), \\ E_{ij}(a)E_{ij}(b) &= E_{ij}(a+b), \\ E_{ij}(a) &= E_{ik}(a)E_{kj}(1)E_{ik}(a)^{-1}E_{kj}(1)^{-1}. \end{aligned}$$

Can you come up with some other equalities?

Exercise 2.32. Find the LU-decompositions of the matrices in Exercise 1.28 that do not involve parameters.

Exercise 2.33. The LU-decompositions is derived from using row operations of third type (and maybe also second type) to get a row echelon form. What do you get by using the similar column operations.

2.2 Onto, One-to-one, and Inverse

Definition 2.2.1. Let $f: X \to Y$ be a map.

1. f is *onto* (or *surjective*) if for any $y \in Y$, there is $x \in X$, such that $f(x) = y$.
2. f is *one-to-one* (or *injective*) if $f(x) = f(x')$ implies $x = x'$.
3. f is a *one-to-one correspondence* (or *bijective*) if it is onto and one-to-one.

The onto property can be regarded as that the equation $f(x) = y$ has solution for all the right side y.

The one-to-one property also means that $x \neq x'$ implies $f(x) \neq f(x')$. The property can be regarded as that uniqueness of the solution of the equation $f(x) = y$.

Proposition 2.2.2. *Let $f: X \to Y$ be a map.*

1. *f is onto if and only if f has right inverse: There is a map $g: Y \to X$, such that $f \circ g = id_Y$.*
2. *f is one-to-one if and only if f has left inverse: There is a map $g: Y \to X$, such that $g \circ f = id_X$.*
3. *f is a* one-to-one correspondence *if and only if it has inverse: There is a map $g: Y \to X$, such that $f \circ g = id_Y$ and $g \circ f = id_X$.*

In the third statement, the map f is invertible, with *inverse map* g denoted $g = f^{-1}$.

Proof. If $f \circ g = id_Y$, then for any $y \in Y$, we have $f(g(y)) = y$. In other words, $x = g(y)$ satisfies $f(x) = y$. Conversely, suppose $f: X \to Y$ is onto. We construct a map $g: Y \to X$ as follows. For any $y \in Y$, by f onto, we can find some $x \in X$ satisfying $f(x) = y$. We choose one such x (strictly speaking, this uses the Axiom of Choice) and define $g(y) = x$. Then the map g satisfies $(f \circ g)(y) = f(x) = y$.

If $g \circ f = id_X$, then we have $g(f(x)) = x$. Therefore

$$f(x) = f(x') \implies g(f(x)) = g(f(x')) \implies x = x'.$$

Conversely, suppose $f: X \to Y$ is one-to-one. We fix an element $x_0 \in X$ and construct a map $g: Y \to X$ as follows. For any $y \in Y$, if $y = f(x)$ for some $x \in X$ (i.e., y lies in the image of f), then we define $g(y) = x$. If we cannot find such x, then we define $g(y) = x_0$. Note that in the first case, if $y = f(x')$ for another $x' \in X$, then by f one-to-one, we have $x' = x$. This shows that g is well defined. For the case $y = f(x)$, our construction of g implies $(g \circ f)(x) = g(f(x)) = x$.

From the first and second parts, we know a map f is onto and one-to-one if and only if there are maps g and h, such that $f \circ g = id$ and $h \circ f = id$. Compared with the definition of invertibility, for the third statement, we only need to show $g = h$. This follows from $g = id \circ g = h \circ f \circ g = h \circ id = h$. □

Example 2.2.1. Consider the map ($f =$) Instructor: Courses $\to$ Professors.

The map is onto means every professor teaches some course. The map g in Proposition 2.2.2 can take a professor (say me) to any one course (say linear algebra) the professor teaches.

The map is one-to-one means any professor either teaches one course, or does not teach any course. This also means that no professor teaches two or more courses. If a professor (say me) teaches one course, then the map g in Proposition 2.2.2 takes the professor to the unique course (say linear algebra) the professor teaches. If a professor does not teach any course, then g may take the professor to any one existing course.

Example 2.2.2. The identity map $I(x) = x \colon X \to X$ is always onto and one-to-one, with $I^{-1} = I$.

The zero map $O(\vec{v}) = \vec{0} \colon V \to W$ in Example 2.1.1 is onto if and only if W is the zero vector space in Example 1.1.1. The zero map is one-to-one if and only if V is the zero vector space.

The coordinate map in Section 1.3.7 is onto and one-to-one, with the linear combination map as the inverse.

The rotation and flipping in Example 2.1.3 are invertible, with $R_\theta^{-1} = R_{-\theta}$ and $F_\theta^{-1} = F_\theta$.

Exercise 2.34. Prove that the composition of onto maps is onto.

Exercise 2.35. Prove that the composition of one-to-one maps is one-to-one.

Exercise 2.36. Prove that the composition of invertible maps is invertible. Moreover, we have $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

Exercise 2.37. Prove that if $f \circ g$ is onto, then f is onto. Prove that if $g \circ f$ is one-to-one, then f is one-to-one.

2.2.1 Onto and One-to-one for Linear Transformation

For a linear transformation $L \colon V \to W$, we may regard the onto and one-to-one properties as the existence and uniqueness of solutions of the equation $L(\vec{x}) = \vec{b}$. For the case V and W are Euclidean spaces, this becomes the existence and uniqueness of solutions of the system of linear equations $A\vec{x} = \vec{b}$. This can be determined by the row echelon form of the matrix A.

Example 2.2.3. For the linear transformation in Example 2.1.12, we have the row echelon form (1.2.3). Since there are non-pivot rows and non-pivot columns, the linear transformation is not onto and not one-to-one.

More generally, by the row echelon form in Example 1.3.6, the linear transformation

$$L\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}x_1+4x_2+7x_3+10x_4\\2x_1+5x_2+8x_3+11x_4\\3x_1+6x_2+9x_3+12x_4\end{pmatrix}:\mathbb{R}^4\to\mathbb{R}^3$$

is onto if and only if $a\neq 9$, or $a=0$ and $b\neq 12$. Moreover, the linear transformation is never one-to-one.

Example 2.2.4. Consider the linear transformation

$$L\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}x_1+4x_2+7x_3\\2x_1+5x_2+8x_3\\3x_1+6x_2+ax_3\end{pmatrix}:\mathbb{R}^3\to\mathbb{R}^3$$

By the row echelon form in Example 1.3.6, the linear transformation is onto if and only if $a\neq 9$, and it is one-to-one if and only if $a\neq 9$. Therefore it is invertible if and only if $a\neq 9$.

Exercise 2.38. Determine whether the linear transformation is onto or one-to-one.

1. $L(x_1,x_2,x_3,x_4)=(x_1+2x_2+3x_3+4x_4,5x_1+6x_2+7x_3+8x_4,9x_1+10x_2+11x_3+12x_4)$.
2. $L(x_1,x_2,x_3,x_4)=(x_1+2x_2+3x_3+4x_4,5x_1+6x_2+7x_3+8x_4,9x_1+10x_2+ax_3+bx_4)$.
3. $L(x_1,x_2,x_3,x_4)=(x_1+2x_2+3x_3+4x_4,3x_1+4x_2+5x_3+6x_4,5x_1+6x_2+7x_3+8x_4,7x_1+8x_2+9x_3+10x_4)$
4. $L(x_1,x_2,x_3,x_4)=(x_1+2x_2+3x_3+4x_4,3x_1+4x_2+5x_3+6x_4,5x_1+6x_2+ax_3+8x_4,7x_1+8x_2+9x_3+bx_4)$.
5. $L(x_1,x_2,x_3)=(x_1+5x_2+9x_3,2x_1+6x_2+10x_3,3x_1+7x_2+11x_3,4x_1+8x_2+12x_3)$.
6. $L(x_1,x_2,x_3)=(x_1+5x_2+9x_3,2x_1+6x_2+10x_3,3x_1+7x_2+11x_3,4x_1+8x_2+ax_3)$.

Example 2.2.5. We claim that the evaluation $L(f(t))=(f(0),f(1),f(2))\colon C^\infty\to\mathbb{R}^3$ in Example 2.1.5 is onto. The idea is to find functions $f_1(t),f_2(t),f_3(t)$, such that $L(f_1(t))=\vec{e}_1,L(f_2(t))=\vec{e}_2,L(f_3(t))=\vec{e}_3$. Then any vector in $\mathbb{R}^3$ is

$$(x_1,x_2,x_3)=L(x_1f_1(t)+x_3f_2(t)+x_3f_3(t)).$$

It is not difficult to find a smooth function $f(t)$ satisfying $f(0)=1$ and $f(t)=0$ for $|t|\geq 1$. Then we may take $f_1(t)=f(t),f_2(t)=f(t-1),f_3(t)=f(t-2)$.

The evaluation is not one-to-one because $L(f(t-3))=(0,0,0)=L(0)$, and $f(t-3)$ is not a zero function.

Example 2.2.6. The derivation operator in Example 2.1.6 is onto due to the Newton-Leibniz formula. For any $g \in C^\infty$, we have $g = f'$ for $f(t) = \int_0^t g(\tau)d\tau \in C^\infty$. It is not one-to-one because all the constant functions are mapped to the zero function.

The integration operator in Example 2.1.6 is not onto because any function $g(t) = \int_0^t f(\tau)d\tau$ must satisfy $g(0) = 0$. The operator is one-to-one because taking derivative of $\int_0^t f_1(\tau)d\tau = \int_0^t f_2(\tau)d\tau$ implies $f_1(t) = f_2(t)$.

In fact, the Newton-Leibniz formula says derivation $\circ$ integration is the identity map. Then by Proposition 2.2.2, the derivation is onto and the integration is one-to-one.

need exercises

Proposition 2.2.3. *Suppose $L\colon V \to W$ is a linear transformation, and V, W are finite dimensional. The following are equivalent.*

1. *L is onto.*
2. *L takes a spanning set of V to a spanning set of W.*
3. *There is a linear transformation $K\colon W \to V$ satisfying $L \circ K = I_W$.*

From the proof below, we note that the equivalence of the first two statements only needs V to be finite dimensional, and the equivalence of the first and third statements only needs W to be finite dimensional.

Proof. Suppose $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ span V. Then

$$\begin{aligned} \vec{x} \in V &\implies \vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n \\ &\implies L(\vec{x}) = x_1L(\vec{v}_1) + x_2L(\vec{v}_2) + \cdots + x_nL(\vec{v}_n). \end{aligned}$$

The equality shows the first two statements are equivalent

$$\begin{aligned} L \text{ is onto} &\iff L(\vec{x}) \text{ can be all the vectors in } W \\ &\iff L(\vec{v}_1), L(\vec{v}_2), \dots, L(\vec{v}_n) \text{ span } W. \end{aligned}$$

By Proposition 2.2.2, we know the third statement implies the first. Conversely, assume L is onto. For a basis $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m$ of W, we can find $\vec{v}_i \in V$ satisfying $L(\vec{v}_i) = \vec{w}_i$. By Proposition 2.1.4, there is a linear transformation $K\colon W \to V$ satisfying $K(\vec{w}_i) = \vec{v}_i$. By $(L \circ K)(\vec{w}_i) = L(K(\vec{w}_i)) = L(\vec{v}_i) = \vec{w}_i$ and Proposition 2.1.2, we get $L \circ K = I_W$. $\square$

Proposition 2.2.4. *Suppose $L\colon V \to W$ is a linear transformation, and W is finite dimensional. The following are equivalent.*

1. *L is one-to-one.*

2. *L takes a linearly independent set in V to a linearly independent set in W.*

3. *$L(\vec{v}) = \vec{0}$ implies $\vec{v} = \vec{0}$.*

4. *There is a linear transformation $K: W \to V$ satisfying $K \circ L = I_V$.*

From the proof below, we note that the equivalence of the first three statements does not need W to be finite dimensional.

Proof. Suppose L is one-to-one and vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ in V are linearly independent. The following shows that $L(\vec{v}_1), L(\vec{v}_2), \dots, L(\vec{v}_n)$ are linearly independent

$$\begin{aligned}
&x_1L(\vec{v}_1) + x_2L(\vec{v}_2) + \cdots + x_nL(\vec{v}_n) = y_1L(\vec{v}_1) + y_2L(\vec{v}_2) + \cdots + y_nL(\vec{v}_n) \\
&\implies L(x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n) = L(y_1\vec{v}_1 + y_2\vec{v}_2 + \cdots + y_n\vec{v}_n) \quad (L \text{ is linear}) \\
&\implies x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = y_1\vec{v}_1 + y_2\vec{v}_2 + \cdots + y_n\vec{v}_n \quad (L \text{ is one-to-one}) \\
&\implies x_1 = y_1,\ x_2 = y_2,\ \dots,\ x_n = y_n. \quad (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \text{ are linearly independent})
\end{aligned}$$

Next we assume the second statement. If $\vec{v} \neq \vec{0}$, then by Example 1.3.10, the single vector $\vec{v}$ is linearly independent. By the assumption, the single vector $L(\vec{v})$ is also linearly independent. Again by Example 1.3.10, this means $L(\vec{v}) \neq \vec{0}$. This proves $\vec{v} \neq \vec{0} \implies L(\vec{v}) \neq \vec{0}$, which is the same as $L(\vec{v}) = \vec{0} \implies \vec{v} = \vec{0}$.

The following proves that the third statement implies the first

$$L(\vec{x}) = L(\vec{y}) \implies L(\vec{x} - \vec{y}) = L(\vec{x}) - L(\vec{y}) = \vec{0} \implies \vec{x} - \vec{y} = \vec{0} \implies \vec{x} = \vec{y}.$$

This completes the proof that the first three statements are equivalent.

By Proposition 2.2.2, we know the fourth statement implies the first. It remains to prove that the first three statements imply the fourth. This makes use of the assumption that W is finite dimensional.

Suppose $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ in V are linearly independent. By the second statement, the vectors $\vec{w}_1 = L(\vec{v}_1), \vec{w}_2 = L(\vec{v}_2), \dots, \vec{w}_n = L(\vec{v}_n)$ in W are also linearly independent. By Proposition 1.3.13, we get $n \leq \dim W$. Since $\dim W$ is finite, this implies that V is also finite dimensional. Therefore V has a basis, which we still denote by $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$. By Theorem 1.3.11, the corresponding linearly independent set $\{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n\}$ can be extended to a basis $\{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n, \vec{w}_{n+1}, \dots, \vec{w}_m\}$ of W. By Proposition 2.1.4, there is a linear transformation $K: W \to V$ satisfying $K(\vec{w}_i) = \vec{v}_i$ for $i \leq n$ and $K(\vec{w}_i) = \vec{0}$ for $n < i \leq m$. By $(K \circ L)(\vec{v}_i) = K(L(\vec{v}_i)) = K(\vec{w}_i) = \vec{v}_i$ for $i \leq n$ and Proposition 2.1.2, we get $K \circ L = I_V$. □

The following is comparable to Proposition 1.3.13. It reflects the intuition that, if every professor teaches some course (see Example 2.2.1), then the number of courses is more than the number of professors. On the other hand, if each professor teaches at most one course, then the number of courses is less than the number of professors.

Proposition 2.2.5. *Suppose $L\colon V \to W$ is a linear transformation between finite dimensional vector spaces.*

1. *If L is onto, then $\dim V \geq \dim W$.*

2. *If L is one-to-one, then $\dim V \leq \dim W$.*

Proof. Suppose $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a basis of V. We denote the image of the basis by $L(\alpha) = \{L(\vec{v}_1), L(\vec{v}_2), \dots, L(\vec{v}_n)\}$.

If L is onto, then by Proposition 2.2.3, $L(\alpha)$ spans W. By the first part of Proposition 1.3.13, we get $\dim V = n \geq \dim W$. If L is one-to-one, then by Proposition 2.2.4, $L(\alpha)$ is linearly independent. By the second part of Proposition 1.3.13, we get $\dim V = n \leq \dim W$. □

The following is the linear transformation version of Theorem 1.3.14.

Theorem 2.2.6. *Suppose $L\colon V \to W$ is a linear transformation between finite dimensional vector spaces. Then any two of the following imply the third.*

- *L is onto.*
- *L is one-to-one.*
- $\dim V = \dim W$.

Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ be a basis of V. The equivalence follows from Theorem 1.3.14, and applying the equivalence of the first two statements in Propositions 2.2.3 and 2.2.4.

Example 2.2.7. For the evaluation $L(f(t)) = (f(0), f(1), f(2))\colon C^\infty \to \mathbb{R}^3$ in Example 2.1.5, we find the functions $f_1(t), f_2(t), f_3(t)$ in Example 2.2.5 satisfying $L(f_1(t)) = \vec{e}_1, L(f_2(t)) = \vec{e}_2, L(f_3(t)) = \vec{e}_3$. This means that $K(x_1, x_2, x_3) = x_1 f_1(t) + x_3 f_2(t) + x_3 f_3(t)$ satisfies $L \circ K = I$. This is actually the reason for L to be onto in Example 2.2.5.

Example 2.2.8. The differential equation $f'' + (1+t^2)f' + tf = b(t)$ in Example 2.1.16 can be interpreted as $L(f(t)) = b(t)$ for a linear transformation $L\colon C^\infty \to C^\infty$. If we regard L as a linear transformation $L\colon P_n \to P_{n+1}$ (restricting L to polynomials), then by Proposition 2.2.5, the restriction linear transformation is not onto. For example, we can find a polynomial $b(t)$ of degree 5, such that $f'' + (1+t^2)f' + tf = b(t)$ cannot be solved for a polynomial $f(t)$ of degree 4.

Exercise 2.39. Show that the linear combination map $L(x_1, x_2, x_3) = x_1 \cos t + x_2 \sin t + x_3 e^t\colon \mathbb{R}^3 \to C^\infty$ in Example 2.1.5 is not onto and is one-to-one.

Exercise 2.40. Show that the multiplication map $f(t) \mapsto a(t)f(t)\colon C^\infty \to C^\infty$ in Example 2.1.6 is onto if and only if $a(t) \neq 0$ everywhere. Show that the map is one-to-one if $a(t) = 0$ at only finitely many places.

Exercise 2.41. Strictly speaking, the second statement of Proposition 2.2.3 can be about one spanning set or all spanning sets of V. Show that the two versions are equivalent. What about the second statement of Proposition 2.2.4?

Exercise 2.42. Suppose α is a basis of V. Prove that a linear transformation $L\colon V \to W$ is onto if and only if $L(\alpha)$ spans W, and L is one-to-one if and only if $L(\alpha)$ is linearly independent.

Exercise 2.43. Prove that a linear transformation is onto if it takes a (not necessarily spanning) set to a spanning set.

Exercise 2.44. Suppose $L\colon V \to W$ is an onto linear transformation. If V is finite dimensional, prove that W is finite dimensional.

Exercise 2.45. Let A be an $m \times n$ matrix. Explain that a system of linear equations $A\vec{x} = \vec{b}$ has solution for all $\vec{b} \in \mathbb{R}^m$ if and only if there is an $n \times m$ matrix B, such that $AB = I_m$. Moreover, the solution is unique if and only if there is B, such that $BA = I_n$.

Exercise 2.46. Suppose $L \circ K$ and K are linear transformations. Prove that if K is onto, then L is also a linear transformation.

Exercise 2.47. Suppose $L \circ K$ and L are linear transformations. Prove that if L is one-to-one, then K is also a linear transformation.

Exercise 2.48. Recall the induced maps f_* and f^* in Exercise 2.14. Prove that if f is onto, then f_* is onto and f^* is one-to-one. Prove that if f is one-to-one, then f_* is one-to-one and f^* is onto.

Exercise 2.49. Suppose L is an onto linear transformation. Prove that two linear transformations K and K' are equal if and only if $K \circ L = K' \circ L$. What does this tell you about the linear transformation $L^*\colon \mathrm{Hom}(V, W) \to \mathrm{Hom}(U, W)$ in Exercise 2.13?

Exercise 2.50. Suppose L is a one-to-one linear transformation. Prove that two linear transformations K and K' are equal if and only if $L \circ K = L \circ K'$. What does this tell you about the linear transformation $L_*\colon \mathrm{Hom}(U, V) \to \mathrm{Hom}(U, W)$ in Exercise 2.12?

2.2.2 Isomorphism

Definition 2.2.7. An invertible linear transformation is an *isomorphism*. If there is an isomorphism between two vector spaces V and W, then we say V and W are *isomorphic*, and denote $V \cong W$.

The isomorphism can be used to translate the linear algebra in one vector space to the linear algebra in another vector space.

Theorem 2.2.8. *If a linear transformation $L\colon V \to W$ is an isomorphism, then the inverse map $L^{-1}\colon W \to V$ is also a linear transformation. Moreover, suppose $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are vectors in V.*

1. *$\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ span V if and only if $L(\vec{v}_1), L(\vec{v}_2), \dots, L(\vec{v}_n)$ span W.*
2. *$\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly independent if and only if $L(\vec{v}_1), L(\vec{v}_2), \dots, L(\vec{v}_n)$ are linearly independent.*
3. *$\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ form a basis of V if and only if $L(\vec{v}_1), L(\vec{v}_2), \dots, L(\vec{v}_n)$ form a basis of W.*

The linearity of L^{-1} follows from Exercise 2.46 or 2.47. The rest of the proposition follows from the second statements in Propositions 2.2.3 and 2.2.4.

Example 2.2.9. Given a basis α of V, we explained in Section 1.3.7 that the α-coordinate map $[\cdot]_\alpha\colon V \to \mathbb{R}^n$ has an inverse. Therefore the α-coordinate map is an isomorphism.

Example 2.2.10. A linear transformation $L\colon \mathbb{R} \to V$ gives a vector $L(1) \in V$. This is a linear map (see Exercise 2.11)

$$L \in \mathrm{Hom}(\mathbb{R}, V) \mapsto L(1) \in V.$$

Conversely, for any $\vec{v} \in V$, we may construct a linear transformation $L(x) = x\vec{v}\colon \mathbb{R} \to V$. The construction gives a map

$$\vec{v} \in V \mapsto (L(x) = x\vec{v}) \in \mathrm{Hom}(\mathbb{R}, V).$$

We can verify that the two maps are inverse to each other. Therefore we get an isomorphism $\mathrm{Hom}(\mathbb{R}, V) \cong V$.

Example 2.2.11. The matrix of linear transformation between Euclidean spaces gives an invertible map

$$L \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) \longleftrightarrow A \in M_{m\times n}, \quad A = (L(\vec{e}_1)\ L(\vec{e}_2)\ \cdots\ L(\vec{e}_n)), \quad L(\vec{x}) = A\vec{x}.$$

The vector space structure on $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ is given by Proposition 2.1.5. Then the addition and scalar multiplication in $M_{m\times n}$ are defined for the purpose of making the map into an isomorphism.

Example 2.2.12. The transpose of matrices is an isomorphism

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \in M_{m\times n} \mapsto A^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix} \in M_{n\times m}.$$

In fact, we have $(A^T)^T = A$, which means that the inverse of the transpose map is the transpose map.

Example 2.2.13 (Lagrange interpolation)**.** Let $t_0, t_1, \dots, t_n$ be $n+1$ distinct numbers. Consider the general evaluation linear transformation (see Example 2.1.5)

$$L(f(t)) = (f(t_0), f(t_1), \dots, f(t_n)) \colon P_n \to \mathbb{R}^{n+1}.$$

We claim that L is onto by finding polynomials $p_i(t)$ satisfying $L(p_i(t)) = \vec{e}_i$ (see Example 2.2.5 and Exercise 2.43). Here we denote vectors in $\mathbb{R}^{n+1}$ by $(x_0, x_1, \dots, x_n)$, and denote the standard basis vectors by $\vec{e}_0, \vec{e}_1, \dots, \vec{e}_n$.

Let

$$g(t) = (t - t_1)(t - t_2)\cdots(t - t_n).$$

Since $t_0, t_1, \dots, t_n$ are distinct, we have

$$g(t_0) = (t_0 - t_1)(t_0 - t_2)\cdots(t_0 - t_n) \neq 0, \quad g(t_1) = g(t_2) = \cdots = g(t_n) = 0.$$

This implies that, if we take

$$p_0(t) = \frac{g(t)}{g(t_0)} = \frac{(t - t_1)(t - t_2)\cdots(t - t_n)}{(t_0 - t_1)(t_0 - t_2)\cdots(t_0 - t_n)},$$

then we have $L(p_0(t)) = \vec{e}_0$. Similarly, we have *Lagrange polynomials*

$$p_i(t) = \frac{(t - t_1)\cdots(t - t_{i-1})(t - t_{i+1})\cdots(t - t_n)}{(t_0 - t_1)\cdots(t_0 - t_{i-1})(t_0 - t_{i+1})\cdots(t_0 - t_n)} = \prod_{0\le j\le n,\ j\neq i} \frac{t - t_j}{t_i - t_j},$$

satisfying $L(p_i(t)) = \vec{e}_i$.

Since L is onto and $\dim P_n = \dim \mathbb{R}^{n+1}$, by Theorem 2.2.6, we conclude that L is an isomorphism. The one-to-one property of L means that a polynomial $f(t)$ of degree n is determined by its values $x_0 = f(t_0), x_1 = f(t_1), \dots, x_n = f(t_n)$ at $n + 1$ distinct places. The formula for $f(t)$ in terms of these values is given by the inverse $L^{-1} \colon \mathbb{R}^{n+1} \to P_n$, called the *Lagrange interpolation*

$$\begin{aligned} f(t) = L^{-1}(x_0, x_1, \dots, x_n) &= \sum_{i=0}^{n} x_i L^{-1}(\vec{e}_i) = \sum_{i=0}^{n} x_i p_i(t) \\ &= \sum_{i=0}^{n} x_i \prod_{0\le j\le n,\ j\neq i} \frac{t - t_j}{t_i - t_j} = \sum_{i=0}^{n} f(t_i) \prod_{0\le j\le n,\ j\neq i} \frac{t - t_j}{t_i - t_j}. \end{aligned}$$

For example, a quadratic polynomial $f(t)$ satisfying $f(-1) = 1, f(0) = 2, f(1) = 3$ is uniquely given by

$$f(t) = 1\frac{t(t-1)}{(-1)\cdot(-2)} + 2\frac{(t+1)(t-1)}{1\cdot(-1)} + 3\frac{(t+1)t}{4\cdot 3} = 2 - \frac{1}{4}t - \frac{5}{4}t^2.$$

Exercise 2.51. Suppose $\dim V = \dim W$ and $L\colon V \to W$ is a linear transformation. Prove that the following are equivalent.

1. L is invertible.

2. L has left inverse: There is $K\colon W \to V$, such that $K \circ L = I$.

3. L has right inverse: There is $K\colon W \to V$, such that $L \circ K = I$.

Moreover, show that the two K in the second and third parts must be the same.

Exercise 2.52. Explain $\dim \mathrm{Hom}(V, W) = \dim V \dim W$.

Exercise 2.53. Explain that the linear transformation

$$f \in P_n \mapsto (f(t_0), f'(t_0), \dots, f^{(n)}(t_0)) \in \mathbb{R}^{n+1}$$

is an isomorphism. What is the inverse isomorphism?

Exercise 2.54. Explain that the linear transformation (the right side has obvious vector space structure)

$$f \in C^\infty \mapsto (f', f(t_0)) \in C^\infty \times \mathbb{R}$$

is an isomorphism. What is the inverse isomorphism?

2.2.3 Invertible Matrix

A matrix A is *invertible* if the corresponding linear transformation $L(\vec{x}) = A\vec{x}$ is invertible (i.e., an isomorphism), and the *inverse matrix* A^{-1} is the matrix of the inverse linear transformation L^{-1}. By Theorem 2.2.6 (and Theorems 1.3.14 and 1.3.15), an invertible matrix must be a square matrix.

A linear transformation K is the inverse of L if $L \circ K = I$ and $K \circ L = I$. Correspondingly, a matrix B is the inverse of A if $AB = BA = I$. Exercise 2.51 shows that, in case of equal dimension, $L \circ K = I$ is equivalent to $K \circ L = I$. Correspondingly, for square matrices, $AB = I$ is equivalent to $BA = I$.

Example 2.2.14. Since the inverse of the identity linear transformation is the identity, the inverse of the identity matrix is the identity matrix: $I_n^{-1} = I_n$.

Example 2.2.15. The rotation R_θ of the plane by angle θ in Example 2.1.3 is invertible, with the inverse $R_\theta^{-1} = R_{-\theta}$ being the rotation by angle $-\theta$. Therefore the matrix of $R_{-\theta}$ is the inverse of the matrix of R_θ

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

One can directly verify that the multiplication of the two matrices is the identity.

The flipping F_ρ in Example 2.1.3 is also invertible, with the inverse $F_\rho^{-1} = F_\rho$ being the flipping itself. Therefore the matrix of F_ρ is the inverse of itself ($\theta = 2\rho$)

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

Exercise 2.55. Construct a 2×3 matrix A and a 3×2 matrix B satisfying $AB = I_2$. Explain that BA can never be I_3.

Exercise 2.56. What are the inverses of elementary matrices?

Exercise 2.57. Suppose A and B are invertible matrices. Prove that $(AB)^{-1} = B^{-1}A^{-1}$.

Exercise 2.58. Prove that the trace defined in Example 2.10 satisfies $\mathrm{tr}AXA^{-1} = \mathrm{tr}X$.

The following summarises many equivalent criteria for invertible matrix (and there will be more).

Proposition 2.2.9. *The following are equivalent for an $n \times n$ matrix A.*

1. *A is invertible.*
2. *$A\vec{x} = \vec{b}$ has solution for all $\vec{b} \in \mathbb{R}^n$.*
3. *The solution of $A\vec{x} = \vec{b}$ is unique.*
4. *The homogeneous system $A\vec{x} = \vec{0}$ has only trivial solution $\vec{x} = \vec{0}$.*
5. *$A\vec{x} = \vec{b}$ has unique solution for all $\vec{b} \in \mathbb{R}^n$.*
6. *The columns of A span $\mathbb{R}^n$.*
7. *The columns of A are linearly independent.*
8. *The columns of A form a basis of $\mathbb{R}^n$.*
9. *All rows of A are pivot.*
10. *All columns of A are pivot.*

11. *The reduced row echelon form of A is the identity matrix I.*

12. *A is the product of elementary matrices.*

Next we try to calculate the inverse matrix.

Let A be the matrix of an invertible linear transformation $L\colon \mathbb{R}^n \to \mathbb{R}^n$. Let $A^{-1} = (\vec{w}_1\ \vec{w}_2\ \cdots\ \vec{w}_n)$ be the matrix of the inverse linear transformation L^{-1}. Then $\vec{w}_i = L^{-1}(\vec{e}_i) = A^{-1}\vec{e}_i$. This implies $A\vec{w}_i = L(\vec{w}_i) = \vec{e}_i$. Therefore the i-th column of A^{-1} is the solution of the system of linear equation $A\vec{x} = \vec{e}_i$. The solution can be calculated by the reduced row echelon form of the augmented matrix

$$(A\ \vec{e}_i) \to (I\ \vec{w}_i).$$

Here the row operations can reduce A to I by Proposition 2.2.9. Then the solution of $A\vec{x} = \vec{e}_i$ is exactly the last column of the reduced row echelon form $(I\ \vec{w}_i)$.

Since the systems of liner equations $A\vec{x} = \vec{e}_1$, $A\vec{x} = \vec{e}_2$, $\dots$, $A\vec{x} = \vec{e}_n$ have the same coefficient matrix A, we may solve these equations simultaneously by combining the row operations

$$(A\ I) = (A\ \vec{e}_1\ \vec{e}_2\ \cdots\ \vec{e}_n) \to (I\ \vec{w}_1\ \vec{w}_2\ \cdots\ \vec{w}_n) = (I\ A^{-1}).$$

We have already used the idea in Example 1.3.17.

Example 2.2.16. The row operations in Example 2.1.20 give

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}.$$

In general, we can directly verify that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad ad \neq bc.$$

Example 2.2.17. The basis in Example 1.3.14 shows that the matrix

$$A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{pmatrix}$$

is invertible. Then we carry out the row operations

$$
\begin{aligned}
(A\ I) &= \begin{pmatrix} 1 & 4 & 7 & 1 & 0 & 0 \\ 2 & 5 & 8 & 0 & 1 & 0 \\ 3 & 6 & 10 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 7 & 1 & 0 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 0 & -6 & -11 & -3 & 0 & 1 \end{pmatrix} \\
&\to \begin{pmatrix} 1 & 4 & 7 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & -\frac{5}{3} & \frac{4}{3} & 0 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{pmatrix} \\
&\to \begin{pmatrix} 1 & 0 & 0 & -\frac{2}{3} & -\frac{2}{3} & 1 \\ 0 & 1 & 0 & -\frac{4}{3} & \frac{11}{3} & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{pmatrix}.
\end{aligned}
$$

Therefore

$$
\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{2}{3} & -\frac{2}{3} & 1 \\ -\frac{4}{3} & \frac{11}{3} & -2 \\ 1 & -2 & 1 \end{pmatrix}.
$$

Example 2.2.18. The row operation in Example 1.3.17 shows that

$$
\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}^{-1} = \frac{1}{3} \begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{pmatrix}.
$$

In terms of linear transformation, the result means that the linear transformation

$$
L \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + x_3 \\ -x_1 + x_3 \\ -x_2 + x_3 \end{pmatrix} : \mathbb{R}^3 \to \mathbb{R}^3
$$

is invertible, and the inverse is

$$
L^{-1} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} x_1 - 2x_2 + x_3 \\ x_1 + x_2 - 2x_3 \\ x_1 + x_2 + x_3 \end{pmatrix} : \mathbb{R}^3 \to \mathbb{R}^3.
$$

Exercise 2.59. What is the inverse of 1×1 matrix (a)?

Exercise 2.60. Verify the formula for the inverse of 2×2 matrix in Example 2.2.16 by multiplying the two matrices together. Moreover, show that the 2×2 matrix is not invertible when $ad = bc$.

Exercise 2.61. Find inverse matrix.

1. $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

2. $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

3. $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

4. $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$.

5. $\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}$.

6. $\begin{pmatrix} 1 & a & b & c \\ 0 & 1 & a & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

7. $\begin{pmatrix} b & c & 1 \\ 1 & 0 & 0 \\ a & 1 & 0 \end{pmatrix}$.

8. $\begin{pmatrix} c & a & b & 1 \\ b & 1 & a & 0 \\ a & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Exercise 2.62. Find the inverse matrix.

1. $\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$.

2. $\begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 1 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & 1 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$.

3. $\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & a_1 \\ 0 & 0 & \cdots & a_1 & a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & a_1 & \cdots & a_{n-2} & a_{n-1} \end{pmatrix}$.

2.3 Matrix of General Linear Transformation

By using bases to identify general vector spaces with Euclidean spaces, we may introduce the matrix of a general linear transformation with respect to bases.

2.3.1 Matrix with Respect to Bases

Let $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ and $\beta = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m\}$ be (ordered) bases of (finite dimensional) vector spaces V and W. Then a linear transformation $L\colon V \to W$ can be translated into a linear transformation $L_{\beta\alpha}$ between Euclidean spaces.

$$\begin{array}{ccc} V & \xrightarrow{L} & W \\ {\scriptstyle [\cdot]_\alpha}\downarrow\cong & & \cong\downarrow{\scriptstyle [\cdot]_\beta} \\ \mathbb{R}^n & \xrightarrow{L_{\beta\alpha}} & \mathbb{R}^n \end{array} \qquad L_{\beta\alpha}([\vec{v}]_\alpha) = [L(\vec{v})]_\beta \text{ for } \vec{v} \in V.$$

Please pay attention to the notation, that $L_{\beta\alpha}$ is the corresponding linear transformation from α to β.

The *matrix* $[L]_{\beta\alpha}$ *of* L *with respect to bases* α *and* β is the matrix of the linear transformation $L_{\beta\alpha}$, introduced in Section 2.1.2. To calculate this matrix, we apply the translation above to a vector $\vec{v}_i \in \alpha$ and use $[\vec{v}]_\alpha = \vec{e}_i$

$$\begin{tikzcd} \vec{v}_i \arrow[r, "L"] \arrow[d] & L(\vec{v}_i) \arrow[d] \\ \vec{e}_i \arrow[r, "L_{\beta\alpha}"] & {[L(\vec{v}_i)]_\beta} \end{tikzcd} \qquad L_{\beta\alpha}(\vec{e}_i) = L_{\beta\alpha}([\vec{v}_i]_\alpha) = [L(\vec{v}_i)]_\beta.$$

Denote $L(\alpha) = \{L(\vec{v}_1), L(\vec{v}_2), \dots, L(\vec{v}_n)\}$. Then we have

$$[L]_{\beta\alpha} = ([L(\vec{v}_1)]_\beta \; [L(\vec{v}_2)]_\beta \; \cdots \; [L(\vec{v}_n)]_\beta) = [L(\alpha)]_\beta.$$

Specifically, suppose $\alpha = \{\vec{v}_1, \vec{v}_2\}$ is a basis of V, and $\beta = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is a basis of W. A linear transformation $L \colon V \to W$ is determined by

$$\begin{aligned} L(\vec{v}_1) &= a_{11}\vec{w}_1 + a_{21}\vec{w}_2 + a_{31}\vec{w}_3, \\ L(\vec{v}_2) &= a_{12}\vec{w}_1 + a_{22}\vec{w}_2 + a_{32}\vec{w}_3. \end{aligned}$$

Then

$$[L(\vec{v}_1)]_\beta = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}, \quad [L(\vec{v}_2)]_\beta = \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}, \quad [L]_{\beta\alpha} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}.$$

Note that the matrix $[L]_{\beta\alpha}$ is obtained by combining all the coefficients in $L(\vec{v}_1)$, $L(\vec{v}_2)$ and then take the transpose.

Example 2.3.1. The orthogonal projection P of $\mathbb{R}^3$ to the plane $x + y + z = 0$ in Example 2.1.13 preserves vectors on the plan and maps vectors orthogonal to the plane to $\vec{0}$. Specifically, with respect to the basis

$$\alpha = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}, \quad \vec{v}_1 = (1, -1, 0),\ \vec{v}_2 = (1, 0, -1),\ \vec{v}_3 = (1, 1, 1),$$

we have

$$P(\vec{v}_1) = \vec{v}_1, \quad P(\vec{v}_2) = \vec{v}_2, \quad P(\vec{v}_3) = \vec{0}.$$

This means that

$$[P]_{\alpha\alpha} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This is much simpler than the matrix with respect to the standard basis ϵ that we obtained in Example 2.1.13.

Example 2.3.2. With respect to the standard monomial bases $\alpha = \{1, t, t^2, t^3\}$ and $\beta = \{1, t, t^2\}$ of P_3 and P_2, the matrix of the derivative linear transformation $D\colon P_3 \to P_2$ is

$$[D]_{\beta\alpha} = [(1)', (t)', (t^2)', (t^3)']_{\{1,t,t^2\}} = [0, 1, 2t, 3t^2]_{\{1,t,t^2\}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

For example, the derivative $(1 + 2t + 3t^2 + 4t^3)' = 2 + 6t + 12t^2$ fits into

$$\begin{pmatrix} 2 \\ 6 \\ 12 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

If we modify the basis β to $\gamma = \{1, t-1, (t-1)^2\}$ in Example 1.3.3, then

$$\begin{aligned} [D]_{\gamma\alpha} &= [0, 1, 2t, 3t^2]_{\{1,t-1,(t-1)^2\}} \\ &= [0, 1, 2(1 + (t-1)), 3(1 + (t-1))^2]_{\{1,t-1,(t-1)^2\}} \\ &= [0, 1, 2 + 2(t-1), 3 + 6(t-1) + 3(t-1)^2]_{\{1,t-1,(t-1)^2\}} \\ &= \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \end{pmatrix}. \end{aligned}$$

Example 2.3.3. The linear transformation in Example 2.1.16

$$L(f) = (1 + t^2)f'' + (1 + t)f' - f\colon P_3 \to P_3$$

satisfies

$$L(1) = -1, \quad L(t) = 1, \quad L(t^2) = 2 + 2t + 3t^2, \quad L(t^3) = 6t + 3t^2 + 8t^3.$$

Therefore

$$[L]_{\{1,t,t^2,t^3\}\{1,t,t^2,t^3\}} = \begin{pmatrix} -1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 8 \end{pmatrix}.$$

To solve the equation $L(f) = t + 2t^3$ in Example 2.1.16, we have row operations

$$\begin{pmatrix} -1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 6 & 1 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 8 & 2 \end{pmatrix} \to \begin{pmatrix} -1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 6 & 1 \\ 0 & 0 & 0 & 8 & 2 \end{pmatrix} \to \begin{pmatrix} -1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This shows that L is not one-to-one and not onto. Moreover, the solution of the differential equation is given by

$$a_3 = \tfrac{1}{4}, \quad a_2 = -a_3 = -\tfrac{1}{4}, \quad a_0 = 2 \cdot \tfrac{1}{4} + a_1 = \tfrac{1}{2} + a_1.$$

In other words, the solution is ($c = a_1$ is arbitrary)

$$f = \tfrac{1}{2} + a_1 + a_1 t - \tfrac{1}{4}t^2 + \tfrac{1}{4}t^3 = c(1+t) + \tfrac{1}{4}(2 - t^2 + t^3).$$

Example 2.3.4. With respect to the standard basis of $M_{2\times 2}$

$$\sigma = \{S_1, S_2, S_3, S_4\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

the transpose linear transformation $\cdot^T \colon M_{2\times 2} \to M_{2\times 2}$ has matrix

$$[\cdot^T]_{\sigma\sigma} = [\sigma^T]_\sigma = [S_1, S_3, S_2, S_4]_{\{S_1,S_2,S_3,S_4\}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Moreover, the linear transformation $A\cdot \colon M_{2\times 2} \to M_{2\times 2}$ of left multiplying by $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ has matrix

$$\begin{aligned} [A\cdot]_{\sigma\sigma} &= [A\sigma]_\sigma = [AS_1, AS_2, AS_3, AS_4]_{\{S_1,S_2,S_3,S_4\}} \\ &= [S_1 + 3S_3, S_2 + 3S_4, 2S_1 + 4S_3, 2S_2 + 4S_4]_{\{S_1,S_2,S_3,S_4\}} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{pmatrix}. \end{aligned}$$

Exercise 2.63. In Example 2.3.2, what is the matrix of the derivative linear transformation if α is changed to $\{1, t+1, (t+1)^2\}$?

Exercise 2.64. Find the matrix of $\int_0^t \colon P_2 \to P_3$, with respect to the usual bases in P_2 and P_3. What about $\int_1^t \colon P_2 \to P_3$?

Exercise 2.65. In Example 2.3.4, find the matrix of the right multiplication by A.

Proposition 2.3.1. *The matrix of linear transformation has the following properties*

$$[I]_{\alpha\alpha} = I, \quad [L+K]_{\beta\alpha} = [L]_{\beta\alpha} + [K]_{\beta\alpha}, \quad [cL]_{\beta\alpha} = c[L]_{\beta\alpha},$$

$$[L \circ K]_{\gamma\alpha} = [L]_{\gamma\beta}[K]_{\beta\alpha}.$$

Proof. The equality $[I]_{\alpha\alpha} = I$ is equivalent to that $I_{\alpha\alpha}$ is the identity linear transformation. This follows from

$$I_{\alpha\alpha}([\vec{v}]_\alpha) = [I(\vec{v})]_\alpha = [\vec{v}]_\alpha.$$

Alternatively, we have

$$[I]_{\alpha\alpha} = [\alpha]_\alpha = I.$$

The equality $[L+K]_{\beta\alpha} = [L]_{\beta\alpha} + [K]_{\beta\alpha}$ is equivalent to the equality $(L+K)_{\beta\alpha} = L_{\beta\alpha} + K_{\beta\alpha}$ for linear transformations, which we can verify by using Proposition 1.3.2

$$\begin{aligned}(L+K)_{\beta\alpha}([\vec{v}]_\alpha) &= [(L+K)(\vec{v})]_\beta = [L(\vec{v}) + K(\vec{v})]_\beta \\ &= [L(\vec{v})]_\beta + [K(\vec{v})]_\beta = L_{\beta\alpha}([\vec{v}]_\alpha) + K_{\beta\alpha}([\vec{v}]_\alpha).\end{aligned}$$

Alternatively, we have (the second equality is true for individual vectors in α)

$$\begin{aligned}[L+K]_{\beta\alpha} &= [(L+K)(\alpha)]_\beta = [L(\alpha) + K(\alpha)]_\beta \\ &= [L(\alpha)]_\beta + [K(\alpha)]_\beta = [L]_{\beta\alpha} + [K]_{\beta\alpha}.\end{aligned}$$

The verification of $[cL]_{\beta\alpha} = c[L]_{\beta\alpha}$ is similar, and is omitted.

The equality $[L \circ K]_{\gamma\alpha} = [L]_{\gamma\beta}[K]_{\beta\alpha}$ is equivalent to $(L \circ K)_{\gamma\alpha} = L_{\gamma\beta} \circ K_{\beta\alpha}$. This follows from

$$\begin{aligned}(L \circ K)_{\gamma\alpha}([\vec{v}]_\alpha) &= [(L \circ K)(\vec{v})]_\gamma = [L(K(\vec{v}))]_\gamma \\ &= L_{\gamma\beta}([K(\vec{v})]_\beta) = L_{\gamma\beta}(K_{\beta\alpha}([\vec{v}]_\alpha)) = (L_{\gamma\beta} \circ K_{\beta\alpha})([\vec{v}]_\alpha).\end{aligned}$$

Alternatively, we have (the third equality is true for individual vectors in $K(\alpha)$)

$$[L \circ K]_{\gamma\alpha} = [(L \circ K)(\alpha)]_\gamma = [L(K(\alpha))]_\gamma = [L]_{\gamma\beta}[K(\alpha)]_\beta = [L]_{\gamma\beta}[K]_{\beta\alpha}. \qquad \square$$

Example 2.3.5 (Vandermonde Matrix). Applying the evaluation linear transformation in Example 2.2.13 to the monomials, we get the matrix of the linear transformation

$$[L]_{\epsilon\{1,t,\dots,t^n\}} = \begin{pmatrix} 1 & t_0 & t_0^2 & \dots & t_0^n \\ 1 & t_1 & t_1^2 & \dots & t_1^n \\ 1 & t_2 & t_2^2 & \dots & t_2^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & t_n & t_n^2 & \dots & t_n^n \end{pmatrix}.$$

This is called the *Vandermonde matrix*. Example 2.2.13 tells us that the matrix is invertible if and only if all t_i are distinct. Moreover, the Lagrangian interpolation is the formula for L^{-1}, and shows that the i-th column of the inverse of the Vandermonde matrix is the coefficients in the polynomial

$$L^{-1}(\vec{e}_i) = \frac{(t-t_1)\cdots(t-t_{i-1})(t-t_{i+1})\cdots(t-t_n)}{(t_0-t_1)\cdots(t_0-t_{i-1})(t_0-t_{i+1})\cdots(t_0-t_n)}.$$

For example, for $n=2$, we have

$$[L^{-1}(\vec{e}_0)]_{\{1,t,t^2\}} = \left[\frac{(t-t_1)(t-t_2)}{(t_0-t_1)(t_0-t_2)}\right]_{\{1,t,t^2\}} = \frac{(t_1t_2, -t_1-t_2, 1)}{(t_0-t_1)(t_0-t_2)},$$

and

$$\begin{pmatrix} 1 & t_0 & t_0^2 \\ 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{t_1t_2}{(t_0-t_1)(t_0-t_2)} & \frac{t_0t_2}{(t_1-t_0)(t_1-t_2)} & \frac{t_0t_1}{(t_2-t_0)(t_2-t_1)} \\ \frac{-t_1-t_2}{(t_0-t_1)(t_0-t_2)} & \frac{-t_0-t_2}{(t_1-t_0)(t_1-t_2)} & \frac{-t_0-t_1}{(t_2-t_0)(t_2-t_1)} \\ \frac{1}{(t_0-t_1)(t_0-t_2)} & \frac{1}{(t_1-t_0)(t_1-t_2)} & \frac{1}{(t_2-t_0)(t_2-t_1)} \end{pmatrix}.$$

Exercise 2.66. In Example 2.3.2, directly verify $[D]_{\gamma\alpha} = [I]_{\gamma\beta}[D]_{\beta\alpha}$.

Exercise 2.67. Prove that $(cL)_{\beta\alpha} = cL_{\beta\alpha}$ and $(L\circ K)_{\gamma\alpha} = L_{\gamma\beta}K_{\beta\alpha}$. This implies the equalities $[cL]_{\beta\alpha} = c[L]_{\beta\alpha}$ and $[L\circ K]_{\gamma\alpha} = [L]_{\gamma\beta}[K]_{\beta\alpha}$ in Proposition 2.3.1.

Exercise 2.68. Prove that L is invertible if and only if $[L]_{\beta\alpha}$ is invertible. Moreover, we have $[L^{-1}]_{\alpha\beta} = [L]_{\beta\alpha}^{-1}$.

Exercise 2.69. Find the matrix of the linear transformation in Exercise 2.53 with respect to the standard basis in P_n and $\mathbb{R}^{n+1}$. Also find the inverse matrix.

Exercise 2.70. The left multiplication in Example 2.3.4 is an isomorphism. Find the matrix of the inverse.

2.3.2 Change of Basis

The matrix $[L]_{\beta\alpha}$ depends on the choice of (ordered) bases α and β. If α' and β' are also bases, then by Proposition 2.3.1, the matrix of L with respect to the new choice is

$$[L]_{\beta'\alpha'} = [I\circ L\circ I]_{\beta'\alpha'} = [I]_{\beta'\beta}[L]_{\beta\alpha}[I]_{\alpha\alpha'}.$$

This shows that the matrix of linear transformation is modified by multiplying matrices $[I]_{\alpha\alpha'}$, $[I]_{\beta'\beta}$ of the identity operator with respect to various bases. Such matrices are the *matrices for the change of basis*, and is simply the coordinates of vectors in one basis with respect to the other basis

$$[I]_{\alpha\alpha'} = [I(\alpha')]_\alpha = [\alpha']_\alpha.$$

Proposition 2.3.1 implies the following properties.

Proposition 2.3.2. *The matrix for the change of basis has the following properties*

$$[I]_{\alpha\alpha} = I, \quad [I]_{\beta\alpha} = [I]_{\alpha\beta}^{-1}, \quad [I]_{\gamma\alpha} = [I]_{\gamma\beta}[I]_{\beta\alpha}.$$

Example 2.3.6. Let ϵ be the standard basis of $\mathbb{R}^n$, and let $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be another basis. Then the matrix for changing from α to ϵ is

$$[I]_{\epsilon\alpha} = [\alpha]_\epsilon = (\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n) = (\alpha).$$

In general, the matrix for changing from α to β is

$$[I]_{\beta\alpha} = [I]_{\beta\epsilon}[I]_{\epsilon\alpha} = [I]_{\epsilon\beta}^{-1}[I]_{\epsilon\alpha} = [\beta]_\epsilon^{-1}[\alpha]_\epsilon = (\beta)^{-1}(\alpha).$$

For example, the matrix for changing from the basis in Example 2.2.17

$$\alpha = \{(1,2,3),(4,5,6),(7,8,10)\}$$

to the basis in Examples 1.3.17, 2.2.18 and 2.3.1

$$\beta = \{(1,-1,0),(1,0,-1),(1,1,1)\}$$

is

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 0 & 0 & 1 \\ -3 & -3 & -5 \\ 6 & 15 & 25 \end{pmatrix}.$$

Example 2.3.7. Consider the basis $\alpha_\theta = \{(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\}$ of unit length vectors on the plane at angles θ and $\theta + \frac{\pi}{2}$. The matrix for the change of basis from α_{θ_1} to α_{θ_2} is obtained from the α_{θ_2}-coordinates of vectors in α_{θ_1}. Since α_{θ_1} is obtained from α_{θ_2} by rotating $\theta = \theta_1 - \theta_2$, the coordinates are the same as the ϵ-coordinates of vectors in α_θ. This means

$$[I]_{\alpha_{\theta_2}\alpha_{\theta_1}} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos(\theta_1-\theta_2) & -\sin(\theta_1-\theta_2) \\ \sin(\theta_1-\theta_2) & \cos(\theta_1-\theta_2) \end{pmatrix}.$$

This is consistent with the formula in Example 2.3.6

$$[I]_{\alpha_{\theta_2}\alpha_{\theta_1}} = (\alpha_{\theta_2})^{-1}(\alpha_{\theta_1}) = \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix}^{-1} \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix}.$$

Example 2.3.8. The matrix for the change from the basis $\alpha = \{1, t, t^2, t^3\}$ of P_3 to another basis $\beta = \{1, t-1, (t-1)^2, (t-1)^3\}$ is

$$\begin{aligned} [I]_{\beta\alpha} &= [1, t, t^2, t^3]_{\{1,t-1,(t-1)^2,(t-1)^3\}} \\ &= [1, 1+(t-1), 1+2(t-1)+(t-1)^2, \\ &\qquad 1+3(t-1)+3(t-1)^2+(t-1)^3]_{\{1,t-1,(t-1)^2,(t-1)^3\}} \\ &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

The inverse of this matrix is

$$\begin{aligned}
[I]_{\alpha\beta} &= [1, t-1, (t-1)^2, (t-1)^3]_{\{1,t,t^2,t^3\}} \\
&= [1, -1+t, 1-2t+t^2, -1+3t-3t^2+t^3]_{\{1,t,t^2,t^3\}} \\
&= \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

We can also use the method outlined before Example 2.2.16 to calculate the inverse. But the method above is simpler.

The equality

$$(t+1)^3 = 1 + 3t + 3t^2 + t^3 = ((t-1)+2)^3 = 8 + 12(t-1) + 6(t-1)^2 + (t-1)^3$$

gives coordinates

$$[(t+1)^3]_\alpha = (1,3,3,1), \quad [(t+1)^3]_\beta = (8,12,6,1).$$

The two coordinates are related by the matrices for the change of basis

$$\begin{pmatrix} 8 \\ 12 \\ 6 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 8 \\ 12 \\ 6 \\ 1 \end{pmatrix}.$$

Exercise 2.71. Use matrices for the change of basis in Example 2.3.8 to find the matrix $[L]_{\{1,t,t^2,t^3\}\{1,t-1,(t-1)^2,(t-1)^3\}}$ of the linear transformation L in Example 2.3.3.

Exercise 2.72. If the basis in the source vector space V is changed by one of three operations in Exercise 1.59, how is the matrix of linear transformation changed? What about the similar change in the target vector space W?

2.3.3 Similar Matrix

For a linear operator $L\colon V \to V$, we usually choose the same basis α for the domain V and the range V. The matrix of the linear operator with respect to the basis α is $[L]_{\alpha\alpha}$. The matrices with respect to different bases are related by

$$[L]_{\beta\beta} = [I]_{\beta\alpha}[L]_{\alpha\alpha}[I]_{\alpha\beta} = [I]_{\alpha\beta}^{-1}[L]_{\alpha\alpha}[I]_{\alpha\beta} = [I]_{\beta\alpha}[L]_{\alpha\alpha}[I]_{\beta\alpha}^{-1}.$$

We say the two matrices $A = [L]_{\alpha\alpha}$ and $B = [L]_{\beta\beta}$ are *similar* in the sense that they are related by

$$B = P^{-1}AP = QAQ^{-1},$$

where P (matrix for changing from β to α) is an invertible matrix with $P^{-1} = Q$ (matrix for changing from α to β).

Example 2.3.9. In Example 2.3.1, we showed that the matrix of the orthogonal projection P in Example 2.1.13 with respect to $\alpha = \{(1,-1,0),(1,0,-1),(1,1,1)\}$ is very simple

$$[P]_{\alpha\alpha} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

On the other hand, by Examples 2.3.6 and 2.2.18, we have

$$[I]_{\epsilon\alpha} = [\alpha]_\epsilon = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}, \quad [I]_{\epsilon\alpha}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then we get the usual matrix of P (with respect the the standard basis ϵ)

$$\begin{aligned} [P]_{\epsilon\epsilon} &= [I]_{\epsilon\alpha}[P]_{\alpha\alpha}[I]_{\epsilon\alpha}^{-1} \\ &= \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}. \end{aligned}$$

The matrix is the same as the one obtained in Example 2.1.13 by another method.

Example 2.3.10. Consider the linear operator $L(f(t)) = tf'(t) + f(t)\colon P_3 \to P_3$. Applying the operator to the bases $\alpha = \{1, t, t^2, t^3\}$, we get

$$L(1) = 1, \quad L(t) = 2t, \quad L(t^2) = 3t^2, \quad L(t^3) = 4t^3.$$

Therefore

$$[L]_{\alpha\alpha} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

Consider another basis $\beta = \{1, t-1, (t-1)^2, (t-1)^3\}$ of P_2. By Example 2.3.8, we have

$$[I]_{\beta\alpha} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad [I]_{\alpha\beta} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore

$$\begin{aligned} [L]_{\beta\beta} &= [I]_{\beta\alpha}[L]_{\alpha\alpha}[I]_{\alpha\beta} \\ &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix}. \end{aligned}$$

We can verify the result by directly applying $L(f) = (tf(t))'$ to vectors in β

$$\begin{aligned}
L(1) &= 1, \\
L(t-1) &= [(t-1) + (t-1)^2]' = 1 + 2(t-1), \\
L((t-1)^2) &= [(t-1)^2 + (t-1)^3]' = 2(t-1) + 3(t-1)^2, \\
L((t-1)^3) &= [(t-1)^3 + (t-1)^4]' = 3(t-1)^2 + 4(t-1)^3.
\end{aligned}$$

Exercise 2.73. Explain that if A is similar to B, then B is similar to A.

Exercise 2.74. Explain that if A is similar to B, and B is similar to C, then A is similar to C.

Exercise 2.75. Find the matrix of the linear operator of $\mathbb{R}^2$ that sends $\vec{v}_1 = (1, 2)$ and $\vec{v}_2 = (3, 4)$ to $2\vec{v}_1$ and $3\vec{v}_2$. What about sending $\vec{v}_1, \vec{v}_2$ to $\vec{v}_2, \vec{v}_1$?

Exercise 2.76. Find the matrix of the reflection of $\mathbb{R}^3$ with respect to the plane $x+y+z = 0$.

Exercise 2.77. Find the matrix of the linear operator of $\mathbb{R}^3$ that circularly sends the basis vectors in Example 2.1.13 to each other

$$(1, -1, 0) \mapsto (1, 0, -1) \mapsto (1, 1, 1) \mapsto (1, -1, 0).$$

2.4 Dual

2.4.1 Dual Space

A function on a vector space V is a map $l \colon V \to \mathbb{R}$. If the map is a linear transformation

$$l(\vec{u} + \vec{v}) = l(\vec{u}) + l(\vec{v}), \quad l(c\vec{u}) = c\, l(\vec{u}),$$

then we call the map a *linear functional*. All the linear functionals on a vector space V form a vector space, called the *dual space*

$$V^* = \mathrm{Hom}(V, \mathbb{R}).$$

An ordered bases $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ of V gives an isomorphism

$$l \in V^* \longleftrightarrow l(\alpha) = (l(\vec{v}_1), l(\vec{v}_2), \dots, l(\vec{v}_n)) = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n. \tag{2.4.1}$$

The $1 \times n$ matrix on the right is the matrix $[l(\alpha)]_1 = [l]_{1\alpha}$ of l with respect to the basis α of V and the basis 1 of $\mathbb{R}$. The formula for l (i.e., the inverse of (2.4.1)) is then given by (2.1.1)

$$l(\vec{x}) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n, \quad [\vec{x}]_\alpha = (x_1, x_2, \dots, x_n). \tag{2.4.2}$$

The isomorphism tells us

$$\dim V^* = \dim V. \tag{2.4.3}$$

Under the isomorphism (2.4.1), the standard basis ϵ of $\mathbb{R}^n$ corresponds to a basis $\alpha^* = \{\vec{v}_1^*, \vec{v}_2^*, \ldots, \vec{v}_n^*\}$ of V^*, called the *dual basis*. The correspondence means $(\vec{v}_i^*(\vec{v}_1), \vec{v}_i^*(\vec{v}_2), \ldots, \vec{v}_i^*(\vec{v}_n)) = \vec{e}_i$, or

$$\vec{v}_i^*(\vec{v}_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

In other words, the linear functional $\vec{v}_i^*$ is the i-th α-coordinate

$$\vec{v}_i^*(x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n) = x_i.$$

For $\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n$, we can also write this as

$$[\vec{x}]_\alpha = (\vec{v}_1^*(\vec{x}), \vec{v}_2^*(\vec{x}), \ldots, \vec{v}_n^*(\vec{x})) = \alpha^*(\vec{x}),$$

or

$$\vec{x} = \vec{v}_1^*(\vec{x})\vec{v}_1 + \vec{v}_2^*(\vec{x})\vec{v}_2 + \cdots + \vec{v}_n^*(\vec{x})\vec{v}_n.$$

Proposition 2.4.1. *Suppose V is a finite dimensional vector space, and $\vec{x}, \vec{y} \in V$. Then $\vec{x} = \vec{y}$ if and only if $l(\vec{x}) = l(\vec{y})$ for all $l \in V^*$.*

For the sufficiency, we take l to be $\vec{v}_i^*$ and get $[\vec{x}]_\alpha = [\vec{y}]_\alpha$. Since the coordinate map is an isomorphism, this implies $\vec{x} = \vec{y}$.

We may also interpret (2.4.2) as

$$[l]_{\alpha^*} = (l(\vec{v}_1), l(\vec{v}_2), \ldots, l(\vec{v}_n)) = l(\alpha),$$

or

$$l = l(\vec{v}_1)\vec{v}_1^* + l(\vec{v}_2)\vec{v}_2^* + \cdots + l(\vec{v}_n)\vec{v}_n^*.$$

Example 2.4.1. The dual basis of the standard basis of Euclidean space is given by

$$\vec{e}_i^*(x_1, x_2, \ldots, x_n) = x_i.$$

The isomorphism (2.4.1) is given by

$$l(x_1, x_2, \ldots, x_n) = a_1x_2 + a_2x_2 + \cdots + a_nx_n \in (\mathbb{R}^n)^* \longleftrightarrow (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n.$$

Example 2.4.2. We want to calculate the dual basis of the basis $\alpha = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \{(1, -1, 0), (1, 0, -1), (1, 1, 1)\}$ in Example 1.3.17. The dual basis vector

$$\vec{v}_1^*(x_1, x_2, x_3) = a_1x_1 + a_2x_2 + a_3x_3$$

is characterised by

$$\begin{aligned}\vec{v}_1^*(\vec{v}_1) &= a_1 - a_2 = 1,\\ \vec{v}_1^*(\vec{v}_2) &= a_1 - a_3 = 0,\\ \vec{v}_1^*(\vec{v}_3) &= a_1 + a_2 + a_3 = 0.\end{aligned}$$

This is a system of linear equations with vectors in α as rows, and $\vec{e}_1$ as the right side. We get the similar systems for the other dual basis vectors $\vec{v}_2^*, \vec{v}_3^*$, with $\vec{e}_2, \vec{e}_3$ as the right sides. Similar to Example 1.3.17, we may solve the three systems at the same time by carrying out the row operations

$$\begin{pmatrix} \vec{v}_1^T & & & \\ \vec{v}_2^T & \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \vec{v}_3^T & & & \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

This gives the dual basis

$$\begin{aligned}\vec{v}_1^*(x_1, x_2, x_3) &= \tfrac{1}{3}(x_1 - 2x_2 + x_3),\\ \vec{v}_2^*(x_1, x_2, x_3) &= \tfrac{1}{3}(x_1 + x_2 - 2x_3),\\ \vec{v}_3^*(x_1, x_2, x_3) &= \tfrac{1}{3}(x_1 + x_2 + x_3).\end{aligned}$$

We note that the right half of the matrix obtained by the row operations is the transpose of the right half of the corresponding matrix in Example 1.3.17. This will be explained by the equality $(A^T)^{-1} = (A^{-1})^T$.

Example 2.4.3. For any number a, the evaluation $E_a(p(t)) = p(a)$ at a is a linear functional on P_n (and on all the other function spaces). We argue that the three evaluations E_0, E_1, E_2 form a basis of the dual space P_2^*.

The key idea already appeared in Example 1.3.12. We argued that $p_1(t) = t(t-1)$, $p_2(t) = t(t-2)$, $p_3 = (t-1)(t-2)$ form a basis of P_2 because their values at $0, 1, 2$ almost form the standard basis of $\mathbb{R}^3$

$$E_0(p_1, p_2, p_3) = (0, 0, 2), \quad E_1(p_1, p_2, p_3) = (0, -1, 0), \quad E_2(p_1, p_2, p_3) = (2, 0, 0).$$

This can be interpreted as the linear transformation $E = (E_0, E_1, E_2)\colon P_2 \to \mathbb{R}^3$ taking $\alpha = \{\frac{1}{2}p_3, -p_2, \frac{1}{2}p_1\}$ to the standard basis of $\mathbb{R}^3$. Since the standard basis is linearly independent, the set α is also linearly independent (Exercise 2.3). Since $\dim P_2 = 3$ is the number of vectors in α, by Theorem 1.3.14, α is a basis of P_2. Moreover, the linear transformation E shows that $\{E_0, E_1, E_2\}$ is the dual basis of α.

Exercise 2.78. If we permute the standard basis of the Euclidean space, how is the dual basis changed?

Exercise 2.79. How is the dual basis changed if we change a basis $\{\vec{v}_1, \dots, \vec{v}_i, \dots, \vec{v}_j, \dots, \vec{v}_n\}$ to the following bases? (see Exercise 1.59)

1. $\{\vec{v}_1, \dots, \vec{v}_j, \dots, \vec{v}_i, \dots, \vec{v}_n\}$.

2. $\{\vec{v}_1, \dots, c\vec{v}_i, \dots, \vec{v}_n\}$, $c \neq 0$.

3. $\{\vec{v}_1, \dots, \vec{v}_i + c\vec{v}_j, \dots, \vec{v}_j, \dots, \vec{v}_n\}$.

Exercise 2.80. Find the dual basis of a basis $\{(a,b),(c,d)\}$ of $\mathbb{R}^2$.

Exercise 2.81. Find the dual basis of the basis $\{(1,2,3),(4,5,6),(7,8,10)\}$ of $\mathbb{R}^3$.

Exercise 2.82. Find the dual basis of the basis $\{1-t, 1-t^2, 1+t+t^2\}$ of P_2.

Exercise 2.83. Find the dual basis of the basis $\{1, t, t^2\}$ of P_2. Moreover, express the dual basis in the form $l(p(t)) = \int_0^1 p(t)\lambda(t)dt$ for suitable $\lambda(t) \in P_2$.

Exercise 2.84. Find the basis of P_2, such that the dual basis is the evaluations at three distinct places t_1, t_2, t_3. Moreover, extend your result to P_n.

Exercise 2.85. Find the basis of P_2, such that the dual basis is the three derivatives at 0: $p(t) \mapsto p(0)$, $p(t) \mapsto p'(0)$, $p(t) \mapsto p''(0)$. Extend to the derivatives up to n-order for P_n, and at a place a other than 0.

2.4.2 Dual Linear Transformation

A linear transformation $L\colon V \to W$ induces the *dual linear transformation*

$$L^*(l) = l \circ L\colon W^* \to V^*, \quad (L^*(l))(\vec{v}) = l(L(\vec{v})).$$

This is the special case $U = \mathbb{R}$ of Exercise 2.13 (and therefore justifies that L^* is a linear transformation). The following shows that the dual linear transformation corresponds to the transpose of matrix.

Proposition 2.4.2. *Suppose α, β are bases of V, W, and α^*, β^* are the corresponding dual bases of V^*, W^*. Then*

$$[L^*]_{\alpha^*\beta^*} = [L]_{\beta\alpha}^T.$$

Proof. For $1 \le i \le m = \dim W$ and $1 \le j \le n = \dim V$, denote (notice ji for L^*)

$$\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}, \quad \beta = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m\}, \quad [L]_{\beta\alpha} = (a_{ij}), \quad [L^*]_{\alpha^*\beta^*} = (a^*_{ji}).$$

Then (recall the 3×2 matrix of L before Example 2.3.1)

$$\begin{aligned} L(\vec{v}_j) &= a_{1j}\vec{w}_1 + a_{2j}\vec{w}_2 + \cdots + a_{mj}\vec{w}_m, \\ L^*(\vec{w}^*_i) &= a^*_{1i}\vec{v}^*_1 + a^*_{2i}\vec{v}^*_2 + \cdots + a^*_{ni}\vec{v}^*_n. \end{aligned}$$

Applying the linear functional $\vec{w}_i^*$ to the first equality, and applying the second equality to $\vec{v}_j$, we get

$$\vec{w}_i^*(L(\vec{v}_j)) = a_{ij}, \quad (L^*(\vec{w}_i^*))(\vec{v}_j) = a_{ji}^*.$$

By the definition of L^*, we have $(L^*(\vec{w}_i^*))(\vec{v}_j) = \vec{w}_i^*(L(\vec{v}_j))$. Therefore $a_{ij} = a_{ji}^*$. This means $[L]_{\beta\alpha}^T = [L^*]_{\alpha^*\beta^*}$. □

By Exercise 2.13, the dual linear transformation has the following properties

$$I^* = I, \quad (L+K)^* = L^* + K^*, \quad (cL)^* = cL^*, \quad (L \circ K)^* = K^* \circ L^*.$$

Then by Proposition 2.4.2, these translate into properties of the transpose of matrix

$$I^T = I, \quad (A+B)^T = A^T + B^T, \quad (cA)^T = cA^T, \quad (AB)^T = B^T A^T.$$

Exercise 2.86. Directly verify that $L^*(l)\colon W^* \to V^*$ is a linear transformation: $L^*(l+k) = L^*(l) + L^*(k)$, $L^*(cl) = cL^*(l)$.

Exercise 2.87. Directly verify that the dual linear transformation has the claimed properties: $I^*(l) = l$, $(L+K)^*(l) = L^*(l) + K^*(l)$, $(cL)^*(l) = cL^*(l)$, $(L \circ K)^*(l) = K^*(L^*(l))$.

By the last statements of Propositions 2.2.3 and 2.2.4, we know that $L \circ K = I$ implies L is onto and K is one-to-one. Applying the dual switches the order into $K^* \circ L^* = I$. This suggests that K^* is onto and L^* is one-to-one, and almost gives the proof of the following result.

Proposition 2.4.3. *Suppose $L\colon V \to W$ is a linear transformation between finite dimensional vector spaces.*

1. *If L is onto, then L^* is one-to-one.*
2. *If L is one-to-one, then L^* is onto.*

Proof. Suppose L is onto, and $l, k \in V^*$ satisfy $L^*(l) = L^*(k)$. Then $l(L(\vec{v})) = L^*(l)(\vec{v}) = L^*(k)(\vec{v}) = k(L(\vec{v}))$ for all $\vec{v} \in V$. Since L is onto, every vector of W is of the form $L(\vec{v})$ for some $\vec{v} \in V$. Therefore the equality means $l(\vec{w}) = k(\vec{w})$ for all $\vec{w} \in W$. This is the definition of $l = k$, and proves the first statement.

The following proves the second statement

$$\begin{aligned} L \text{ is one-to-one} &\implies K \circ L = I && \text{(Proposition 2.2.4)} \\ &\implies L^* \circ K^* = I && ((L \circ K)^* = K^* \circ L^*) \\ &\implies L^* \text{ is onto.} && \text{(Proposition 2.2.3)} \end{aligned}$$

In fact, we may also use the similar idea to prove the first statement. □

Exercise 2.88. Let A be an $m \times n$ matrix. What is the relation between the existence and the uniqueness of solutions of the following two systems of linear equations?

1. $A\vec{x} = \vec{b}$: m equations in n variables.
2. $A^T\vec{y} = \vec{c}$: n equations in m variables.

Exercise 2.89. By Proposition 2.4.3, what can you say about the pivots of the row echelon forms of a matrix A and its transpose A^T? Note that row operation on A^T can also be regarded as column operation on A.

Exercise 2.90. Do we really need to assume finite dimension in Proposition 2.4.3?

2.4.3 Double Dual

Any vector $\vec{v} \in V$ induces a function on the dual space V^*

$$\vec{v}^{**}(l) = l(\vec{v}).$$

As the special case $W = \mathbb{R}$ of Exercise 2.11, the function $\vec{v}^{**}$ is a linear functional on V^*. Therefore $\vec{v}^{**}$ is a vector in the *double dual* $V^{**} = (V^*)^*$ of the dual space V^*. This gives the natural double dual map

$$\vec{v} \in V \mapsto \vec{v}^{**} \in V^{**}.$$

The following shows that the double dual map is a linear transformation

$$(a\vec{v} + b\vec{w})^{**}(l) = l(a\vec{v} + b\vec{w}) = al(\vec{v}) + bl(\vec{w}) = a\vec{v}^{**}(l) + b\vec{w}^{**}(l) = (a\vec{v}^{**} + b\vec{w}^{**})(l).$$

Proposition 2.4.1 can be interpreted as $\vec{v}^{**}(l) = \vec{w}^{**}(l)$ for all l implying $\vec{v} = \vec{w}$. Therefore the double dual map $V \mapsto V^{**}$ is one-to-one. By using (2.4.3) twice, we get $\dim V^{**} = \dim V$. Then by Theorem 2.2.6, we conclude that the double dual map is an isomorphism.

Proposition 2.4.4. *The double dual of a finite dimensional vector space is naturally isomorphic to itself.*

A linear transformation $L\colon V \to W$ induces a linear transformation $L^*\colon W^* \to V^*$, which further induces the *double dual linear transformation*

$$L^{**}\colon V^{**} \to W^{**}.$$

Let us compare with the natural isomorphism in Proposition 2.4.4

$$\begin{array}{ccc} V & \xrightarrow{L} & W \\ \cong\big\downarrow & & \big\downarrow\cong \\ V^{**} & \xrightarrow{L^{**}} & W^{**} \end{array}$$

The following shows that L^{**} can be identified with L under the natural isomorphism ($\vec{v} \in V$ and $l \in W^*$)

$$L^{**}(\vec{v}^{**})(l) = \vec{v}^{**}(L^*(l)) = (L^*(l))(\vec{v}) = l(L(\vec{v})) = (L(\vec{v}))^{**}(l).$$

Therefore like Proposition 2.4.4, the double dual L^{**} of a linear transformation L is naturally identified with the linear transformation L itself.

Let $A = [L]_{\beta\alpha}$ be the matrix of a linear transformation. Then by Proposition 2.4.2, we have $A^T = [L^*]_{\alpha^*\beta^*}$, and $(A^T)^T = [L^{**}]_{\beta^{**}\alpha^{**}}$ (Exercise 2.91 implicitly used). The natural identification of L^{**} and L is then an elaborately way of explaining $(A^T)^T = A$.

Combining the natural identification of L^{**} and L with Proposition 2.4.3 gives the following more complete result.

Theorem 2.4.5. *Suppose $L\colon V \to W$ is a linear transformation between finite dimensional vector spaces.*

1. *L is onto if and only if L^* is one-to-one.*

2. *L is one-to-one if and only if L^* is onto.*

Exercise 2.91. For a basis $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ of V, the notation $\alpha^{**} = \{\vec{v}_1^{**}, \vec{v}_2^{**}, \dots, \vec{v}_n^{**}\}$ has two possible meanings.

1. $(\alpha^*)^*$: First get dual basis α^* of V^*. Then get the dual of the dual basis $(\alpha^*)^*$ of $(V^*)^*$.

2. α^{**}: The image under the natural transformation $V \to V^{**}$.

Explain that the two meanings are the same.

2.4.4 Dual Pairing

A function $b\colon V \times W \to \mathbb{R}$ is *bilinear* if it is linear in V and also linear in W

$$\begin{aligned} b(x_1\vec{v}_1 + x_2\vec{v}_2, \vec{w}) &= x_1 b(\vec{v}_1, \vec{w}) + x_2 b(\vec{v}_2, \vec{w}), \\ b(\vec{v}, y_1\vec{w}_1 + y_2\vec{w}_2) &= y_1 b(\vec{v}, \vec{w}_1) + y_2 b(\vec{v}, \vec{w}_2). \end{aligned}$$

Let $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ and $\beta = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n\}$ be bases of V and W. Then the bilinear property imples

$$b(x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m,\ y_1\vec{w}_1 + y_2\vec{w}_2 + \cdots + y_n\vec{w}_n) = \sum_{\substack{1 \le i \le m \\ 1 \le j \le n}} x_i y_j b(\vec{v}_i, \vec{w}_j).$$

Denote the *matrix of bilinear function* with respect to the bases

$$[b]_{\alpha\beta} = (b_{ij}), \quad b_{ij} = b(\vec{v}_i, \vec{w}_j).$$

Then we have

$$b(\vec{x},\vec{y}) = \sum_{i,j} b_{ij}x_iy_j = [\vec{x}]_\alpha^T[b]_{\alpha\beta}[\vec{y}]_\beta.$$

We can define the linear combination of bilinear functions in the obvious way

$$(c_1\, b_1 + c_2\, b_2)(\vec{x},\vec{y}) = c_1\, b_1(\vec{x},\vec{y}) + c_1\, b_2(\vec{x},\vec{y}).$$

This makes all the bilinear functions on $V \times W$ into a vector space. It is also easy to see that

$$[c_1\, b_1 + c_2\, b_2]_{\alpha\beta} = c_1[b_1]_{\alpha\beta} + c_2[b_2]_{\alpha\beta}.$$

Therefore the vector space of bilinear functions is isomorphic to the vector space of $m \times n$ matrices, $m = \dim V$, $n = \dim W$.

For a bilinear function $b(\vec{x},\vec{y})$ on $V \times W$, the linearity in V gives a map

$$\vec{y} \in W \mapsto b(\cdot,\vec{y}) \in V^*.$$

Then the linearity in W implies that the map is a linear transformation. Conversely, a linear transformation $L\colon W \to V^*$, gives a bilinear function

$$b(\vec{x},\vec{y}) = L(\vec{y})(\vec{x}).$$

Here $L(\vec{y})$ is a linear functional on V and can be applied to $\vec{x}$. This gives an isomorphism between the vector space of all bilinear functions on $V \times W$ and the vector space $\mathrm{Hom}(W, V^*)$.

Due to the symmetry in V and W, we also have the isomorphism between the vector space of all bilinear functions on $V \times W$ and the vector space $\mathrm{Hom}(V, W^*)$. One direction is given by

$$\vec{x} \in V \mapsto K(\vec{x}) = b(\vec{x},\cdot) \in W^*.$$

The converse is given by

$$K \in \mathrm{Hom}(V, W^*) \mapsto b(\vec{x},\vec{y}) = K(\vec{x})(\vec{y}).$$

Example 2.4.4. The *evaluation pairing*

$$e(\vec{x}, l) = l(\vec{x})\colon V \times V^* \to \mathbb{R}$$

is a bilinear function. The corresponding linear transformation in $\mathrm{Hom}(V^*, V^*)$ is the identity. The corresponding linear transformation in $\mathrm{Hom}(V, (V^*)^*)$ is the double dual map $\vec{v} \mapsto \vec{v}^{**}$.

Let α and β be bases of V and V^*. Then $[e]_{\alpha\beta} = I$ if and only if β is the dual basis of α.

Exercise 2.92. Explain that two bilinear functions on $V \times W$ are equal if and only if they are equal on two spanning sets of V and W.

Exercise 2.93. How is the matrix of a bilinear function changed when the bases are changed?

Exercise 2.94. Prove that $L \in \mathrm{Hom}(W, V^*)$ and $K \in \mathrm{Hom}(V, W^*)$ give the same bilinear function on $V \times W$ if and only if $K = L^*$, subject to the isomorphism in Proposition 2.4.4.

Exercise 2.95. A bilinear function b on $V \times W$ corresponds to $L \in \mathrm{Hom}(W, V^*)$ and $K \in \mathrm{Hom}(V, W^*)$. Let α, β be bases of V, W, and α^*, β^* be the dual bases. How are the matrices $[b]_{\alpha\beta}$, $[L]_{\alpha^*\beta}$, $[K]_{\beta^*\alpha}$ related?

Exercise 2.96. For a bilinear function $b(\vec{v}, \vec{w})$ on $V \times W$, $b^t(\vec{w}, \vec{v}) = b(\vec{v}, \vec{w})$ is a bilinear function on $W \times V$. Let α, β be bases of V, W. How are the matrices $[b]_{\alpha\beta}$ and $[b^t]_{\beta\alpha}$ related? Moreover, the linear functions b and b^t correspond to four linear transformations. How are these linear transformations related?

Exercise 2.97. Let $b(\vec{x}, \vec{y})$ be a bilinear function on $V \times W$. Let $L\colon U \to V$ be a linear transformation.

1. Explain that $b(L(\vec{z}), \vec{y})$ is a bilinear function on $U \times W$.
2. Given bases for U, V, W, how are the matrices of $b(\vec{x}, \vec{y})$ and $b(L(\vec{z}), \vec{y})$ related?
3. The two bilinear functions correspond to four linear transformations. How are these linear transformations related?

Finally, please study the same problem for $b(\vec{x}, L(\vec{z}))$.

Definition 2.4.6. A bilinear function $b\colon V \times W \to \mathbb{R}$ is a *dual pairing* if both induced linear transformations $V \to W^*$ and $W \to V^*$ are isomorphisms.

By Exercise 2.94, the two linear transformations can be regarded as dual to each other. Therefore we only need one of them to be isomorphic. Moreover, by Proposition 2.4.5, the dual pairing is equivalent to both $V \to W^*$ and $W \to V^*$ being onto, and is also equivalent to both being one-to-one.

The evaluation pairing in Example 2.4.4 is a basic example of dual pairing. Using the same idea, a basis $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ of V and a basis $\beta = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n\}$ of W are *dual bases* with respect to the dual pairing b if

$$b(\vec{v}_i, \vec{w}_j) = \delta_{ij}, \text{ or } [b]_{\alpha\beta} = I.$$

This is equivalent to that the corresponding $V \to W^*$ maps the basis α of V to the dual basis β^* of W^*, and is also equivalent to that the corresponding $W \to V^*$ maps the basis β of W to the dual basis α^* of V^*. The dual bases with respect to the bilinear function also give

$$\begin{aligned} \vec{x} &= b(\vec{x}, \vec{w}_1)\vec{v}_1 + b(\vec{x}, \vec{w}_2)\vec{v}_2 + \cdots + b(\vec{x}, \vec{w}_n)\vec{v}_n && \text{for any } \vec{x} \in V, \\ \vec{y} &= b(\vec{v}_1, \vec{y})\vec{w}_1 + b(\vec{v}_2, \vec{y})\vec{w}_2 + \cdots + b(\vec{v}_n, \vec{y})\vec{w}_n && \text{for any } \vec{y} \in W. \end{aligned}$$

Exercise 2.98. Prove that a bilinear function is a dual pairing if and only if its matrix with respect to some bases is invertible.

Chapter 3

Subspace

As human civilisation became more sophisticated, they found it necessary to extend their number systems. The ancient Greeks found that the rational numbers $\mathbb{Q}$ was not sufficient for describing lengths in geometry, and the problem was solved later by the Arabs who extended the rational numbers to the real numbers $\mathbb{R}$. Then the Italians found it useful to take the square root of -1 in their search for the roots of cubic equations, and the idea led to the extension of real numbers to complex numbers $\mathbb{C}$.

In extending the number system, we still wish to preserve the key features of the old system. This means that the inclusions $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ are compatible with the four arithmetic operations. In other words, $2+3=5$ is an equality of rational numbers, and is also an equality of real (or complex) numbers. In this sense, we may call $\mathbb{Q}$ a sub-number system of $\mathbb{R}$ and $\mathbb{C}$.

3.1 Definition

3.1.1 Subspace

Definition 3.1.1. A subset H of a vector space V is a *subspace* if it satisfies

$$\vec{u}, \vec{v} \in H,\ a, b \in \mathbb{R} \implies a\vec{u} + b\vec{v} \in H.$$

Using the addition and scalar multiplication of V, the subset H is also a vector space. One should imagine that a subspace is a flat and infinite (with the only exception of the *trivial subspace* $\{\vec{0}\}$) subset passing through the origin.

The smallest subspace is the trivial subspace. The biggest subspace is the whole space V itself. Polynomials of degree ≤ 3 is a subspace of polynomials of degree ≤ 5. All polynomials is a subspace of all functions. Although $\mathbb{R}^3$ can be identified with a subspace of $\mathbb{R}^5$ (in many different ways), $\mathbb{R}^3$ is *not* a subspace of $\mathbb{R}^5$.

Proposition 3.1.2. *If H is a subspace of a finite dimensional vector space V, then* $\dim H \leq \dim V$. *Moreover,* $H = V$ *if and only if* $\dim H = \dim V$.

Proof. Let $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be a basis of H. Then α is a linearly independent set in V. By Proposition 1.3.13, we have $\dim H = k \leq \dim V$. Moreover, if $k = \dim H = \dim V$, then by Theorem 1.3.14, the linear independence of α implies that α also spans V. Since α spans H, we get $H = V$. □

Exercise 3.1. Determine whether the subset is a subspace of $\mathbb{R}^2$.

1. $\{(x, 0) \colon x \in \mathbb{R}\}$.
2. $\{(x, y) \colon x + y = 0\}$.
3. $\{(x, y) \colon 2x - 3y = 0\}$.
4. $\{(x, y) \colon 2x - 3y = 1\}$.
5. $\{(x, y) \colon xy = 0\}$.
6. $\{(x, y) \colon x, y \in \mathbb{Q}\}$.

Exercise 3.2. Determine whether the subset is a subspace of $\mathbb{R}^3$.

1. $\{(x, 0, z) \colon x, z \in \mathbb{R}\}$.
2. $\{(x, y, z) \colon x + y + z = 0\}$.
3. $\{(x, y, z) \colon x + y + z = 1\}$.
4. $\{(x, y, z) \colon x + y + z = 0, x + 2y + 3z = 0\}$.

Exercise 3.3. Determine whether the subset is a subspace of P_n.

1. even polynomials.
2. polynomials satisfying $f(1) = 0$.
3. polynomials satisfying $f(0) = 1$.
4. polynomials satisfying $f'(0) = f(1)$.

Exercise 3.4. Determine whether the subset is a subspace of C^∞.

1. odd functions.
2. functions satisfying $f'' + f = 0$.
3. functions satisfying $f'' + f = 1$.
4. functions satisfying $f'(0) = f(1)$.
5. functions satisfying $\lim_{t\to\infty} f(t) = 0$.
6. functions satisfying $\lim_{t\to\infty} f(t) = 1$.
7. functions such that $\lim_{t\to\infty} f(t)$ diverges.
8. functions satisfying $\int_0^1 f(t)dt = 0$.

Exercise 3.5. Determine whether the subset is a subspace of the space of all sequences (x_n).

1. x_n converges.
2. x_n diverges.
3. The series $\sum x_n$ converges.
4. The series $\sum x_n$ absolutely converges.

Exercise 3.6. If H is a subspace of V and V is a subspace of H, what can you conclude?

Exercise 3.7. Prove that H is a subspace if and only if $\vec{0} \in H$ and $a\vec{u} + \vec{v} \in H$ for any $a \in \mathbb{R}$ and $\vec{u}, \vec{v} \in H$.

Exercise 3.8. Suppose H is a subspace of V, and $\vec{v} \in V$. Prove that $\vec{v}+H = \{\vec{v}+\vec{h} \colon \vec{h} \in H\}$ is still a subspace if and only if $\vec{v} \in H$.

Exercise 3.9. Suppose H is a subspace of V. Prove that the *inclusion* $i(\vec{h}) = \vec{h} \colon H \to V$ is a one-to-one linear transformation.

For any linear transformation $L \colon V \to W$, the *restriction* $L|_H = L \circ i \colon H \to W$ is still a linear transformation.

Exercise 3.10. Suppose H and H' are subspaces of V.

1. Prove that the sum $H + H' = \{\vec{h} + \vec{h}' \colon \vec{h} \in H,\ \vec{h}' \in H'\}$ is a subspace.
2. Prove that the intersection $H \cap H'$ is a subspace.

When is the union $H \cup H'$ a subspace?

3.1.2 Span

The *span* of a set of vectors is the collection of all linear combinations

$$\begin{aligned}\mathrm{Span}\alpha &= \mathrm{Span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \\ &= \{x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n \colon x_i \in \mathbb{R}\} \\ &= \mathbb{R}\vec{v}_1 + \mathbb{R}\vec{v}_2 + \cdots + \mathbb{R}\vec{v}_n.\end{aligned}$$

By Proposition 1.2.1, a linear combination of linear combinations is still a linear combination. This means $\mathrm{Span}\alpha$ is a subspace.

The span of a single nonzero vector is the straight line in the direction of the vector. If the vector is zero, then the span is reduced to the origin.

The span of two non-parallel vectors is the 2-dimensional plane containing the origin and the two vectors (or containing the parallelogram formed by the two vectors, see Figure 1.2.1). If two vectors are parallel, then the span is reduced to a line in the direction of the two vectors.

Exercise 3.11. Prove that $\mathrm{Span}\alpha$ is the smallest subspace containing α.

Exercise 3.12. Prove that $\alpha \subset \beta$ implies $\mathrm{Span}\alpha \subset \mathrm{Span}\beta$.

Exercise 3.13. Prove that $\vec{v}$ is a linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ if and only if

$$\mathbb{R}\vec{v}_1 + \mathbb{R}\vec{v}_2 + \cdots + \mathbb{R}\vec{v}_n + \mathbb{R}\vec{v} = \mathbb{R}\vec{v}_1 + \mathbb{R}\vec{v}_2 + \cdots + \mathbb{R}\vec{v}_n.$$

Exercise 3.14. Prove that if one vector is a linear combination of the other vectors, then deleting the vector does not change the span.

Exercise 3.15. Prove that $\text{Span}\alpha \subset \text{Span}\beta$ if and only if vectors in α are linear combinations of vectors in β. In particular, $\text{Span}\alpha = \text{Span}\beta$ if and only if vectors in α are linear combinations of vectors in β, and vectors in β are linear combinations of vectors in α.

By the definition, the subspace $\text{Span}\alpha$ is already spanned by α. By Theorem 1.3.10, we get a basis of $\text{Span}\alpha$ by finding a maximal linearly independent set in α.

Example 3.1.1. To find a basis for the span of

$$\vec{v}_1 = (1,2,3), \quad \vec{v}_2 = (4,5,6), \quad \vec{v}_3 = (7,8,9), \quad \vec{v}_4 = (10,11,12),$$

we consider the row operations in Example 1.2.1

$$(\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4) = \begin{pmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 7 & 10 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If we restrict the row operations to the first two columns, then we find that $\vec{v}_1, \vec{v}_2$ are linearly independent. If we restrict the row operations to the first three columns, then we see that adding $\vec{v}_3$ gives linearly dependent set $\vec{v}_1, \vec{v}_2, \vec{v}_3$, because the third column is not pivot. By the same reason, $\vec{v}_1, \vec{v}_2, \vec{v}_4$ are also linearly dependent. Therefore $\vec{v}_1, \vec{v}_2$ form a maximal linearly independent set among $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$. By Theorem 1.3.10, $\vec{v}_1, \vec{v}_2$ form a basis of $\mathbb{R}\vec{v}_1 + \mathbb{R}\vec{v}_2 + \mathbb{R}\vec{v}_3 + \mathbb{R}\vec{v}_4$.

In general, given $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \in \mathbb{R}^m$, we carry out row operations on the matrix $(\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n)$. Then the pivot columns in $(\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n)$ form a basis of $\mathbb{R}\vec{v}_1 + \mathbb{R}\vec{v}_2 + \cdots + \mathbb{R}\vec{v}_n$. In particular, the dimension of the span is the number of pivots after row operations on $(\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n)$.

Exercise 3.16. Show that $\{\vec{v}_1, \vec{v}_3\}$ and $\{\vec{v}_1, \vec{v}_4\}$ are also bases in Example 3.1.1.

Exercise 3.17. Find a basis of the span.

1. $(1,2,3,4), (2,3,4,5), (3,4,5,6)$.
2. $(1,5,9), (2,6,10), (3,7,11), (4,8,12)$.
3. $(1,1,0,0), (1,0,1,0), (1,0,0,1), (0,1,1,0), (0,1,0,1), (0,0,1,1)$.

4. $\begin{pmatrix}1&1\\0&0\end{pmatrix},\begin{pmatrix}1&0\\1&0\end{pmatrix},\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}0&1\\0&1\end{pmatrix},\begin{pmatrix}0&0\\1&1\end{pmatrix}$.

5. $1-t, t-t^3, 1-t^3, t^3-t^5, t-t^5$.

By Exercise 1.32, the span has the following property.

Proposition 3.1.3. *The following operations do not change the span.*

1. $\{\vec{v}_1,\dots,\vec{v}_i,\dots,\vec{v}_j,\dots,\vec{v}_n\}\to\{\vec{v}_1,\dots,\vec{v}_j,\dots,\vec{v}_i,\dots,\vec{v}_n\}$.

2. $\{\vec{v}_1,\dots,\vec{v}_i,\dots,\vec{v}_n\}\to\{\vec{v}_1,\dots,c\vec{v}_i,\dots,\vec{v}_n\}$, $c\neq 0$.

3. $\{\vec{v}_1,\dots,\vec{v}_i,\dots,\vec{v}_j,\dots,\vec{v}_n\}\to\{\vec{v}_1,\dots,\vec{v}_i+c\vec{v}_j,\dots,\vec{v}_j,\dots,\vec{v}_n\}$.

For vectors in a Euclidean space, the operations in Proposition 3.1.3 are *column operations* on the matrix $(\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n)$. The proposition basically says that column operations do not change the span. We can take advantage of the proposition to find another way of calculating a basis of $\mathrm{Span}\alpha$.

Example 3.1.2. For the four vectors in Example 3.1.1, we carry out column operations on the matrix $(\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4)$

$$\begin{pmatrix}1&4&7&10\\2&5&8&11\\3&6&9&12\end{pmatrix}\xrightarrow[C_2-C_1]{\substack{C_4-C_3\\C_3-C_2}}\begin{pmatrix}1&3&3&3\\2&3&3&3\\3&3&3&3\end{pmatrix}\xrightarrow{\substack{C_4-C_3\\C_3-C_2\\\frac{1}{3}C_2}}\begin{pmatrix}1&1&0&0\\2&1&0&0\\3&1&0&0\end{pmatrix}\xrightarrow{\substack{C_1-C_2\\C_1\leftrightarrow C_2}}\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&2&0&0\end{pmatrix}.$$

The result is a *column echelon form*. By Proposition 3.1.3, we get

$$\begin{aligned}\mathbb{R}\vec{v}_1+\mathbb{R}\vec{v}_2+\mathbb{R}\vec{v}_3+\mathbb{R}\vec{v}_4&=\mathbb{R}(1,1,1)+\mathbb{R}(0,1,2)+\mathbb{R}(0,0,0)+\mathbb{R}(0,0,0)\\&=\mathbb{R}(1,1,1)+\mathbb{R}(0,1,2).\end{aligned}$$

The two pivot columns $(1,1,1),(0,1,2)$ of the column echelon form are always linearly independent, and therefore form a basis of $\mathbb{R}\vec{v}_1+\mathbb{R}\vec{v}_2+\mathbb{R}\vec{v}_3+\mathbb{R}\vec{v}_4$.

Example 3.1.3. By taking the transpose of the row operation in Example 3.1.1, we get the column operation

$$\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\\10&11&12\end{pmatrix}\to\begin{pmatrix}1&0&0\\4&-3&-6\\7&-6&-12\\10&-9&-18\end{pmatrix}\to\begin{pmatrix}1&0&0\\4&1&0\\7&2&0\\10&3&0\end{pmatrix}.$$

We find that $(1,4,7,10),(0,1,2,3)$ form a basis of $\mathbb{R}(1,4,7,10)+\mathbb{R}(2,5,8,11)+\mathbb{R}(3,6,9,12)$.

In general, given $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \in \mathbb{R}^m$, we use column operations to find a column echelon form of $(\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n)$. Then the pivot columns in the column echelon form (*not* the columns in the original matrix $(\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n)$) is a basis of $\mathbb{R}\vec{v}_1 + \mathbb{R}\vec{v}_2 + \cdots + \mathbb{R}\vec{v}_n$. In particular, the dimension of the span is the number of pivots after column operations on $(\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n)$. This is also the same as the number of pivots after row operations on the transpose $(\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n)^T$.

Comparing the dimension of the span obtained by two ways of calculating the span, we conclude that applying row operations to A and A^T give the same number of pivots.

Exercise 3.18. Explain how Proposition 3.1.3 follows from Exercise 1.32.

Exercise 3.19. List all the 2×3 column echelon forms.

Exercise 3.20. Explain that the nonzero columns in a column echelon form are linearly independent.

Exercise 3.21. Explain that, if the columns of an $n \times n$ matrix is a basis of $\mathbb{R}^n$, then the rows of the matrix is also a basis of $\mathbb{R}^n$.

Exercise 3.22. Use column operations to find a basis of the span in Exercise 3.17.

3.1.3 Calculation of Extension to Basis

The column echelon form can also be used to extend linearly independent vectors to a basis. By the proof of Theorem 1.3.11, this can be achieved by finding vectors not in the span.

Example 3.1.4. In Example 1.3.15, we use row operations to find that the vectors $\vec{v}_1 = (1,4,7,11), \vec{v}_2 = (2,5,8,12), \vec{v}_3 = (3,6,10,10)$ in $\mathbb{R}^4$ are linearly independent. We may also use column operations to get

$$(\vec{v}_1\ \vec{v}_2\ \vec{v}_3) = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \\ 11 & 12 & 10 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 4 & -3 & -6 \\ 7 & -6 & -11 \\ 11 & -10 & -23 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 4 & -3 & 0 \\ 7 & -6 & 1 \\ 11 & -10 & -3 \end{pmatrix}.$$

This shows that $(1,4,7,11), (0,-3,-6,-10), (0,0,1,-3)$ form a basis of $\mathbb{R}\vec{v}_1 + \mathbb{R}\vec{v}_2 + \mathbb{R}\vec{v}_3$. In particular, the span has dimension 3. By Theorem 1.3.14, we find that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are also linearly independent.

It is obvious that $(1,4,7,11), (0,-3,-6,-10), (0,0,1,-3), \vec{e}_4 = (0,0,0,1)$ form a basis of $\mathbb{R}^4$. In fact, the same column operations (applied to the first three columns only) gives

$$(\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{e}_4) = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 10 & 0 \\ 11 & 12 & 10 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & -3 & -6 & 0 \\ 7 & -6 & -11 & 0 \\ 11 & -10 & -23 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & -3 & 0 & 0 \\ 7 & -6 & 1 & 0 \\ 11 & -10 & -3 & 1 \end{pmatrix}.$$

Then $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{e}_4$ and $(1,4,7,11), (0,-3,-6,-10), (0,0,1,-3), (0,0,0,1)$ span the same vector space, which is $\mathbb{R}^4$. By Theorem 1.3.14, adding $\vec{e}_4$ gives a basis $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{e}_4$ of $\mathbb{R}^4$.

In Example 1.3.15, we extended $\vec{v}_1, \vec{v}_2, \vec{v}_3$ to a basis by a different method. The reader should compare the two methods.

Example 3.1.4 suggests the following practical way of extending a linearly independent set in $\mathbb{R}^n$ to a basis of $\mathbb{R}^n$. Suppose column operations on three linearly independent vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^5$ give

$$(\vec{v}_1\ \vec{v}_2\ \vec{v}_3) \xrightarrow{\text{col op}} \begin{pmatrix} \bullet & 0 & 0 \\ * & 0 & 0 \\ * & \bullet & 0 \\ * & * & \bullet \\ * & * & * \end{pmatrix}.$$

We may add $\vec{u}_1 = (0, \bullet, *, *, *)$ and $\vec{u}_2 = (0,0,0,0,\bullet)$ to create pivots in the second and the fifth columns

$$(\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{u}_1\ \vec{u}_2) \xrightarrow[\text{first 3 col}]{\text{col op on}} \begin{pmatrix} \bullet & 0 & 0 & 0 & 0 \\ * & 0 & 0 & \bullet & 0 \\ * & \bullet & 0 & * & 0 \\ * & * & \bullet & * & 0 \\ * & * & * & * & \bullet \end{pmatrix} \xrightarrow[\text{col}]{\text{exchange}} \begin{pmatrix} \bullet & 0 & 0 & 0 & 0 \\ * & \bullet & 0 & 0 & 0 \\ * & * & \bullet & 0 & 0 \\ * & * & * & \bullet & 0 \\ * & * & * & * & \bullet \end{pmatrix}.$$

Then $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{u}_1, \vec{u}_2$ form a basis of $\mathbb{R}^5$.

Exercise 3.23. Extend the basis you find in Exercise 3.22 to a basis of the whole vector space.

3.2 Range and Kernel

A linear transformation $L\colon V \to W$ induces two subspaces. The *range* (or *image*) is

$$\text{Ran}L = L(V) = \{L(\vec{v})\colon \text{all } \vec{v} \in V\} \subset W.$$

The following verifies this is a subspace

$$\begin{aligned} \vec{w}, \vec{w}\,' \in L(V) &\implies \vec{w} = L(\vec{v}), \vec{w}\,' = L(\vec{v}\,') \text{ for some } \vec{v}, \vec{v}\,' \in V \\ &\implies a\vec{w} + b\vec{w}\,' = aL(\vec{v}) + bL(\vec{v}\,') = L(a\vec{v} + b\vec{v}\,') \in L(V). \end{aligned}$$

The *kernel* is the *preimage* of the zero vector

$$\mathrm{Ker}L = L^{-1}(\vec{0}) = \{\vec{v}\colon \vec{v} \in V \text{ and } L(\vec{v}) = \vec{0}\} \subset V.$$

The following verifies this is a subspace

$$\begin{aligned}\vec{v}, \vec{v}\,' \in \mathrm{Ker}L \implies & L(\vec{v}) = \vec{0},\ L(\vec{v}\,') = \vec{0} \\ \implies & L(a\vec{v} + b\vec{v}\,') = aL(\vec{v}) + bL(\vec{v}\,') = a\vec{0} + b\vec{0} = \vec{0}.\end{aligned}$$

Exercise 3.24. Prove that $\mathrm{Ran}(L \circ K) \subset \mathrm{Ran}L$. Moreover, if K is onto, then $\mathrm{Ran}(L \circ K) = \mathrm{Ran}L$.

Exercise 3.25. Prove that $\mathrm{Ker}(L \circ K) \supset \mathrm{Ker}K$. Moreover, if L is one-to-one, then $\mathrm{Ker}(L \circ K) = \mathrm{Ker}K$.

Exercise 3.26. Suppose $L\colon V \to W$ is a linear transformation, and $H \subset V$ is a subspace. Prove that $L(H) = \{L(\vec{v})\colon \text{all } \vec{v} \in H\}$ is a subspace.

Exercise 3.27. Suppose $L\colon V \to W$ is a linear transformation, and $H \subset W$ is a subspace. Prove that $L^{-1}(H) = \{\vec{v} \in V\colon L(\vec{v}) \in H\}$ is a subspace.

Exercise 3.28. Prove that $L(H) \cap \mathrm{Ker}K = L(H \cap \mathrm{Ker}(K \circ L))$.

3.2.1 Range

The range is actually defined for any map $f\colon X \to Y$

$$\mathrm{Ran}f = f(X) = \{f(x)\colon \text{all } x \in X\} \subset Y.$$

For the map Instructor: Courses $\to$ Professors, the range is all the professors who teach some courses.

The map is onto if and only if $f(X) = Y$. This suggests that we may consider the same map with smaller target

$$\tilde{f}\colon X \to f(X), \quad \tilde{f}(x) = f(x).$$

For the Instructor map, this means $\widetilde{\text{Instructor}}$: Courses $\to$ Teaching Professors. The advantage of the modification is the following.

Proposition 3.2.1. *For any map $f\colon X \to Y$, the corresponding map $\tilde{f}\colon X \to f(X)$ has the following properties.*

1. *$\tilde{f}$ is onto.*
2. *$\tilde{f}$ is one-to-one if and only if f is one-to-one.*

Exercise 3.29. Prove Proposition 3.2.1.

Exercise 3.30. Prove that $\mathrm{Ran}(f \circ g) \subset \mathrm{Ran}f$. Moreover, if g is onto, then $\mathrm{Ran}(f \circ g) = \mathrm{Ran}f$.

Specialising to a linear transformation $L\colon V \to W$, if $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ spans V, then any vector of V is a linear combination of α, and L is given by (2.1.1) for all vectors in V. This implies that the range subspace is a span

$$\mathrm{Ran}L = L(V) = \mathrm{Span}L(\alpha).$$

If $L\colon \mathbb{R}^n \to \mathbb{R}^m$ is actually a linear transformation of Euclidean spaces, with $m \times n$ matrix A, then the interpretation above shows that the range of L is the span of the column vectors $L(\vec{e}_i)$ of A, called the *column space*

$$\mathrm{Ran}L = \mathrm{Col}A \subset \mathbb{R}^m.$$

Of course we can also consider the span of the rows of A and get the *row space*. The row and column spaces are clearly related by the transpose of the matrix

$$\mathrm{Row}A = \mathrm{Col}A^T \subset \mathbb{R}^n.$$

By Proposition 2.4.2 and the natural identification between $(\mathbb{R}^n)^*$ and $\mathbb{R}^n$ in Example 2.4.1, the row space corresponds to the range space of the dual linear transformation L^*.

Example 3.2.1. The derivative of a polynomial of degree n is a polynomial of degree $n-1$. Therefore we have linear transformation $D(f) = f'\colon P_n \to P_m$ for $m \ge n-1$. By integrating polynomial (i.e., anti-derivative), we get $\mathrm{Ran}D = P_{n-1} \subset P_m$. The linear transformation is onto if and only if $m = n-1$.

Example 3.2.2. Consider the linear transformation $L(f) = f'' + (1+t^2)f' + tf\colon P_3 \to P_4$ in Example 2.3.3. The row operations in the earlier example shows that the following form a basis of $\mathrm{Ran}L$

$$L(1) = t, \quad L(t) = 1 + 2t^2, \quad L(t^2) = 2 + 2t + 3t^3, \quad L(t^3) = 6t + 3t^2 + 4t^4.$$

Alternatively, we may carry out the column operations

$$\begin{pmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 6 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & -4 & 3 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & -4 & 0 \\ 0 & 0 & 3 & \frac{9}{4} \\ 0 & 0 & 0 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & -4 & 0 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 0 & 16 \end{pmatrix}.$$

This shows that $1+2t^2, t, -4t^2+3t^3, 9t^3+16t^4$ form a basis of $\mathrm{Ran}L$. It is also easy to see that adding t^4 gives a basis of P_4.

Example 3.2.3. Consider the linear transformation $L(A) = A + A^T : M_{n\times n} \to M_{n\times n}$. By $(A+A^T)^T = A^T+(A^T)^T = A^T+A = A+A^T$, any $X = A+A^T \in \text{Ran}L$ satisfies $X^T = X$. Such matrices are called *symmetric* because they are of the form (for $n = 3$, for example)

$$X = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}.$$

Conversely, if $X = X^T$, then for $A = \frac{1}{2}X$, we have

$$L(A) = A + A^T = \frac{1}{2}X + \frac{1}{2}X^T = \frac{1}{2}X + \frac{1}{2}X = X.$$

This shows that any symmetric matrix lies in $\text{Ran}L$. Therefore the range of L consists of exactly all the symmetric matrices. A basis of 3×3 symmetric matrices is given by

$$\begin{aligned} \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix} &= a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &\quad + d\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + e\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + f\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned}$$

Exercise 3.31. Explain that $A\vec{x} = \vec{b}$ has solution if and only if $\vec{b} \in \text{Col}A$.

Exercise 3.32. Suppose $L: V \to W$ is a linear transformation.

1. Prove the modified linear transformation (see Proposition 3.2.1) $\tilde{L}: V \to L(V)$ is an onto linear transformation.
2. Let $i: L(V) \to W$ be the inclusion linear transformation in Exercise 3.9. Show that $L = i \circ \tilde{L}$.

Exercise 3.33. Suppose $L: V \to W$ is a linear transformation. Prove that L is one-to-one if and only if $\tilde{L}: V \to L(V)$ is an isomorphism.

Exercise 3.34. Show that the range of the linear transformation $L(A) = A - A^T : M_{n\times n} \to M_{n\times n}$ consists of matrices X satisfying $X^T = -X$. These are called *skew-symmetric* matrices.

Exercise 3.35. Find the dimensions of the subspaces of symmetric and skew-symmetric matrices.

3.2.2 Rank

The span of a set of vectors, the range of a linear transformation, and the column space of a matrix are different presentations of the same concept. Their size, which is their dimension, is the *rank*

$$\text{rank}\alpha = \dim \text{Span}\alpha, \quad \text{rank}L = \dim \text{Ran}L, \quad \text{rank}A = \dim \text{Col}A.$$

By the calculation of basis of span subspace in Section 3.1.1, the rank of a matrix A is the number of pivots in the row echelon form, and is also the number of pivots in the column echelon form. Since column operation on A is the same as row operation on A^T, we have

$$\text{rank}A^T = \text{rank}A.$$

By Proposition 2.4.2, this means

$$\text{rank}L^* = \text{rank}L.$$

Since the number of pivots of an $m \times n$ matrix is always no more than m and n, we have

$$\text{rank}A_{m\times n} \leq \min\{m, n\}.$$

If the equality holds, then the matrix has *full rank*. This means either $\text{rank}A_{m\times n} = m$ (all rows are pivot), or $\text{rank}A_{m\times n} = n$ (all columns are pivot). By Propositions 1.3.4, 1.3.6, 2.2.9, we have the following.

Proposition 3.2.2. *Let A be an $m\times n$ matrix. Then* $\text{rank}A \leq \min\{m, n\}$. *Moreover,*

1. *$A\vec{x} = \vec{b}$ has solution for all $\vec{b}$ if and only if* $\text{rank}A = m$.

2. *The solution of $A\vec{x} = \vec{b}$ is unique if and only if* $\text{rank}A = n$.

3. *A is invertible if and only if* $\text{rank}A = m = n$.

The following is the same result for set of vectors. The first statement actually follows from Proposition 3.1.2, and the second statement may be obtained by applying Theorem 1.3.14 to $V = \text{Span}\alpha$.

Proposition 3.2.3. *Let α be a set of n vectors in V. Then* $\text{rank}\alpha \leq \min\{n, \dim V\}$, *Moreover,*

1. *α spans V if and only if* $\text{rank}\alpha = \dim V$.

2. *α is linearly independent if* $\text{rank}\alpha = n$.

3. *α is a basis of V if and only if* $\text{rank}\alpha = \dim V = n$.

The following is the same result for linear transformation. The first statement actually follows from Proposition 3.1.2, and the second statement may be obtained by applying Theorem 2.2.6 to $\tilde{L}\colon V \to \text{Ran}L$ in Exercises 3.32 and 3.33.

Proposition 3.2.4. *Let* $L: V \to W$ *be a linear transformation. Then* $\text{rank}L \le \min\{\dim V, \dim W\}$*. Moreover,*

1. *L is onto if and only if* $\text{rank}L = \dim W$*.*
2. *L is one-to-one if and only if* $\text{rank}L = \dim V$*.*
3. *L is invertible if and only if* $\text{rank}L = \dim V = \dim W$*.*

Exercise 3.36. What is the rank of the vector set in Exercise 3.17?

Exercise 3.37. If a set of vectors is enlarged, how is the rank changed?

Exercise 3.38. Suppose the columns of an $m \times n$ matrix A are linearly independent. Prove that there are n rows of A that are also linearly independent. Similarly, if the rows of A are linearly independent, then there are n columns of A that are also linearly independent.

Exercise 3.39. Directly prove Proposition 3.2.3.

Exercise 3.40. Consider a composition $U \xrightarrow{K} V \xrightarrow{L} W$. Let $L|_{K(U)}: K(U) \to W$ be the restriction of L to the subspace $K(U)$.

1. Prove $\text{Ran}(L \circ K) = \text{Ran}L|_{K(U)} \subset \text{Ran}L$.
2. Prove $\text{rank}(L \circ K) \le \min\{\text{rank}L, \text{rank}K\}$.
3. Prove $\text{rank}(L \circ K) = \text{rank}L$ when K is onto.
4. Prove $\text{rank}(L \circ K) = \text{rank}K$ when L is one-to-one.
5. Translate the second part into a fact about matrices.

3.2.3 Kernel

By the third part of Proposition 2.2.4, we know that a linear transformation L is one-to-one if and only if $\text{Ker}L = \{\vec{0}\}$. In contrast, the linear transformation is onto if and only if $\text{Ran}L$ is the whole target space.

For a linear transformation $L(\vec{x}) = A\vec{x}: \mathbb{R}^n \to \mathbb{R}^m$ between Euclidean spaces, the kernel is all the solutions of the homogeneous system of linear equations, called the *null space*

$$\text{Nul}A = \text{Ker}L = \{\vec{v}: \text{all } \vec{v} \in \mathbb{R}^n \text{ satisfying } A\vec{v} = \vec{0}\}.$$

The uniqueness of solution means $\text{Nul}A = \{\vec{0}\}$. In contrast, the existence of solution (for all right side) means $\text{Col}A = \mathbb{R}^m$.

Example 3.2.4. To find the null space of

$$A = \begin{pmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{pmatrix},$$

we use the row operation in Example 1.2.1 and continue to get the reduced row echelon form

$$A \longrightarrow \begin{pmatrix} 1 & 4 & 7 & 10 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

As pointed out in Section 1.2.4, this is the same as the general solution of the homogeneous system $A\vec{x} = \vec{0}$. We express the solution in vector form

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_3 + 2x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} = x_3\vec{v}_1 + x_4\vec{v}_2.$$

Since the free variables x_3, x_4 can be arbitrary, this shows that $\text{Nul}A = \mathbb{R}\vec{v}_1 + \mathbb{R}\vec{v}_2$. The following further shows that $\vec{v}_1, \vec{v}_2$ are linearly independent

$$\begin{aligned} x_3\vec{v}_1 + x_4\vec{v}_2 = \vec{0} &\iff \text{the solution } \vec{x} = \vec{0} \\ &\implies \text{coordinates of the solution } x_3 = x_4 = 0. \end{aligned}$$

In general, the solution of a homogeneous system $A\vec{x} = \vec{0}$ is $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$, where $c_1, c_2, \dots, c_k$ are all the free variables. This implies $\text{Nul}A = \mathbb{R}\vec{v}_1 + \mathbb{R}\vec{v}_2 + \cdots + \mathbb{R}\vec{v}_k$. Moreover, the argument in the example shows that the vectors are always linearly independent. Therefore $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ form a basis of $\text{Nul}A$.

The *nullity* $\dim \text{Nul}A$ of a matrix is the dimension of the null space. The calculation of the null space shows that the nullity is the number of non-pivot columns (corresponding to free variables) of A. Since $\text{rank}A = \dim \text{Col}A$ is the number of pivot columns (corresponding to non-free variables) of A, we conclude that

$$\dim \text{Nul}A + \text{rank}A = \text{number of columns of } A.$$

Translated into linear transformations, this means the following.

Theorem 3.2.5. *If $L\colon V \to W$ is a linear transformation, then* $\dim \text{Ker}L + \text{rank}L = \dim V$.

Example 3.2.5. For the linear transformation $L(f) = f''\colon P_5 \to P_3$, we have

$$\text{Ker}L = \{f \in P_5 \colon f'' = 0\} = \{a + bt \colon a, b \in \mathbb{R}\}.$$

The monomials $1, t$ form a basis of the kernel, and $\dim \text{Ker}L = 2$. Since $L(P_5) = P_3$ is onto, we have $\text{rank}L = \dim L(P_5) = \dim P_3 = 4$. Then

$$\dim \text{Ker}L + \text{rank}L = 2 + 4 = 6 = \dim P_5.$$

This confirms Theorem 3.2.5.

Example 3.2.6. Consider the linear transformation

$$L(f) = (1+t^2)f'' + (1+t)f' - f \colon P_3 \to P_3$$

in Examples 2.1.16 and 2.3.3. The row operations in Example 2.3.3 show that $\text{rank}L = 3$. Therefore $\dim \text{Ker}L = \dim P_3 - \text{rank}L = 1$. Since we already know $L(1+t) = 0$, we conclude that $\text{Ker}L = \mathbb{R}(1+t)$.

Example 3.2.7. By Exercise 2.20, the left multiplication by an $m \times n$ matrix A is a linear transformation

$$L_A(X) = AX \colon M_{n\times k} \to M_{m\times k}.$$

Let $X = (\vec{x}_1 \ \vec{x}_2 \ \cdots \ \vec{x}_k)$. Then $AX = (A\vec{x}_1 \ A\vec{x}_2 \ \cdots \ A\vec{x}_k)$. Therefore $Y \in \text{Ran}L_A$ if and only if all columns of Y lie in $\text{Col}A$, and $X \in \text{Ker}L_A$ if and only if all columns of X lie in $\text{Nul}A$.

Let $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$ be a basis of $\text{Col}A$ ($r = \text{rank}A$). Then for the special case $k = 2$, the following is a basis of $\text{Ran}L_A$

$$(\vec{v}_1 \ \vec{0}),\ (\vec{0} \ \vec{v}_1),\ (\vec{v}_2 \ \vec{0}),\ (\vec{0} \ \vec{v}_2),\ \dots,\ (\vec{v}_r \ \vec{0}),\ (\vec{0} \ \vec{v}_r).$$

Therefore $\dim \text{Ran}L_A = 2r = 2\dim \text{Col}A$. Similarly, we have $\dim \text{Ker}L_A = 2r = 2\dim \text{Nul}A$.

In general, we have

$$\begin{aligned}\dim \text{Ran}L_A &= k \dim \text{Col}A = k\,\text{rank}A,\\ \dim \text{Ker}L_A &= k \dim \text{Nul}A = k(n - \text{rank}A).\end{aligned}$$

Example 3.2.8. In Example 3.2.3, we saw the range of linear transformation $L(A) = A + A^T \colon M_{n\times n} \to M_{n\times n}$ is exactly all symmetric matrices. The kernel of the linear transformation consists those A satisfying $A + A^T = O$, or $A^T = -A$. These are the skew-symmetric matrices. See Exercises 3.34 and 3.35. We have

$$\begin{aligned}\text{rank}L &= \dim\{\text{symmetric matrices}\}\\ &= 1 + 2 + \cdots + n = \frac{1}{2}n(n+1),\end{aligned}$$

and

$$\begin{aligned}\dim\{\text{skew-symmetric matrices}\} &= \dim M_{n\times n} - \text{rank}L\\ &= n^2 - \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1).\end{aligned}$$

Exercise 3.41. Use Theorem 3.2.5 to show that $L\colon V \to W$ is one-to-one if and only if $\text{rank} L = \dim V$ (the second statement of Proposition 3.2.4).

Exercise 3.42. An $m \times n$ matrix A induces four subspaces $\text{Col}A, \text{Row}A, \text{Nul}A, \text{Nul}A^T$.

1. Which are subspaces of $\mathbb{R}^m$? Which are subspaces of $\mathbb{R}^n$?
2. Which basis can you calculate by using row operations on A?
3. Which basis can you calculate by using column operations on A?
4. What are the dimensions in terms of the rank of A?

Exercise 3.43. Find a basis of the kernel of the linear transformation given by the matrix (some appeared in Exercise 1.28).

1. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 5 & 6 & 7 & 8 \\ 7 & 8 & 9 & 10 \end{pmatrix}$.
2. $\begin{pmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \\ 3 & 5 & 7 & 9 \\ 4 & 6 & 8 & 10 \end{pmatrix}$.
3. $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 1 \\ 4 & 1 & 2 \end{pmatrix}$.
4. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \end{pmatrix}$.
5. $\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 1 & 2 \\ 3 & 1 & 2 & 3 \end{pmatrix}$.
6. $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$.

Exercise 3.44. Find the rank and nullity.

1. $L(x_1, x_2, x_3, x_4) = (x_1 + x_2, x_2 + x_3, x_3 + x_4, x_4 + x_1)$.
2. $L(x_1, x_2, x_3, x_4) = (x_1 + x_2 + x_3, x_2 + x_3 + x_4, x_3 + x_4 + x_1, x_4 + x_1 + x_2)$.
3. $L(x_1, x_2, x_3, x_4) = (x_1 - x_2, x_1 - x_3, x_1 - x_4, x_2 - x_3, x_2 - x_4, x_3 - x_4)$.
4. $L(x_1, x_2, \dots, x_n) = (x_i - x_j)_{1 \le i < j \le n}$.
5. $L(x_1, x_2, \dots, x_n) = (x_i + x_j)_{1 \le i < j \le n}$.

Exercise 3.45. Find the rank and nullity.

1. $L(f) = f'' + (1 + t^2)f' + tf \colon P_3 \to P_4$.
2. $L(f) = f'' + (1 + t^2)f' + tf \colon P_3 \to P_5$.
3. $L(f) = f'' + (1 + t^2)f' + tf \colon P_n \to P_{n+1}$.

Exercise 3.46. Find the dimensions of the range and the kernel of right multiplication by an $m \times n$ matrix A

$$R_A(X) = XA \colon M_{k \times m} \to M_{k \times n}.$$

3.2.4 General Solution of Linear Equation

Let $L\colon V \to W$ be a linear transformation, and $\vec{b} \in W$. Then $L(\vec{x}) = \vec{b}$ has solution (i.e., there is $\vec{x}_0 \in V$ satisfying the equation) if and only if $\vec{b} \in \text{Ran}L$. Next, we try to find the collection of all solutions (i.e., the preimage of $\vec{b}$)

$$L^{-1}(\vec{b}) = \{\vec{x} \in V : L(\vec{x}) = \vec{b}\}.$$

By

$$L(\vec{x}) = \vec{b} \iff L(\vec{x} - \vec{x}_0) = \vec{b} - \vec{b} = \vec{0} \iff \vec{x} - \vec{x}_0 \in \text{Ker}L,$$

we conclude that

$$L^{-1}(\vec{b}) = \vec{x}_0 + \text{Ker}L = \{\vec{x}_0 + \vec{v} : L(\vec{v}) = \vec{0}\}.$$

In terms of system of linear equations, this means that the solution of $A\vec{x} = \vec{b}$ (in case $\vec{b} \in \text{Col}A$) is of the form $\vec{x}_0 + \vec{v}$, where $\vec{x}_0$ is one special solution, and $\vec{v} \in \text{Nul}A$ is any solution of the homogeneous system $A\vec{x} = \vec{0}$.

Geometrically, the kernel is a subspace. The collection of all solutions is obtained by shifting the subspace by one special solution $\vec{x}_0$.

We note the range (and $\vec{x}_0$) manifests the existence, while the kernel manifests the variations in the solution. In particular, the uniqueness of solution means no variation, or the triviality of the kernel.

Example 3.2.9. The system of linear equations

$$\begin{aligned} x_1 + 4x_2 + 7x_3 + 10x_4 &= 1, \\ 2x_1 + 5x_2 + 8x_3 + 11x_4 &= 1, \\ 3x_1 + 6x_2 + 9x_3 + 12x_4 &= 1, \end{aligned}$$

has an obvious solution $\vec{x}_0 = \frac{1}{3}(-1, 1, 0, 0)$. In Example 3.2.4, we found that $\vec{v}_1 = (1, -2, 1, 0)$ and $\vec{v}_2 = (2, -3, 0, 1)$ form a basis of the kernel. Therefore the general solution is (c_1, c_2 are arbitrary)

$$\vec{x} = \vec{x}_0 + c_1\vec{v}_1 + c_2\vec{v}_2 = \frac{1}{3}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_1\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} + c_1 + 2c_2 \\ \frac{1}{3} - 2c_1 - 3c_2 \\ c_1 \\ c_2 \end{pmatrix}.$$

Geometrically, the solutions is the plane $\mathbb{R}(1, -2, 1, 0) + \mathbb{R}(1, -2, 1, 0)$ shifted by $\vec{x}_0 = \frac{1}{3}(-1, 1, 0, 0)$.

We may also use another obvious solution $\frac{1}{3}(0, -1, 1, 0)$ and get an alternative formula for the general solution

$$\vec{x} = \frac{1}{3}\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + c_1\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 + 2c_2 \\ -\frac{1}{3} - 2c_1 - 3c_2 \\ \frac{1}{3} + c_1 \\ c_2 \end{pmatrix}.$$

Example 3.2.10. For the linear transformation in Examples 2.1.16 and 2.3.3

$$L(f) = (1+t^2)f'' + (1+t)f' - f\colon P_3 \to P_3,$$

we know from Example 3.2.6 that the kernel is $\mathbb{R}(1+t)$. Moreover, by $L(1), L(t^2), L(t^3)$ in Example 2.3.3, we also know $L(2-t^2+t^3) = 4t+8t^3$. Therefore $\frac{1}{4}(2-t^2+t^3)$ is a special solution of the differential equation in Example 2.1.16. Then we get the general solution in Example 2.3.3

$$f = \tfrac{1}{4}(2-t^2+t^3) + c(1+t).$$

Example 3.2.11. The general solution of the linear differential equation $f' = \sin t$ is

$$f = -\cos t + C.$$

Here $f_0 = -\cos t$ is one special solution, and the arbitrary constants C form the kernel of the derivative linear transform

$$\mathrm{Ker}(f \mapsto f') = \{C\colon C \in \mathbb{R}\} = \mathbb{R}1.$$

Similarly, $f'' = \sin t$ has a special solution $f_0 = -\sin t$. Moreover, we have (see Example 3.2.5)

$$\mathrm{Ker}(f \mapsto f'') = \{C + Dt\colon C, D \in \mathbb{R}\} = \mathbb{R}1 + \mathbb{R}t.$$

Therefore the general solution of $f'' = \sin t$ is $f = -\sin t + C + Dt$.

Example 3.2.12. The left side of a *linear differential equation* of order n (see Example 2.1.15)

$$L(f) = \frac{d^n f}{dt^n} + a_1(t)\frac{d^{n-1}f}{dt^{n-1}} + a_2(t)\frac{d^{n-2}f}{dt^{n-2}} + \cdots + a_{n-1}(t)\frac{df}{dt} + a_n(t)f = b(t)$$

is a linear transformation $C^\infty \to C^\infty$. A fundamental theorem in the theory of differential equations says that $\dim \mathrm{Ker}L = n$. Therefore to solve the differential equation, we need to find one special function f_0 satisfying $L(f_0) = b(t)$ and n linearly independent functions $f_1, f_2, \ldots, f_n$ satisfying $L(f_i) = 0$. Then the general solution is

$$f = f_0 + c_1 f_1 + c_2 f_2 + \cdots + c_n f_n.$$

Take the second order differential equation $f''+f = e^t$ as an example. We try the special solution $f_0 = ae^t$ and find $(ae^t)''+ae^t = 2ae^t = e^t$ implying $a = \frac{1}{2}$. Therefore $f_0 = \frac{1}{2}e^t$ is a solution. Moreover, we know that both $f_1 = \cos t$ and $f_2 = \sin t$ satisfy the homogeneous equation $f'' + f = 0$. By Example 1.3.11, f_1 and f_2 are linearly independent. This leads to the general solution of the differential equation

$$f = \tfrac{1}{2}e^t + c_1\cos t + c_2 \sin t, \quad c_1, c_2 \in \mathbb{R}.$$

Exercise 3.47. Find a basis of the kernel of $L(f) = f''+3f'-4f$ by trying functions of the form $f(t) = e^{at}$. Then find general solution.

1. $f'' + 3f' - 4f = 1 + t$.
2. $f'' + 3f' - 4f = e^t$.
3. $f'' + 3f' - 4f = \cos t + 2\sin t$.
4. $f'' + 3f' - 4f = 1 + t + e^t$.

3.3 Sum of Subspace

The sum of subspaces generalises the span. The direct sum of subspaces generalises the linear independence. Subspace, sum, and direct sum are the deeper linear algebra concepts that replace vector, span, and linear independence.

3.3.1 Sum and Direct Sum

Definition 3.3.1. The *sum* of subspaces $H_1, H_2, \dots, H_k \subset V$ is

$$H_1 + H_2 + \cdots + H_k = \{\vec{h}_1 + \vec{h}_2 + \cdots + \vec{h}_k \colon \vec{h}_i \in H_i\}.$$

The sum is *direct* if

$$\vec{h}_1 + \vec{h}_2 + \cdots + \vec{h}_k = \vec{h}'_1 + \vec{h}'_2 + \cdots + \vec{h}'_k,\ \vec{h}_i, \vec{h}'_i \in H_i \implies \vec{h}_i = \vec{h}'_i.$$

We indicate the direct sum by writing $H_1 \oplus H_2 \oplus \cdots \oplus H_k$.

If $H_i = \mathbb{R}\vec{v}_i$, then the sum is the span of $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$. If $\vec{v}_i \neq \vec{0}$, then the direct sum means that α is linearly independent.

Example 3.3.1. Let P_n^{even} be all the even polynomials in P_n and P_n^{odd} be all the odd polynomials in P_n. Then $P_n = P_n^{\text{even}} \oplus P_n^{\text{odd}}$.

Example 3.3.2. For $k = 1$, we have the sum H_1 of single vector space H_1. The single sum is always direct.

For $H_1 + H_2$ to be direct, we require

$$\vec{h}_1 + \vec{h}_2 = \vec{h}'_1 + \vec{h}'_2 \implies \vec{h}_1 = \vec{h}'_1,\ \vec{h}_2 = \vec{h}'_2.$$

Let $\vec{v}_1 = \vec{h}_1 - \vec{h}'_1 \in H_1$ and $\vec{v}_2 = \vec{h}_2 - \vec{h}'_2 \in H_2$. Then the condition becomes

$$\vec{v}_1 + \vec{v}_2 = \vec{0} \implies \vec{v}_1 = \vec{v}_2 = \vec{0}.$$

The equality on the left means $\vec{v}_1 = -\vec{v}_2$, which is a vector in $H_1 \cap H_2$. Therefore the condition above means exactly $H_1 \cap H_2 = \{\vec{0}\}$. This is the criterion for the sum $H_1 + H_2$ to be direct.

Example 3.3.3 (Abstract Direct Sum)**.** Let V and W be vector spaces. Construct a vector space $V \oplus W$ to be the set $V \times W = \{(\vec{v}, \vec{w}) \colon \vec{v} \in V,\ \vec{w} \in W\}$, together with addition and scalar multiplication

$$(\vec{v}_1, \vec{w}_1) + (\vec{v}_2, \vec{w}_2) = (\vec{v}_1 + \vec{v}_2, \vec{w}_1 + \vec{w}_2), \quad a(\vec{v}, \vec{w}) = (a\vec{v}, a\vec{w}).$$

It is easy to verify that $V\oplus W$ is a vector space. Moreover, V and W are isomorphic to the following subspaces of $V\oplus W$

$$V\cong V\oplus\vec{0}=\{(\vec{v},\vec{0}_W)\colon \vec{v}\in V\},\quad W\cong\vec{0}\oplus W=\{(\vec{0}_V,\vec{w})\colon \vec{w}\in W\}.$$

Since any vector in $V\oplus W$ can be uniquely expressed as $(\vec{v},\vec{w})=(\vec{v},\vec{0}_W)+(\vec{0}_V,\vec{w})$, we find that $V\oplus W$ is the direct sum of two subspaces

$$V\oplus W=\{(\vec{v},\vec{0}_W)\colon \vec{v}\in V\}\oplus\{(\vec{0}_V,\vec{w})\colon \vec{w}\in W\}.$$

This is the reason why $V\oplus W$ is called *abstract direct sum*.

The construction allows us to write $\mathbb{R}^m\oplus\mathbb{R}^n=\mathbb{R}^{m+n}$. Strictly speaking, $\mathbb{R}^m$ and $\mathbb{R}^n$ are not subspaces of $\mathbb{R}^{m+n}$. The equality means

1. $\mathbb{R}^m$ is isomorphic to the subspace of vectors in $\mathbb{R}^{m+n}$ with the last n coordinates vanishing.

2. $\mathbb{R}^n$ is isomorphic to the subspace of vectors in $\mathbb{R}^{m+n}$ with the first m coordinates vanishing.

3. $\mathbb{R}^{m+n}$ is the direct sum of these two subspaces.

Exercise 3.48. Prove that

$$H_1+H_2=H_2+H_1,\quad (H_1+H_2)+H_3=H_1+H_2+H_3=H_1+(H_2+H_3).$$

Exercise 3.49. Prove that an intersection $H_1\cap H_2\cap\cdots\cap H_k$ of subspaces is a subspace.

Exercise 3.50. Prove that $H_1+H_2+\cdots+H_k$ is the smallest subspace containing all H_i.

Exercise 3.51. Prove that $\text{Span}\alpha+\text{Span}\beta=\text{Span}(\alpha\cup\beta)$.

Exercise 3.52. Prove that $\text{Span}(\alpha\cap\beta)\subset(\text{Span}\alpha)\cap(\text{Span}\beta)$. Show that the two sides may or may not equal.

Exercise 3.53. We may regard a subspace H as a sum of single subspace. Explain that the single sum is always direct.

Exercise 3.54. If a sum is direct, prove that the sum of a selection of subspaces is also direct.

Exercise 3.55. Prove that a sum $H_1+H_2+\cdots+H_k$ is direct if and only if the sum expression for $\vec{0}$ is unique

$$\vec{h}_1+\vec{h}_2+\cdots+\vec{h}_k=\vec{0},\ \vec{h}_i\in H_i \implies \vec{h}_1=\vec{h}_2=\cdots=\vec{h}_k=\vec{0}.$$

The generalises Proposition 1.3.7.

Exercise 3.56. Prove that a sum $H_1 + H_2 + \cdots + H_k$ is direct if and only if

$$\vec{h}_1 + \vec{h}_2 + \cdots + \vec{h}_{k-1} \in H_k,\ \vec{h}_i \in H_i \implies \vec{h}_1 = \vec{h}_2 = \cdots = \vec{h}_{k-1} = \vec{0}.$$

Explain that this generalises Proposition 1.3.8.

Exercise 3.57. Show that $M_{n\times n}$ is the direct sum of the subspace of symmetric matrices (see Example 3.2.3) and the subspace of skew-symmetric matrices (see Exercise 3.34). In other words, any square matrix is the sum of a unique symmetric matrix and a unique skew-symmetric matrix.

Exercise 3.58. Let H, H' be subspaces of V. We have the sum $H + H' \subset V$ and we also have the abstract direct sum $H \oplus H'$ from Example 3.3.3. Prove that $L(\vec{h}, \vec{h}') = \vec{h} + \vec{h}' \colon H \oplus H' \to H + H'$ is an onto linear transformation, and $\text{Ker}L$ is isomorphic to $H \cap H'$.

Exercise 3.59. Use Exercise 3.58 to prove $\dim(H + H') = \dim H + \dim H' - \dim(H \cap H')$.

In general, a sum of sums of subspaces is a sum. For example, we have

$$(H_1 + H_2) + H_3 + (H_4 + H_5) = H_1 + H_2 + H_3 + H_4 + H_5.$$

We will show that the sum on the right is direct if and only if $H_1 + H_2$, H_3, $H_4 + H_5$, $(H_1 + H_2) + H_3 + (H_4 + H_5)$ are direct sums.

To state the general result, we consider n sums

$$H_i = +_j H_{ij} = +_{j=1}^{k_i} H_{ij} = H_{i1} + H_{i2} + \cdots + H_{ik_i}, \quad i = 1, 2, \dots, n.$$

Then we consider the sum

$$H = +_i(+_j H_{ij}) = +_{i=1}^{n} H_i = H_1 + H_2 + \cdots + H_n,$$

and consider the further splitting of the sum

$$\begin{aligned} H = +_{ij} H_{ij} = H_{11} + H_{12} + \cdots + H_{1k_1} + H_{21} + H_{22} + \cdots + H_{2k_2} \\ + \cdots\cdots + H_{n1} + H_{n2} + \cdots + H_{nk_n}. \end{aligned}$$

Proposition 3.3.2. *The sum $+_{ij} H_{ij}$ is direct if and only if the sum $+_i(+_j H_{ij})$ is direct and the sum $+_j H_{ij}$ is direct for each i.*

Proof. Suppose $H = +_{ij} H_{ij}$ is a direct sum. To prove that $H = +_i H_i = +_i(+_j H_{ij})$ is direct, we consider a vector $\vec{h} = \sum_i \vec{h}_i = \vec{h}_1 + \vec{h}_2 + \cdots + \vec{h}_n$, $\vec{h}_i \in H_i$, in the sum. By $H_i = +_j H_{ij}$, we have $\vec{h}_i = \sum_j \vec{h}_{ij} = \vec{h}_{i1} + \vec{h}_{i2} + \cdots + \vec{h}_{ik_i}$, $\vec{h}_{ij} \in H_{ij}$. Then $\vec{h} = \sum_{ij} \vec{h}_{ij}$. Since $H = +_{ij} H_{ij}$ is direct, we find that $\vec{h}_{ij}$ are uniquely determined by $\vec{h}$. This implies that $\vec{h}_i$ are also uniquely determined by $\vec{h}$. This proves that $H = +_i H_i$ is direct.

Next we further prove that $H_i = +_j H_{ij}$ is also direct. We consider a vector $\vec{h} = \sum_j \vec{h}_{ij} = \vec{h}_{i1} + \vec{h}_{i2} + \cdots + \vec{h}_{ik_i}$, $\vec{h}_{ij} \in H_{ij}$, in the sum. By taking $\vec{h}_{i'j} = \vec{0}$ for all $i' \neq i$, we form the double sum $\vec{h} = \sum_{ij} \vec{h}_{ij}$. Since $H = +_{ij} H_{ij}$ is direct, all $\vec{h}_{pj}$, $p = i$ or $p = i'$, are uniquely determined by $\vec{h}$. In particular, $\vec{h}_{i1}, \vec{h}_{i2}, \dots, \vec{h}_{ik_i}$ are uniquely determined by $\vec{h}$. This proves that $H_i = +_j H_{ij}$ is direct.

Conversely, suppose the sum $+_i(+_j H_{ij})$ is direct and the sum $+_j H_{ij}$ is direct for each i. To prove that $H = +_{ij} H_{ij}$ is direct, we consider a vector $\vec{h} = \sum_{ij} \vec{h}_{ij}$, $\vec{h}_{ij} \in H_{ij}$, in the sum. We have $\vec{h} = \sum_i \vec{h}_i$ for $\vec{h}_i = \sum_j \vec{h}_{ij} \in +_j H_{ij}$. Since $H = +_i(+_j H_{ij})$ is direct, we find that $\vec{h}_i$ are uniquely determined by $\vec{h}$. Since $H_i = +_j H_{ij}$ is direct, we also find that $\vec{h}_{ij}$ are uniquely determined by $\vec{h}_i$. Therefore all $\vec{h}_{ij}$ are uniquely determined by $\vec{h}$. This proves that $+_{ij} H_{ij}$ is direct. $\square$

For the special case $H_{ij} = \mathbb{R}\vec{v}_{ij}$ is spanned by a single non-zero vector, the equality $+_i(+_j H_{ij}) = +_{ij} H_{ij}$ means that, if $\alpha_i = \{\vec{v}_{i1}, \vec{v}_{i2}, \dots, \vec{v}_{ik_i}\}$ spans H_i for each i, then the union $\alpha = \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_k = \{\vec{v}_{ij} \colon \text{all } i, j\}$ spans the sum $H = H_1 + H_2 + \cdots + H_k$. If α_i is a basis of H_i, then by Propositions 1.3.13, 3.3.2 and Theorem 1.3.14, we get the following.

Proposition 3.3.3. *If $H_1, H_2, \dots, H_k$ are finite dimensional subspaces, then*

$$\dim(H_1 + H_2 + \cdots + H_k) \leq \dim H_1 + \dim H_2 + \cdots + \dim H_k.$$

Moreover, the sum is direct if and only if the equality holds.

Exercise 3.60. Suppose α_i are linearly independent. Prove that the sum $\text{Span}\alpha_1 + \text{Span}\alpha_2 + \cdots + \text{Span}\alpha_k$ is direct if and only if $\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_k$ is linearly independent.

Exercise 3.61. Suppose α_i is a basis of H_i. Prove that $\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_k$ is a basis of $H_1 + H_2 + \cdots + H_k$ if and only if the sum $H_1 + H_2 + \cdots + H_k$ is direct.

3.3.2 Projection

A direct sum $V = H \oplus H'$ induces a map by picking the first term in the unique expression

$$P(\vec{v}) = \vec{h}, \quad \text{if } \vec{v} = \vec{h} + \vec{h}',\ \vec{h} \in H,\ \vec{h}' \in H'.$$

The direct sum implies that P is a well defined linear transformation satisfying $P^2 = P$. See Exercise 3.62.

Definition 3.3.4. A linear operator $P \colon V \to V$ is a *projection* if $P^2 = P$.

Conversely, given any projection P, we have $\vec{v} = P(\vec{v}) + (I - P)(\vec{v})$. By $P(I - P)(\vec{v}) = (P - P^2)(\vec{v}) = \vec{0}$, we have $P(\vec{v}) \in \text{Ran}P$ and $(I - P)(\vec{v}) \in \text{Ker}P$. Therefore $V = \text{Ran}P + \text{Ker}P$. On the other hand, if $\vec{v} = \vec{h} + \vec{h}'$ with $\vec{h} \in \text{Ran}P$ and $\vec{h}' \in \text{Ker}P$, then $\vec{h} = P(\vec{w})$ for some $\vec{w} \in V$, and

$$\begin{aligned}
P(\vec{v}) &= P(\vec{h}) + P(\vec{h}') \\
&= P(\vec{h}) && (\vec{h}' \in \text{Ker}P) \\
&= P^2(\vec{w}) && (\vec{h} = P(\vec{w})) \\
&= P(\vec{w}) && (P^2 = P) \\
&= \vec{h}.
\end{aligned}$$

This shows that $\vec{h}$ is unique. Therefore the decomposition $\vec{v} = \vec{h} + \vec{h}'$ is also unique, and we have a direct sum

$$V = \text{Ran}P \oplus \text{Ker}P.$$

We conclude that there is a one-to-one correspondence between projections of V and decompositions of V into direct sums of two subspaces.

Example 3.3.4. With respect to the direct sum $P_n = P_n^{\text{even}} \oplus P_n^{\text{odd}}$ in Example 3.3.1, the projection to even polynomials is given by $f(t) \mapsto \frac{1}{2}(f(t) + f(-t))$.

Example 3.3.5. In C^∞, we consider subspaces

$$H = \mathbb{R}1 = \{\text{constant functions}\}, \quad H' = \{f \colon f(0) = 0\}.$$

Since any function $f(t) = f(0) + (f(t) - f(0))$ with $f(0) \in H$ and $f(t) - f(0) \in H'$, we have $C^\infty = H + H'$. Since $H \cap H'$ consists of zero function only, we have direct sum $C^\infty = H \oplus H'$. Moreover, the projection to H is $f(t) \mapsto f(0)$ and the projection to H' is $f(t) \mapsto f(t) - f(0)$.

Exercise 3.62. Given a direct sum $V = H \oplus H'$, verify that $P(\vec{v}) = \vec{h}$ is well defined, is a linear operator, and satisfies $P^2 = P$.

Exercise 3.63. Directly verify that the matrix A of the orthogonal projection in Example 2.1.13 satisfies $A^2 = A$.

Exercise 3.64. For the orthogonal projection P in Example 2.1.13, explain that $I - P$ is also a projection. What is the subspace corresponding to $I - P$?

Exercise 3.65. If P is a projection, prove that $Q = I - P$ is also a projection satisfying

$$P + Q = I, \quad PQ = QP = O.$$

Moreover, P and Q induce the same direct sum decomposition

$$V = H \oplus H', \quad H = \mathrm{Ran}P = \mathrm{Ker}Q, \quad H' = \mathrm{Ran}Q = \mathrm{Ker}P,$$

with the only exception that the order of H and H' are switched.

Exercise 3.66. Find the formula for the projections given by the direct sum in Exercise 3.57

$$M_{n\times n} = \{\text{symmetric matrix}\} \oplus \{\text{skew-symmetric matrix}\}.$$

Exercise 3.67. Consider subspaces in C^∞

$$H = \mathbb{R}1 \oplus \mathbb{R}t = \{\text{polynomials of degree} \le 1\}, \quad H' = \{f \colon f(0) = f(1) = 0\}.$$

Show that we have a direct sum $C^\infty = H \oplus H'$, and find the corresponding projections. Moreover, generalise to higher order polynomials and evaluation at more points (and not necessarily including 0 or 1).

Suppose we have a direct sum

$$V = H_1 \oplus H_2 \oplus \cdots \oplus H_k.$$

Then the direct sum $V = H_i \oplus (\oplus_{j\ne i} H_j)$ corresponds to a projection $P_i \colon V \to H_i \subset V$. It is easy to see that the unique decomposition

$$\vec{v} = \vec{h}_1 + \vec{h}_2 + \cdots + \vec{h}_k, \quad \vec{h}_i \in H_i,$$

gives (and is given by) $\vec{h}_i = P_i(\vec{v})$. The interpretation immediately implies

$$P_1 + P_2 + \cdots + P_k = I, \quad P_iP_j = O \text{ for } i \ne j.$$

Conversely, given linear operators P_i satisfying the above, we get $P_i = P_iI = P_iP_1 + P_iP_2 + \cdots + P_iP_k = P_i^2$. Therefore P_i is a projection. Moreover, if

$$\vec{v} = \vec{h}_1 + \vec{h}_2 + \cdots + \vec{h}_k, \quad \vec{h}_i = P_i(\vec{w}_i) \in H_i = \mathrm{Ran}P_i,$$

then

$$P_i(\vec{v}) = P_iP_1(\vec{w}_1) + P_iP_2(\vec{w}_2) + \cdots + P_iP_k(\vec{w}_k) = P_i^2(\vec{w}_i) = P_i(\vec{w}_i) = \vec{h}_i.$$

This implies the uniqueness of $\vec{h}_i$, and we get a direct sum.

Proposition 3.3.5. *There is a one-to-one correspondence between direct sum decompositions of a vector space V and collections of projections P_i satisfying*

$$P_1 + P_2 + \cdots + P_k = I, \quad P_iP_j = O \textit{ for } i \ne j.$$

Example 3.3.6. The basis in Examples 1.3.17 and 2.1.13

$$\vec{v}_1 = (1,-1,0), \quad \vec{v}_2 = (1,0,-1), \quad \vec{v}_3 = (1,1,1),$$

gives a direct sum $\mathbb{R}^3 = \mathbb{R}\vec{v}_1 \oplus \mathbb{R}\vec{v}_2 \oplus \mathbb{R}\vec{v}_3$. Then we have three projections P_1, P_2, P_3 corresponding to three 1-dimensional subspaces

$$\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 \implies P_1(\vec{b}) = x_1\vec{v}_1,\ P_2(\vec{b}) = x_2\vec{v}_2,\ P_3(\vec{b}) = x_3\vec{v}_3.$$

The calculation of the projections becomes the calculation of the formulae of x_1, x_2, x_3 in terms of $\vec{b}$. In other words, we need to solve the equation $A\vec{x} = \vec{b}$ for the matrix A in Example 2.2.18. The solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1}\vec{b} = \frac{1}{3}\begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

Therefore

$$\begin{aligned} P_1(\vec{b}) &= \tfrac{1}{3}(b_1 - 2b_2 + b_3)(1,-1,0), \\ P_2(\vec{b}) &= \tfrac{1}{3}(b_1 + b_2 - 2b_3)(1,0,-1), \\ P_3(\vec{b}) &= \tfrac{1}{3}(b_1 + b_2 + b_3)(1,1,1). \end{aligned}$$

The matrices of the projections are

$$[P_1] = \tfrac{1}{3}\begin{pmatrix} 1 & -2 & 1 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad [P_2] = \tfrac{1}{3}\begin{pmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ -1 & -1 & 2 \end{pmatrix}, \quad [P_3] = \tfrac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

We also note that the projection to the subspace $\mathbb{R}\vec{v}_1 \oplus \mathbb{R}\vec{v}_2$ in the direct sum $\mathbb{R}^3 = \mathbb{R}\vec{v}_1 \oplus \mathbb{R}\vec{v}_2 \oplus \mathbb{R}\vec{v}_3$ is $P_1 + P_2$ (see Exercise 3.68)

$$\vec{b} = (x_1\vec{v}_1 + x_2\vec{v}_2) + x_3\vec{v}_3 \mapsto x_1\vec{v}_1 + x_2\vec{v}_2 = P_1(\vec{b}) + P_2(\vec{b}).$$

The matrix of the projection is $[P_1] + [P_2]$, which can also be calculated as follows

$$[P_1] + [P_2] = I - [P_3] = \tfrac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Exercise 3.68. Suppose the direct sum $V = H_1 \oplus H_2 \oplus H_3$ corresponds to the projections P_1, P_2, P_3. Prove that the projection to $H_1 \oplus H_2$ in the direct sum $V = (H_1 \oplus H_2) \oplus H_3$ is $P_1 + P_2$.

Exercise 3.69. For the direct sum given by the basis in Example 2.2.17

$$\mathbb{R}^3 = \mathbb{R}(1,2,3) \oplus \mathbb{R}(4,5,6) \oplus \mathbb{R}(7,8,10),$$

find the projections to the three lines. Then find the projection to the plane $\mathbb{R}(1,2,3) \oplus \mathbb{R}(4,5,6)$.

Exercise 3.70. For the direct sum given by modifying the basis in Example 2.1.13

$$\mathbb{R}^3 = \mathbb{R}(1,-1,0) \oplus \mathbb{R}(1,0,-1) \oplus \mathbb{R}(1,0,0),$$

find the projection to the plane $\mathbb{R}(1,-1,0) \oplus \mathbb{R}(1,0,-1)$.

Exercise 3.71. In Example 3.1.4, a set of linearly independent vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is extended to a basis by adding $\vec{e}_4 = (0,0,0,1)$. Find the projections related to the direct sum $\mathbb{R}^4 = \mathrm{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \oplus \mathbb{R}\vec{e}_4$.

Exercise 3.72. We know the rank of

$$A = \begin{pmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{pmatrix}$$

is 2, and $A\vec{e}_1, A\vec{e}_2$ are linearly independent. Show that we have a direct sum $\mathbb{R}^4 = \mathrm{Nul}A \oplus \mathrm{Span}\{\vec{e}_1, \vec{e}_2\}$. Moreover, find the projections corresponding to the direct sum.

Exercise 3.73. The basis $t(t-1), t(t-2), (t-1)(t-2)$ in Example 1.3.12 gives a direct sum $P_2 = \mathrm{Span}\{t(t-1), t(t-2)\} \oplus \mathbb{R}(t-1)(t-2)$. Find the corresponding projections.

3.3.3 Blocks of Linear Transformation

Suppose $L\colon V \to W$ is a linear transformation. Suppose $\alpha = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and $\beta = \{\vec{w}_1, \vec{w}_2\}$ are bases of V and W. Then the matrix of

$$L\colon V = \mathbb{R}\vec{v}_1 \oplus \mathbb{R}\vec{v}_2 \oplus \mathbb{R}\vec{v}_3 \to W = \mathbb{R}\vec{w}_1 \oplus \mathbb{R}\vec{w}_2$$

means

$$[L]_{\beta\alpha} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}, \quad \begin{aligned} L(\vec{v}_1) &= a_{11}\vec{w}_1 + a_{21}\vec{w}_2, \\ L(\vec{v}_2) &= a_{12}\vec{w}_1 + a_{22}\vec{w}_2, \\ L(\vec{v}_3) &= a_{13}\vec{w}_1 + a_{23}\vec{w}_2. \end{aligned}$$

Let $P_1\colon W \to W$ be the projection to $\mathbb{R}\vec{w}_1$. Then $P_1L|_{\mathbb{R}\vec{v}_2}(x\vec{v}_2) = a_{12}x\vec{w}_1$. This means that a_{12} is the 1×1 matrix of the linear transformation $L_{12} = P_1L|_{\mathbb{R}\vec{v}_2}\colon \mathbb{R}\vec{v}_2 \to \mathbb{R}\vec{w}_1$. Similarly, a_{ij} is the 1×1 matrix of the linear transformation $L_{ij}\colon \mathbb{R}\vec{v}_j \to \mathbb{R}\vec{w}_i$ obtained by restricting L to the direct sum components, and we may write

$$L = \begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \end{pmatrix}.$$

In general, a linear transformation $L\colon V_1 \oplus V_2 \oplus \cdots \oplus V_n \to W_1 \oplus W_2 \oplus \cdots \oplus W_m$ has the *block matrix*

$$L = \begin{pmatrix} L_{11} & L_{12} & \dots & L_{1n} \\ L_{21} & L_{22} & \dots & L_{2n} \\ \vdots & \vdots & & \vdots \\ L_{m1} & L_{m2} & \dots & L_{mn} \end{pmatrix}, \quad L_{ij} = P_iL|_{V_j}\colon V_j \to W_i \subset W.$$

Similar to the vertical expression of vectors in Euclidean spaces, we should write

$$\begin{pmatrix} \vec{w}_1 \\ \vec{w}_2 \\ \vdots \\ \vec{w}_m \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} & \dots & L_{1n} \\ L_{21} & L_{22} & \dots & L_{2n} \\ \vdots & \vdots & & \vdots \\ L_{m1} & L_{m2} & \dots & L_{mn} \end{pmatrix} \begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \end{pmatrix},$$

which means

$$\vec{w}_i = L_{i1}(\vec{v}_1) + L_{i2}(\vec{v}_2) + \cdots + L_{in}(\vec{v}_n)$$

, and

$$L(\vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_n) = \vec{w}_1 + \vec{w}_2 + \cdots + \vec{w}_m.$$

Example 3.3.7. The linear transformation $L\colon \mathbb{R}^4 \to \mathbb{R}^3$ given by the matrix

$$[L] = \begin{pmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{pmatrix}$$

can be decomposed as

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} : \mathbb{R}^1 \oplus \mathbb{R}^3 \to \mathbb{R}^2 \oplus \mathbb{R}^1,$$

with matrices

$$[L_{11}] = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad [L_{12}] = \begin{pmatrix} 4 & 7 & 10 \\ 5 & 8 & 11 \end{pmatrix}, \quad [L_{21}] = (3), \quad [L_{22}] = (6 \quad 9 \quad 12).$$

Example 3.3.8. Suppose a projection $P\colon V \to V$ corresponds to a direct sum $V = H \oplus H'$. Then

$$P = \begin{pmatrix} I & O \\ O & O \end{pmatrix}$$

with respect to the direct sum.

Example 3.3.9. The *direct sum* $L = L_1 \oplus L_2 \oplus \cdots \oplus L_n$ of linear transformations $L_i\colon V_i \to W_i$ is the *diagonal block matrix*

$$L = \begin{pmatrix} L_1 & O & \dots & O \\ O & L_2 & \dots & O \\ \vdots & \vdots & & \vdots \\ O & O & \dots & L_n \end{pmatrix} : V_1 \oplus V_2 \oplus \cdots \oplus V_n \to W_1 \oplus W_2 \oplus \cdots \oplus W_n,$$

given by

$$L(\vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_n) = L_1(\vec{v}_1) + L_2(\vec{v}_2) + \cdots + L_n(\vec{v}_n), \quad \vec{v}_i \in V_i, \quad L_i(\vec{v}_i) \in W_i.$$

For example, the identity on $V_1 \oplus \cdots \oplus V_n$ is the direct sum of identities

$$I = \begin{pmatrix} I_{V_1} & O & \dots & O \\ O & I_{V_2} & \dots & O \\ \vdots & \vdots & & \vdots \\ O & O & \dots & I_{V_n} \end{pmatrix}.$$

Exercise 3.74. What is the block matrix for switching the factors in a direct sum $V \oplus W \to W \oplus V$?

The operations of block matrices are similar to the usual matrices, as long as the direct sums match. For example, for linear transformations $V_1 \oplus V_2 \oplus V_3 \xrightarrow{L,K} W_1 \oplus W_2$, we have

$$\begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \end{pmatrix} + \begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \end{pmatrix} = \begin{pmatrix} L_{11}+K_{11} & L_{12}+K_{12} & L_{13}+K_{13} \\ L_{21}+K_{21} & L_{22}+K_{22} & L_{23}+K_{23} \end{pmatrix},$$

$$a \begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \end{pmatrix} = \begin{pmatrix} aL_{11} & aL_{12} & aL_{13} \\ aL_{21} & aL_{22} & aL_{23} \end{pmatrix}.$$

For the composition of linear tranformations $U_1 \oplus U_2 \xrightarrow{K} V_1 \oplus V_2 \xrightarrow{L} W_1 \oplus W_2 \oplus W_3$, we have

$$\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \\ L_{31} & L_{32} \end{pmatrix} \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} = \begin{pmatrix} L_{11}K_{11}+L_{12}K_{21} & L_{11}K_{12}+L_{12}K_{22} \\ L_{21}K_{11}+L_{22}K_{21} & L_{21}K_{12}+L_{22}K_{22} \\ L_{31}K_{11}+L_{32}K_{21} & L_{31}K_{12}+L_{32}K_{22} \end{pmatrix}.$$

Example 3.3.10. We have

$$\begin{pmatrix} I & L \\ O & I \end{pmatrix} \begin{pmatrix} I & K \\ O & I \end{pmatrix} = \begin{pmatrix} I & L+K \\ O & I \end{pmatrix}.$$

In particular, this implies

$$\begin{pmatrix} I & L \\ O & I \end{pmatrix}^{-1} = \begin{pmatrix} I & -L \\ O & I \end{pmatrix}.$$

Exercise 3.75. For invertible L and K, find the inverses of $\begin{pmatrix} L & M \\ O & K \end{pmatrix}$, $\begin{pmatrix} L & O \\ M & K \end{pmatrix}$, $\begin{pmatrix} O & L \\ K & M \end{pmatrix}$.

Exercise 3.76. Find the n-th power of

$$J = \begin{pmatrix} \lambda I & L & O & \dots & O \\ O & \lambda I & L & \dots & O \\ O & O & \lambda I & \dots & O \\ \vdots & \vdots & \vdots & & \vdots \\ O & O & O & \dots & L \\ O & O & O & \dots & \lambda I \end{pmatrix}.$$

Exercise 3.77. Use block matrix to explain that

$$\mathrm{Hom}(V_1 \oplus V_2 \oplus \cdots \oplus V_n, W) = \mathrm{Hom}(V_1, W) \oplus \mathrm{Hom}(V_2, W) \oplus \cdots \oplus \mathrm{Hom}(V_n, W),$$
$$\mathrm{Hom}(V, W_1 \oplus W_2 \oplus \cdots \oplus W_n) = \mathrm{Hom}(V, W_1) \oplus \mathrm{Hom}(V, W_2) \oplus \cdots \oplus \mathrm{Hom}(V, W_n).$$

3.4 Quotient Space

For a subspace H of V, the *quotient space* V/H measures the "difference" between H and V. This is achieved by ignoring the differences in H. When the difference between H and V is realised as a subspace of V, the subspace is the *direct summand* of H in V.

3.4.1 Construction of the Quotient

Given a subspace $H \subset V$, we regard two vectors in V to be *equivalent* if they differ by a vector in H

$$\vec{v} \sim \vec{w} \iff \vec{v} - \vec{w} \in H.$$

The equivalence relation has the following three properties.

1. Reflexivity: $\vec{v} \sim \vec{v}$.
2. Symmetry: $\vec{v} \sim \vec{w} \implies \vec{w} \sim \vec{v}$.
3. Transitivity: $\vec{u} \sim \vec{v}$ and $\vec{v} \sim \vec{w} \implies \vec{u} \sim \vec{w}$.

The reflexivity follows from $\vec{v} - \vec{v} = \vec{0} \in H$. The symmetry follows from $\vec{w} - \vec{v} = -(\vec{v} - \vec{w}) \in H$. The transitivity follows from $\vec{u} - \vec{w} = (\vec{u} - \vec{v}) + (\vec{v} - \vec{w}) \in H$.

The *equivalence class* of a vector $\vec{v}$ is all the vectors equivalent to $\vec{v}$

$$\bar{v} = \{\vec{u} \colon \vec{u} - \vec{v} \in H\} = \{\vec{v} + \vec{h} \colon \vec{h} \in H\} = \vec{v} + H.$$

For example, Section 3.2.4 shows that all the solutions of a linear equation $L(\vec{x}) = \vec{b}$ form an equivalence class with respect to $H = \mathrm{Ker}L$.

Definition 3.4.1. Let H be a subspace of V. The *quotient space* is the collection of all equivalence classes

$$\bar{V} = V/H = \{\vec{v} + H \colon \vec{v} \in V\},$$

together with the addition and scalar multiplication

$$(\vec{u} + H) + (\vec{v} + H) = (\vec{u} + \vec{v}) + H, \quad a(\vec{u} + H) = a\vec{u} + H.$$

Moreover, we have the *quotient map*

$$\pi(\vec{v}) = \bar{v} = \vec{v} + H \colon V \to \bar{V}.$$

The operations in the quotient space can also be written as $\bar{u}+\bar{v}=\overline{u+v}$ and $a\bar{u}=\overline{au}$.

The following shows that the addition is well defined

$$\begin{aligned} \vec{u}\sim\vec{u}',\ \vec{v}\sim\vec{v}' &\iff \vec{u}-\vec{u}',\vec{v}-\vec{v}'\in H \\ &\implies (\vec{u}+\vec{v})-(\vec{u}'+\vec{v}')=(\vec{u}-\vec{u}')+(\vec{v}-\vec{v}')\in H \\ &\iff \vec{u}+\vec{v}\sim\vec{u}'+\vec{v}'. \end{aligned}$$

We can similarly show that the scalar multiplication is also well defined.

We still need to verify the axioms for vector spaces. The commutativity and associativity of the addition in $\bar{V}$ follow from the commutativity and associativity of the addition in V. The zero vector $\bar{0}=\vec{0}+H=H$. The negative vector $-(\vec{v}+H)=-\vec{v}+H$. The axioms for the scalar multiplications can be similarly verified.

Proposition 3.4.2. *The quotient map $\pi\colon V\to\bar{V}$ is an onto linear transformation with kernel H.*

Proof. The onto property of π is tautology. The linearity of π follows from the definition of the vector space operations in $\bar{V}$. In fact, we can say that the operations in $\bar{V}$ are defined for the purpose of making π a linear transformation. Moreover, the kernel of π consists of $\vec{v}$ satisfying $\vec{v}\sim\vec{0}$, which means $\vec{v}=\vec{v}-\vec{0}\in H$. □

Proposition 3.4.2 and Theorem 3.2.5 imply

$$\dim V/H=\text{rank}\pi=\dim V-\dim\text{Ker}\pi=\dim V-\dim H.$$

Example 3.4.1. Let $V=\mathbb{R}^2$ and $H=\mathbb{R}\vec{e}_1=\mathbb{R}\times 0=\{(x,0)\colon x\in\mathbb{R}\}$. Then

$$(a,b)+H=\{(a+x,b)\colon x\in\mathbb{R}=\{(x,b)\colon x\in\mathbb{R}\}$$

are all the horizontal lines. See Figure 3.4.1. These horizontal lines are in one-to-one correspondence with the y-coordinate

$$(a,b)+H\in\mathbb{R}^2/H\longleftrightarrow b\in\mathbb{R}.$$

This identifies the quotient space $\mathbb{R}^2/H$ with $\mathbb{R}$. The identification is a linear transformation because it simply picks the second coordinate. Therefore we have an isomorphism $\mathbb{R}^2/H\cong\mathbb{R}$ of vector spaces.

Exercise 3.78. For subsets X,Y of a vector space V, define

$$X+Y=\{\vec{u}+\vec{v}\colon \vec{u}\in X,\vec{v}\in Y\},\quad aX=\{a\vec{u}\colon \vec{u}\in X\}.$$

Verify the following properties similar to some axioms of vector space.

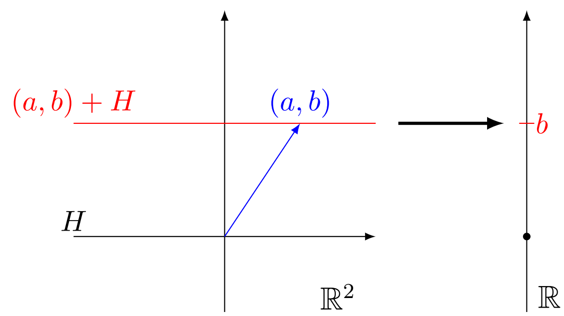

Figure 3.4.1: Quotient space $\mathbb{R}^2/\mathbb{R}\times 0$.

1. $X+Y=Y+X$.
2. $(X+Y)+Z=X+(Y+Z)$.
3. $\{\vec{0}\}+X=X=X+\{\vec{0}\}$.
4. $1X=X$.
5. $(ab)X=a(bX)$.
6. $a(X+Y)=aX+aY$.

Exercise 3.79. Prove that a subset H of a vector space is a subspace if and only if $H+H=H$ and $aH=H$ for $a\neq 0$.

Exercise 3.80. A subset A of a vector space is an *affine subspace* if $aA+(1-a)A=A$ for any $a\in\mathbb{R}$.

1. Prove that sum of two affine subspaces is an affine subspace.
2. Prove that a finite subset is an affine subspace if and only if it is a single vector.
3. Prove that an affine subspace A is a vector subspace if and only if $\vec{0}\in A$.
4. Prove that A is an affine subspace if and only if $A=\vec{v}+H$ for a vector $\vec{v}$ and a subspace H.

Exercise 3.81. An *equivalence relation* on a set X is a collection of ordered pairs $x\sim y$ (regarded as elements in $X\times X$) satisfying the reflexivity, symmetry, and transitivity. The equivalence class of $x\in X$ is

$$\bar{x}=\{y\in X\colon y\sim x\}\subset X.$$

Prove the following.

1. For any $x,y\in X$, either $\bar{x}=\bar{y}$ or $\bar{x}\cap\bar{y}=\emptyset$.
2. $X=\cup_{x\in X}\bar{x}$.

If we choose one element from each equivalence class, and let I be the set of all such elements, then the two properties imply $X = \sqcup_{x\in I}\bar{x}$ is a decomposition of X into a *disjoint union* of non-empty subsets.

Exercise 3.82. Suppose $X = \sqcup_{i\in I}X_i$ is a *partition* (i.e., disjoint union of non-empty subsets). Define $x \sim y \iff x$ and y are in the same subset X_i. Prove that $x \sim y$ is an equivalence relation, and the equivalence classes are exactly X_i.

Exercise 3.83. Let $f\colon X \to Y$ be a map. Define $x \sim x' \iff f(x) = f(x')$. Prove that $x \sim x'$ is an equivalence relation, and the equivalence classes are exactly the preimages $f^{-1}(y) = \{x \in X : f(x) = y\}$ for $y \in f(X)$ (otherwise the preimage is empty).

3.4.2 Universal Property

The quotient map $\pi\colon V \to \bar{V}$ is a linear transformation constructed for the purpose of ignoring (or vanishing on) H. The map is *universal* because it can be used to construct all linear transformations on V that vanish on H.

Theorem 3.4.3. *Suppose H is a subspace of V, and $\pi\colon V \to \bar{V} = V/H$ is the quotient map. Then a linear transformation $L\colon V \to W$ satisfies $L(\vec{h}) = \vec{0}$ for all $\vec{h} \in H$ (i.e., $H \subset \mathrm{Ker}L$) if and only if it is the composition $L = \bar{L}\circ\pi$ for a linear transformation $\bar{L}\colon \bar{V} \to W$.*

The linear transformation $\bar{L}$ can be described by the following *commutative diagram*.

$$\begin{tikzcd} V \arrow[rr, "L"] \arrow[dr, "\pi"'] & & W \\ & V/H \arrow[ur, "\bar{L}"'] & \end{tikzcd}$$

Proof. If $L = \bar{L}\circ\pi$, then $H \subset \mathrm{Ker}L$ by

$$\vec{h} \in H \implies \pi(\vec{h}) = \vec{0} \implies L(\vec{h}) = \bar{L}(\pi(\vec{h})) = \bar{L}(\vec{0}) = \vec{0}.$$

Conversely, if $H \subset \mathrm{Ker}L$, then the following shows that $\bar{L}(\bar{v}) = L(\vec{v})$ is well defined

$$\bar{u} = \bar{v} \iff \vec{u} - \vec{v} \in H \implies L(\vec{u}) - L(\vec{v}) = L(\vec{u} - \vec{v}) = \vec{0} \iff L(\vec{u}) = L(\vec{v}).$$

The following verifies that $\bar{L}$ is a linear transformation

$$\bar{L}(a\bar{u} + b\bar{v}) = \bar{L}(\overline{au + bv}) = L(a\vec{u} + b\vec{v}) = aL(\vec{u}) + bL(\vec{v}) = a\bar{L}(\bar{u}) + b\bar{L}(\bar{v}).$$

The following verifies $L = \bar{L}\circ\pi$

$$L(\vec{v}) = \bar{L}(\bar{v}) = \bar{L}(\pi(\vec{v})) = (\bar{L}\circ\pi)(\vec{v}). \qquad \square$$

Any linear transformation $L\colon V \to W$ vanishes on the kernel $\mathrm{Ker}L$. We may take $H = \mathrm{Ker}L$ in Proposition 3.4.3 and get

$$\begin{tikzcd} V \arrow[rr, "L"] \arrow[dr, "{\pi,\text{ onto}}"'] & & W \\ & V/\mathrm{Ker}L \arrow[ur, "{\bar{L},\text{ one-to-one}}"'] & \end{tikzcd}$$

Here the one-to-one property of $\bar{L}$ follows from

$$\bar{L}(\bar{v}) = \vec{0} \iff \vec{v} \in \mathrm{Ker}L \iff \bar{v} = \vec{v} + \mathrm{Ker}L = \mathrm{Ker}L = \bar{0}.$$

If we further know that L is onto, then we get the following property.

Proposition 3.4.4. *If a linear transformation $L\colon V \to W$ is onto, then $\bar{L}\colon V/\mathrm{Ker}L \cong W$ is an isomorphism.*

Example 3.4.2. The picking of the second coordinate $(x, y) \in \mathbb{R}^2 \to y \in \mathbb{R}$ is an onto linear transformation with kernel $H = \mathbb{R}\vec{e}_1 = \mathbb{R} \times 0$. By Proposition 3.4.4, we get $\mathbb{R}^2/H \cong \mathbb{R}$. This is the isomorphism in Example 3.4.1.

In general, if $H = \mathbb{R}^k \times \vec{0}$ is the subspace of $\mathbb{R}^n$, such that the last $n-k$ coordinates vanish, then $\mathbb{R}^n/H \cong \mathbb{R}^{n-k}$ by picking the last $n-k$ coordinates.

Example 3.4.3. The linear functional $l(x, y, z) = x + y + z\colon \mathbb{R}^3 \to \mathbb{R}$ is onto, and its kernel $H = \{(x, y, z)\colon x + y + z = 0\}$ is the plane in Example 2.1.13. Therefore $\bar{l}\colon \mathbb{R}^3/H \to \mathbb{R}$ is an isomorphism. The equivalence classes are the planes

$$(a, 0, 0) + H = \{(a + x, y, z)\colon x + y + z = 0\} = \{(x, y, z)\colon x + y + z = a\} = l^{-1}(a)$$

parallel to H.

Example 3.4.4. The orthogonal projection P of $\mathbb{R}^3$ in Example 2.1.13 is onto the range $H = \{(x, y, z)\colon x + y + z = 0\}$. The geometrical meaning of P shows that $\mathrm{Ker}P = \mathbb{R}(1, 1, 1)$ is the line in direction $(1, 1, 1)$ and passing through the origin. Then by Proposition 3.4.2, we have $\mathbb{R}^3/\mathbb{R}(1, 1, 1) \cong H$. Note that the vectors in the quotient space $\mathbb{R}^3/\mathbb{R}(1, 1, 1)$ are defined as all the lines in direction $(1, 1, 1)$ (but not necessarily passing through the origin). The isomorphism identifies the collection of such lines with the plane H.

Example 3.4.5. The derivative map $D(f) = f'\colon C^\infty \to C^\infty$ is onto, and the kernel is all the constant functions $\mathrm{Ker}D = \{C\colon C \in \mathbb{R}\} = \mathbb{R}$. This induces an isomorphism $C^\infty/\mathbb{R} \cong C^\infty$.

The second order derivative map $D_2(f) = f''\colon C^\infty \to C^\infty$ vanishes on constant functions. By Theorem 3.4.3, we have $D_2 = \bar{D}_2 \circ D$. Of course we know $\bar{D}_2 = D$ and $D_2 = D^2$.

Exercise 3.84. Prove that the map $\bar{L}$ in Theorem 3.4.3 is one-to-one if and only if $H = \mathrm{Ker}L$.

Exercise 3.85. Use Exercise 3.58, Proposition 3.4.4 and $\dim V/H = \dim V - \dim H$ to prove Proposition **??**.

Exercise 3.86. Show that the linear transformation by the matrix is onto. Then explain the implication in terms of quotient space.

1. $\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$.

2. $\begin{pmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \end{pmatrix}$.

3. $\begin{pmatrix} a_1 & -1 & 0 & \cdots & 0 \\ a_2 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_n & 0 & 0 & \cdots & -1 \end{pmatrix}$.

Exercise 3.87. Explain that the quotient space $M_{n\times n}/\{\text{symmetric spaces}\}$ is isomorphic to the vector space of all skew-symmetric matrices.

Exercise 3.88. Show that
$$H = \{f \in C^\infty \colon f(0) = 0\}$$
is a subspace of C^∞, and $C^\infty/H \cong \mathbb{R}$.

Exercise 3.89. Let $k \le n$ and $t_1, t_2, \dots, t_k$ be distinct. Let
$$H = (t-t_1)(t-t_2)\cdots(t-t_k)P_n = \{(t-t_1)(t-t_2)\cdots(t-t_k)f(t)\colon f \in P_{n-k}\}.$$
Show that H is a subspace of P_n and the evaluations at $t_1, t_2, \dots, t_k$ gives an isomorphism between P_n/H and $\mathbb{R}^k$.

Exercise 3.90. Show that
$$H = \{f \in C^\infty \colon f(0) = f'(0) = 0\}$$
is a subspace of C^∞, and $C^\infty/H \cong \mathbb{R}^2$.

Exercise 3.91. For fixed t_0, the map
$$f \in C^\infty \mapsto (f(t_0), f'(t_0), \dots, f^{(k)}(t_0)) \in \mathbb{R}^{n+1}$$
can be regarded as the n-th order Taylor expansion at t_0. Prove that the Taylor expansion is an onto linear transformation. Find the kernel of the linear transformation and interpret your result in terms of quotient space.

Exercise 3.92. Suppose $\sim$ is an equivalence relation on a set X. Define the *quotient set* $\bar{X} = X/\sim$ to be the collection of equivalence classes.

1. Prove that the *quotient map* $\pi(x) = \bar{x} \colon X \to \bar{X}$ is onto.

2. Prove that a map $f \colon X \to Y$ satisfies $x \sim x'$ implying $f(x) = f(x')$ if and only if it is the composition of a map $\bar{f} \colon \bar{X} \to Y$ with the quotient map.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Y \\ & X/\sim \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

3.4.3 Direct Summand

Definition 3.4.5. A *direct summand* of a subspace H in the whole space V is a subspace H' satisfying $V = H \oplus H'$.

A direct summand fills the gap between H and V, similar to that 3 fills the gap between 2 and 5 by $5 = 2 + 3$. The following shows that a direct summand also "internalises" the quotient space.

Proposition 3.4.6. *A subspace H' is a direct summand of H in V if and only if the composition $H' \subset V \to V/H$ is an isomorphism.*

Proof. The proposition is the consequence of the following two claims and the $k = 2$ case of Proposition 3.3.3 (see the remark after the earlier proposition)

1. $V = H + H'$ if and only if the composition $H' \subset V \to V/H$ is onto.

2. The kernel of the composition $H' \subset V \to V/H$ is $H \cap H'$.

For the first claim, we note that $H' \subset V \to V/H$ is onto means that for any $\vec{v} \in V$, there is $\vec{h}' \in H'$, such that $\vec{v} + H = \vec{h}' + H$, or $\vec{v} - \vec{h}' \in H$. Therefore onto means any $\vec{v} \in V$ can be expressed as $\vec{h} + \vec{h}'$ for some $\vec{h} \in H$ and $\vec{h}' \in H'$. This is exactly $V = H + H'$.

For the second claim, we note that the kernel of the composition is

$$\{\vec{h}' \in H' \colon \pi(\vec{h}') = \vec{0}\} = \{\vec{h}' \in H' \colon \vec{h}' \in \mathrm{Ker}\pi = H\} = H \cap H'. \qquad \square$$

Example 3.4.6. A direct summand of $H = \mathbb{R}\vec{e}_1 = \mathbb{R} \times 0$ in $\mathbb{R}^2$ is a 1-dimensional subspace $H' = \mathbb{R}\vec{v}$, such that $\mathbb{R}^2 = \mathbb{R}\vec{e}_1 \oplus \mathbb{R}\vec{v}$. In other words, $\vec{e}_1$ and $\vec{v}$ form a basis of $\mathbb{R}^2$. The condition means exactly that the second coordinate of $\vec{v}$ is nonzero.

Therefore, by multiplying a non-zero scalar to $\vec{v}$ (which does not change H'), we may assume $H' = \mathbb{R}(a, 1)$. Since different a gives different line $\mathbb{R}(a, 1)$, this gives a one-to-one correspondence

$$\{\text{direct summand } \mathbb{R}(a, 1) \text{ of } \mathbb{R} \times 0 \text{ in } \mathbb{R}^2\} \longleftrightarrow a \in \mathbb{R}.$$

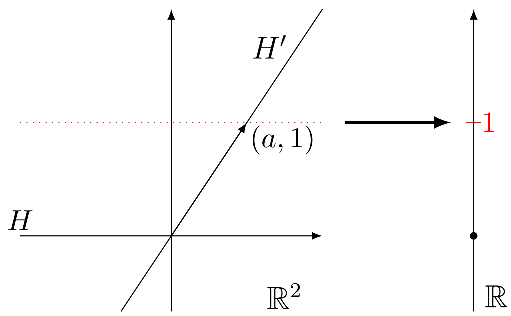

Figure 3.4.2: Direct summands of $\mathbb{R} \times 0$ in $\mathbb{R}^2$.

Example 3.4.7. By Example 3.4.5, the derivative induces an isomorphism $C^\infty/\mathbb{R} \cong C^\infty$, where $\mathbb{R}$ is the subspace of all constant functions. By Proposition 3.4.6, a direct summand of constant functions $\mathbb{R}$ in C^∞ is then a subspace $H \subset C^\infty$, such that $D|_H \colon f \in H \to f' \in C^\infty$ is an isomorphism. For any fixed t_0, we may choose

$$H(t_0) = \{f \in C^\infty \colon f(t_0) = 0\}.$$

For any $g \in C^\infty$, we have $f(t) = \int_{t_0}^t g(\tau)d\tau$ satisfying $f' = g$ and $f(t_0) = 0$. This shows that $D|_H$ is onto. Since $f' = 0$ and $f(t_0) = 0$ implies $f = 0$, we also know that the kernel of $D|_H$ is trivial. Therefore $D|_H$ is an isomorphism, and $H(t_0)$ is a direct summand.

Exercise 3.93. What is the dimension of a direct summand?

Exercise 3.94. Describe all the direct summands of $\mathbb{R}^k \times \vec{0}$ in $\mathbb{R}^n$.

Exercise 3.95. Is it true that any direct summand of $\mathbb{R}$ in C^∞ is $H(t_0)$ in Example 3.4.7 for some t_0?

Exercise 3.96. Suppose α is a basis of H and $\alpha \cup \beta$ is a basis of V. Prove that β spans a direct summand of H in V. Moreover, all the direct summands are obtained in this way.

A direct summand is comparable to an extension of a linearly independent set to a basis of the whole space.

Exercise 3.97. Prove that direct summands of a subspace H in V are in one-to-one correspondence with projections P of V satisfying $P(V) = H$.

Exercise 3.98. A *splitting* of a linear transformation $L\colon V \to W$ is a linear transformation $K\colon W \to V$ satisfying $L \circ K = I$. Let $H = \text{Ker}L$.

1. Prove that L has a splitting if and only if L is onto. By Proposition 3.4.4, L induces an isomorphism $\bar{L}\colon V/H \cong W$.

2. Prove that K is a splitting of L if and only if $K(W)$ is a direct summand of H in V.

3. Prove that splittings of L are in one-to-one correspondence with direct summands of H in V.

Exercise 3.99. Suppose K is a splitting of L. Prove that $K \circ L$ is a projection. Then discuss the relation between two interpretations of direct summands in Exercises 3.97 and 3.98.

Exercise 3.100. Suppose H' and H'' are two direct summands of H in V. Prove that there is a self isomorphism $L\colon V \to V$, such that $L(H) = H$ and $L(H') = H''$. Moreover, prove that it is possible to further require that L satisfies the following, and such L is unique.

1. L fixes H: $L(\vec{h}) = \vec{h}$ for all $\vec{h} \in H$.

2. L is natural: $\vec{h}' + H = L(\vec{h}') + H$ for all $\vec{h}' \in H'$.

Exercise 3.101. Suppose $V = H \oplus H'$. Prove that

$$A \in \text{Hom}(H', H) \mapsto H'' = \{(A(\vec{h}'), \vec{h}')\colon \vec{h}' \in H'\}$$

is a one-to-one correspondence to all direct summands H'' of H in V. This extends Example 3.4.6.

Suppose H' and H'' are direct summands of H in V. Then by Proposition 3.4.6, both natural linear transformations $H' \subset V \to V/H$ and $H'' \subset V \to V/H$ are isomorphisms. Combining the two isomorphisms, we find that H' and H'' are naturally isomorphic. Since the direct summand is unique up to natural isomorphism, we denote the direct summand by $V \ominus H$.

Exercise 3.102. Prove that $H + H' = (H \ominus (H \cap H')) \oplus (H \cap H') \oplus (H' \ominus (H \cap H'))$. Then prove that

$$\dim(H + H') + \dim(H \cap H') = \dim H + \dim H'.$$

Chapter 4

Inner Product

The inner product introduces geometry (such as length, angle, area, volume, etc.) into a vector space. Orthogonality can be introduced in an inner product space, as the most linearly independent (or direct sum) scenario. Moreover, we have the related concepts of orthogonal projection and orthogonal complement. The inner product also induces natural isomorphism between a vector space and its dual space.

4.1 Inner Product

4.1.1 Definition

Definition 4.1.1. An *inner product* on a real vector space V is a function

$$\langle \vec{u}, \vec{v} \rangle \colon V \times V \to \mathbb{R},$$

such that the following are satisfied.

1. Bilinearity: $\langle a\vec{u} + b\vec{u}', \vec{v} \rangle = a\langle \vec{u}, \vec{v} \rangle + b\langle \vec{u}', \vec{v} \rangle$, $\langle \vec{u}, a\vec{v} + b\vec{v} \rangle = a\langle \vec{u}, \vec{v} \rangle + b\langle \vec{u}, \vec{v}' \rangle$.

2. Symmetry: $\langle \vec{v}, \vec{u} \rangle = \langle \vec{u}, \vec{v} \rangle$.

3. Positivity: $\langle \vec{u}, \vec{u} \rangle \geq 0$ and $\langle \vec{u}, \vec{u} \rangle = 0$ if and only if $\vec{u} = \vec{0}$.

An *inner product space* is a vector space equipped with an inner product.

Example 4.1.1. The *dot product* on the Euclidean space is

$$(x_1, x_2, \ldots, x_n) \cdot (y_1, y_2, \ldots, y_n) = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

If we use the convention of expressing Euclidean vectors as vertical $n \times 1$ matrices, then we have

$$\vec{x} \cdot \vec{y} = \vec{x}^T\vec{y}.$$

This is especially convenient when the dot product is combined with matrices. For example, for matrices $A = (\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_m)$ and $B = (\vec{w}_1 \ \vec{w}_2 \ \cdots \ \vec{w}_n)$, where all column vectors are in the same Euclidean space $\mathbb{R}^k$, we have (A^T is $m \times k$, and B is $k \times n$)

$$A^T B = \begin{pmatrix} \vec{v}_1^T \\ \vec{v}_2^T \\ \vdots \\ \vec{v}_m^T \end{pmatrix} (\vec{w}_1 \ \vec{w}_2 \ \cdots \ \vec{w}_n) = \begin{pmatrix} \vec{v}_1 \cdot \vec{w}_1 & \vec{v}_1 \cdot \vec{w}_2 & \dots & \vec{v}_1 \cdot \vec{w}_n \\ \vec{v}_2 \cdot \vec{w}_1 & \vec{v}_2 \cdot \vec{w}_2 & \dots & \vec{v}_2 \cdot \vec{w}_n \\ \vdots & \vdots & & \vdots \\ \vec{v}_m \cdot \vec{w}_1 & \vec{v}_m \cdot \vec{w}_2 & \dots & \vec{v}_m \cdot \vec{w}_n \end{pmatrix}.$$

In particular, we have

$$A^T \vec{x} = \begin{pmatrix} \vec{v}_1 \cdot \vec{x} \\ \vec{v}_2 \cdot \vec{x} \\ \vdots \\ \vec{v}_m \cdot \vec{x} \end{pmatrix}.$$

Example 4.1.2. The dot product is not the only inner product on the Euclidean space. For example, if all $a_i > 0$, then the following is also an inner product

$$\langle \vec{x}, \vec{y} \rangle = a_1 x_1 y_1 + a_2 x_2 y_2 + \cdots + a_n x_n y_n.$$

For general discussion of inner products on Euclidean spaces, see Section 4.1.3.

Example 4.1.3. On the vector space P_n of polynomials of degree $\leq n$, we may introduce the inner product

$$\langle f, g \rangle = \int_0^1 f(t) g(t) dt.$$

This is also an inner product on the vector space $C[0,1]$ of all continuous functions on $[0,1]$, or the vector space of continuous periodic functions on $\mathbb{R}$ of period 1.

More generally, if $K(t) > 0$, then $\langle f, g \rangle = \int_0^1 f(t) g(t) K(t) dt$ is an inner product.

Example 4.1.4. On the vector space $M_{m \times n}$ of $m \times n$ matrices, we use the trace introduced in Exercise 2.10 to define

$$\langle A, B \rangle = \mathrm{tr} A^T B = \sum_{i,j} a_{ij} b_{ij}, \quad A = (a_{ij}), B = (b_{ij}).$$

By Exercises 2.10 and 2.22, the symmetry and bilinear conditions are satisfied. By $\langle A, B \rangle = \sum_{i,j} a_{ij}^2 \geq 0$, the positivity condition is satisfied. Therefore $\mathrm{tr} A^T B$ is an inner product on $M_{m \times n}$.

In fact, if we use the usual isomorphism between $M_{m \times n}$ and $\mathbb{R}^{mn}$, the inner product is translated into the dot product on the Euclidean space.

Exercise 4.1. Suppose $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$ are two inner products on V. Prove that for any $a, b > 0$, $a\langle\ ,\ \rangle_1 + b\langle\ ,\ \rangle_2$ is also an inner product.

Exercise 4.2. Prove that $\vec{u}$ satisfies $\langle \vec{u}, \vec{v} \rangle = 0$ for all $\vec{v}$ if and only if $\vec{u} = \vec{0}$.

Exercise 4.3. Prove that $\vec{v}_1 = \vec{v}_2$ if and only if $\langle \vec{u}, \vec{v}_1 \rangle = \langle \vec{u}, \vec{v}_2 \rangle$ for all $\vec{u}$. In other words, two vectors are equal if and only if their inner products with all vectors are equal.

Exercise 4.4. Let W be an inner product space. Prove that two linear transformations $L, K \colon V \to W$ are equal if and only if $\langle \vec{w}, L(\vec{v}) \rangle = \langle \vec{w}, K(\vec{v}) \rangle$ for all $\vec{v} \in V$ and $\vec{w} \in W$.

Exercise 4.5. Prove that two matrices A and B are equal if and only if $\vec{x} \cdot A\vec{y} = \vec{x} \cdot B\vec{y}$ (i.e., $\vec{x}^T A\vec{y} = \vec{x}^T B\vec{y}$) for all $\vec{x}$ and $\vec{y}$.

Exercise 4.6. Use the formula for the product of matrices in Example 4.1.1 to show that $(AB)^T = B^T A^T$.

Exercise 4.7. Show that $\langle f, g \rangle = f(0)g(0) + f(1)g(1) + \cdots + f(n)g(n)$ is an inner product on P_n.

4.1.2 Geometry

The usual Euclidean length is given by the Pythagorian theorem

$$\|\vec{x}\| = \sqrt{x_1^2 + \cdots + x_n^2} = \sqrt{\vec{x} \cdot \vec{x}}.$$

In general, the *length* (or *norm*) with respect to an inner product is

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}.$$

We may take the square root because of the positivity property.

Inspired by the geometry in $\mathbb{R}^2$, we define the *angle* θ between two nonzero vectors $\vec{u}, \vec{v}$ by

$$\cos\theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\|\,\|\vec{v}\|}.$$

Two vectors are *orthogonal* if the angle between them is $\frac{1}{2}\pi$. By $\cos\frac{1}{2}\pi = 0$, we define $\vec{u}$ and $\vec{v}$ to be orthogonal, and denote $\vec{u} \perp \vec{v}$, if $\langle \vec{u}, \vec{v} \rangle = 0$.

For the definition of angle to make sense, however, we need the following result.

Proposition 4.1.2 (Cauchy-Schwarz Inequality). $|\langle \vec{u}, \vec{v} \rangle| \le \|\vec{u}\|\,\|\vec{v}\|$.

Proof. For any real number t, we have

$$0 \le \langle \vec{u} + t\vec{v}, \vec{u} + t\vec{v} \rangle = \langle \vec{u}, \vec{u} \rangle + 2t\langle \vec{u}, \vec{v} \rangle + t^2\langle \vec{v}, \vec{v} \rangle.$$

For the quadratic function of t to be always non-negative, the coefficients must satisfy

$$(\langle \vec{u}, \vec{v}\rangle)^2 \le \langle \vec{u}, \vec{u}\rangle\langle \vec{v}, \vec{v}\rangle.$$

This is the same as $|\langle \vec{u}, \vec{v}\rangle| \le \|\vec{u}\|\,\|\vec{v}\|$. □

Knowing the angle, we may compute the area of the parallelogram spanned by the two vectors

$$\begin{aligned}\text{Area}(\vec{u}, \vec{v}) &= \|\vec{u}\|\,\|\vec{v}\|\sin\theta \\ &= \|\vec{u}\|\,\|\vec{v}\|\sqrt{1-\left(\frac{\langle \vec{u}, \vec{v}\rangle}{\|\vec{u}\|\,\|\vec{v}\|}\right)^2} = \sqrt{\|\vec{u}\|^2\|\vec{v}\|^2-(\langle \vec{u}, \vec{v}\rangle)^2}.\end{aligned}$$

Again, we can take the square root due to the Cauchy-Schwarz inequality.

Proposition 4.1.3. *The vector length has the following properties.*

1. *Positivity:* $\|\vec{u}\| \ge 0$, *and* $\|\vec{u}\| = 0$ *if and only if* $\vec{u} = \vec{0}$.
2. *Scaling:* $\|a\vec{u}\| = |a|\,\|\vec{u}\|$.
3. *Triangle inequality:* $\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|$.

The first two properties are easy to verify, and the triangle inequality is a consequence of the Cauchy-Schwarz inequality

$$\begin{aligned}\|\vec{u}+\vec{v}\|^2 &= \langle \vec{u}+\vec{v}, \vec{u}+\vec{v}\rangle \\ &= \langle \vec{u}, \vec{u}\rangle + \langle \vec{u}, \vec{v}\rangle + \langle \vec{v}, \vec{u}\rangle + \langle \vec{v}, \vec{v}\rangle \\ &\le \|\vec{u}\|^2 + \|\vec{u}\|\,\|\vec{v}\| + \|\vec{v}\|\,\|\vec{u}\| + \|\vec{v}\|^2 \\ &= (\|\vec{u}\| + \|\vec{v}\|)^2.\end{aligned}$$

By the scaling property in Proposition 4.1.3, if $\vec{v} \ne \vec{0}$, then by dividing the length, we get a unit length vector (i.e., length 1)

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}.$$

Note that $\vec{u}$ indicates the *direction* of the vector $\vec{v}$ by "forgetting" its length. In fact, all the directions in the inner product space form the *unit sphere*

$$S_1 = \{\vec{u} \in V : \|\vec{u}\| = 1\} = \{\vec{u} \in V : \vec{u}\cdot\vec{u} = 1\}.$$

Any nonzero vector has unique *polar decomposition*

$$\vec{v} = r\vec{u}, \text{ where } r = \|\vec{v}\| > 0 \text{ and } \|\vec{u}\| = 1.$$

Example 4.1.5. With respect to the dot product, the lengths of $(1,1,1)$ and $(1,2,3)$ are

$$\|(1,1,1)\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}, \quad \|(1,2,3)\| = \sqrt{1^2+2^2+3^2} = \sqrt{14}.$$

Their polar decompositions are

$$(1,1,1) = \sqrt{3}(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}), \quad (1,2,3) = \sqrt{14}(\tfrac{1}{\sqrt{14}}, \tfrac{2}{\sqrt{14}}, \tfrac{3}{\sqrt{14}}).$$

The angle between the two vectors is given by

$$\cos\theta = \frac{1\cdot 1 + 1\cdot 2 + 1\cdot 3}{\|(1,1,1)\|\,\|(1,2,3)\|} = \frac{6}{\sqrt{42}}.$$

Therefore the angle is $\arccos\frac{6}{\sqrt{42}} = 0.1234\pi = 22.2077°$.

Example 4.1.6. Consider the triangle with vertices $\vec{a} = (1,-1,0), \vec{b} = (2,0,1), \vec{c} = (2,1,3)$. The area is half of the parallelogram spanned by $\vec{u} = \vec{b} - \vec{a} = (1,1,1)$ and $\vec{v} = \vec{c} - \vec{a} = (1,2,3)$

$$\frac{1}{2}\sqrt{\|(1,1,1)\|^2\|(1,2,3)\|^2 - ((1,1,1)\cdot(1,2,3))^2} = \frac{1}{2}\sqrt{3\cdot 14 - 6^2} = \sqrt{\frac{3}{2}}.$$

Example 4.1.7. By the inner product in Example 4.1.3, the lengths of 1 and t are

$$\|1\| = \sqrt{\int_0^1 dt} = 1, \quad \|t\| = \sqrt{\int_0^1 t^2 dt} = \frac{1}{\sqrt{3}}.$$

Therefore 1 has the unit length, and t has polar decomposition $t = \frac{1}{\sqrt{3}}(\sqrt{3}t)$. The angle between 1 and t is given by

$$\cos\theta = \frac{\int_0^1 tdt}{\|1\|\,\|t\|} = \frac{\sqrt{3}}{2}.$$

Therefore the angle is $\frac{1}{6}\pi$. Moreover, the area of the parallelogram spanned by 1 and t is

$$\sqrt{\int_0^1 dt \int_0^1 t^2 dt - \left(\int_0^1 tdt\right)^2} = \frac{1}{2\sqrt{3}}.$$

Exercise 4.8. Show that the area of the triangle with vertices $(0,0), (a,b), (c,d)$ is $\frac{1}{2}|ad-bc|$. More generally, the area of the triangle vertices $\vec{0}, \vec{x}, \vec{y} \in \mathbb{R}^n$ is $\frac{1}{2}\sqrt{\sum_{1\le i<j\le n}(x_iy_j - x_jy_i)^2}$.

Exercise 4.9. In Example 4.1.6, we calculated the area of the triangle by subtracting $\vec{a}$. By the obvious symmetry, we can also calculate the area by subtracting $\vec{b}$ or $\vec{c}$. Please verify that the alternative calculations give the same results. Can you provide an argument for the general case.

Exercise 4.10. Prove that the *distance* $d(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\|$ in an inner product space has the following properties.

1. Positivity: $d(\vec{u},\vec{v}) \geq 0$, and $d(\vec{u},\vec{v}) = 0$ if and only if $\vec{u} = \vec{v}$.
2. Symmetry: $d(\vec{u},\vec{v}) = d(\vec{v},\vec{u})$.
3. Triangle inequality: $d(\vec{u},\vec{w}) \leq d(\vec{u},\vec{v}) + d(\vec{v},\vec{w})$.

Exercise 4.11. Show that the area of the parallelogram spanned by two vectors is zero if and only if the two vectors are parallel.

Exercise 4.12. Prove the *polarisation identity*

$$\langle \vec{u},\vec{v}\rangle = \frac{1}{4}(\|\vec{u}+\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2) = \frac{1}{2}(\|\vec{u}+\vec{v}\|^2 - \|\vec{u}\|^2 - \|\vec{v}\|^2).$$

Exercise 4.13. Prove that two inner products $\langle \cdot,\cdot\rangle_1$ and $\langle \cdot,\cdot\rangle_2$ are equal if and only if they induce the same length: $\langle \vec{x},\vec{x}\rangle_1 = \langle \vec{x},\vec{x}\rangle_2$.

Exercise 4.14. Prove the *parallelogram identity*

$$\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = 2(\|\vec{u}\|^2 + \|\vec{v}\|^2).$$

Exercise 4.15. Find the length of vectors and the angle between vectors.

1. $(1,0)$, $(1,1)$.
2. $(1,0,1)$, $(1,1,0)$.
3. $(1,2,3)$, $(2,3,4)$.
4. $(1,0,1,0)$, $(0,1,0,1)$.
5. $(0,1,2,3)$, $(4,5,6,7)$.
6. $(1,1,1,1)$, $(1,-1,1,-1)$.

Exercise 4.16. Find the area of the triangle with the given vertices.

1. $(1,0)$, $(0,1)$, $(1,1)$.
2. $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.
3. $(1,2,3)$, $(2,3,4)$, $(3,4,5)$.
4. $(1,1,0)$, $(1,0,1)$, $(0,1,1)$.
5. $(1,0,1,0)$, $(0,1,0,1)$, $(1,0,0,1)$.
6. $(0,1,2,3)$, $(4,5,6,7)$, $(8,9,10,11)$.

Exercise 4.17. Find the length of vectors and the angle between vectors.

1. $(1,0,1,0,\dots)$, $(0,1,0,1,\dots)$.
2. $(1,2,3,\dots,n)$, $(n,n-1,n-2,\dots,1)$.
3. $(1,1,1,1,\dots)$, $(1,-1,1,-1,\dots)$.
4. $(1,1,1,1,\dots)$, $(1,2,3,4,\dots)$.

Exercise 4.18. Redo Exercises 4.15, 4.16, 4.17 with respect to the inner product

$$\langle \vec{x},\vec{y}\rangle = x_1y_1 + 2x_2y_2 + \cdots + nx_ny_n.$$

Exercise 4.19. Find the area of the triangle with given vertices, with respect to the inner product in Example 4.1.3.

1. $1, t, t^2$.
2. $0, \sin t, \cos t$.
3. $1, a^t, b^t$.
4. $1-t, t-t^2, t^2-1$.

Exercise 4.20. Redo Exercise 4.19 with respect to the inner product

$$\langle f, g\rangle = \int_0^1 tf(t)g(t)dt.$$

Exercise 4.21. Redo Exercise 4.19 with respect to the inner product

$$\langle f, g\rangle = \int_{-1}^1 f(t)g(t)dt.$$

4.1.3 Positive Definite Matrix

An inner product $\langle\cdot,\cdot\rangle$ on a finite dimensional vector space V is a bilinear function. Let $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be a basis of V. Then as explained in Section 2.4.4, the bilinear function is given by a matrix

$$\langle\vec{x}, \vec{y}\rangle = [\vec{x}]_\alpha^T A[\vec{y}]_\alpha, \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad a_{ij} = \langle\vec{v}_i, \vec{v}_j\rangle.$$

For the special case $V = \mathbb{R}^n$ and α is the standard basis, the general inner product may be compared with the dot product

$$\langle\vec{x}, \vec{y}\rangle = \sum_{1\le i,j\le n} a_{ij}x_ix_j = \vec{x}^T A\vec{y} = \vec{x}\cdot A\vec{y}, \quad a_{ij} = \langle\vec{e}_i, \vec{e}_j\rangle.$$

The symmetry property means that the matrix is symmetric

$$A^T = A, \quad a_{ij} = a_{ji}.$$

The positivity property is more subtle.

Definition 4.1.4. A matrix A is *positive definite* if it is a symmetric matrix, and $\vec{x}^T A\vec{x} > 0$ for any $\vec{x} \ne \vec{0}$.

Given a basis of V, the inner products on V are in one-to-one correspondence with positive definite matrices.

Example 4.1.8. A diagonal matrix

$$A = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix}$$

is positive definite if and only if all $a_i > 0$. See Example 4.1.2.

Example 4.1.9. For $A = \begin{pmatrix} 1 & 2 \\ 2 & a \end{pmatrix}$, we have

$$\vec{x}^T A\vec{x} = x_1^2 + 4x_1x_2 + ax_2^2 = (x_1 + 2x_2)^2 + (a-4)x_2^2.$$

Therefore A is positive definite if and only if $a > 4$.

Exercise 4.22. Prove that $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is positive definite if and only if $a > 0$ and $ac > b^2$.

Exercise 4.23. For symmetric matrices A and B, prove that $\begin{pmatrix} A & O \\ O & B \end{pmatrix}$ is positive definite if and only if A and B are positive definite.

Exercise 4.24. Suppose A and B are positive definite, and $a, b > 0$. Prove that $aA + bB$ is positive definite.

Exercise 4.25. Prove that positive definite matrices are invertible.

Exercise 4.26. If A is positive definite and P is invertible, prove that $P^T AP$ is positive definite.

Exercise 4.27. If A is invertible, prove that $A^T A$ is positive definite. In fact, we only need A to be one-to-one (i.e., solution of $A\vec{x} = \vec{b}$ is unique).

Exercise 4.28. Suppose A is symmetric and $q(\vec{x}) = \vec{x} \cdot A\vec{x}$. Prove the *polarisation identity*

$$\vec{x} \cdot A\vec{y} = \frac{1}{4}(q(\vec{x}+\vec{y}) - q(\vec{x}-\vec{y})) = \frac{1}{2}(q(\vec{x}+\vec{y})q(\vec{x}) - q(\vec{y})).$$

Exercise 4.29. Prove that two symmetric matrices A and B are equal if and only if $\vec{x} \cdot A\vec{x} = \vec{x} \cdot B\vec{x}$ for all $\vec{x}$.

In general, we may determine the positive definite property of a symmetric matrix $A = (a_{ij})$ by the process of *completing the square* similar to Example 4.1.9. Suppose

$a_{11} \neq 0$, then we gather all the terms involving x_1 and get

$$\begin{aligned}
\vec{x}^T A\vec{x} &= a_{11}x_1^2 + 2a_{12}x_1x_2 + \cdots + 2a_{1n}x_1x_n + \sum_{2\le i,j\le n} a_{ij}x_ix_j \\
&= a_{11}\left[x_1^2 + 2x_1 \cdot \frac{1}{a_{11}}(a_{12}x_{12} + \cdots + a_{1n}x_{1n}) + \frac{1}{a_{11}^2}(a_{12}x_{12} + \cdots + a_{1n}x_{1n})^2\right] \\
&\quad - \frac{1}{a_{11}}(a_{12}x_{12} + \cdots + a_{1n}x_{1n})^2 + \sum_{2\le i,j\le n} a_{ij}x_ix_j \\
&= a_{11}\left[x_1 + \frac{a_{12}}{a_{11}}x_{12} + \cdots + \frac{a_{1n}}{a_{11}}x_{1n}\right]^2 + \sum_{2\le i,j\le n} a'_{ij}x_ix_j.
\end{aligned}$$

Here the matrix

$$A' = (a'_{ij}), \quad a'_{ij} = a_{ij} - \frac{a_{1i}a_{1j}}{a_{11}},$$

is obtained by the "symmetric row and column operations"

$$\begin{aligned}
A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}
&\xrightarrow{R_i - \frac{a_{i1}}{a_{11}}R_1} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a'_{n2} & \cdots & a'_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & * \\ O & A' \end{pmatrix} \\
&\xrightarrow{C_i - \frac{a_{1i}}{a_{11}}C_1} \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a'_{n2} & \cdots & a'_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & O \\ O & A' \end{pmatrix}.
\end{aligned}$$

In fact, the row operations are already sufficient for getting A'. We find that

$$\vec{x}^T A\vec{x} = b_1y_1^2 + \vec{x}'^T A'\vec{x}',$$

where

$$y_1 = x_1 + \frac{a_{12}}{a_{11}}x_{12} + \cdots + \frac{a_{1n}}{a_{11}}x_{1n},\ b_1 = a_{11},\ \vec{x}' = (x_2, \dots, x_n).$$

We note that $\vec{x}'^T A'\vec{x}'$ does not involve x_1. Therefore the process of completing the square can continue until we get

$$\vec{x}^T A\vec{x} = b_1y_1^2 + b_2y_2^2 + \cdots + b_ny_n^2, \quad y_i = x_i + c_{i(i+1)}x_{i+1} + \cdots + c_{in}x_n.$$

Then $\vec{x}^T A\vec{x}$ is positive definite if and only if all $b_i > 0$.

Example 4.1.10. By the row operations

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 0 & -2 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 0 & 0 & -2 \end{pmatrix},$$

we get $\vec{x}^A\vec{x} = y_1^2 - 2y_2^2 - 2y_3^2$ after completing the square. The matrix A is therefore not positive definite.

By the row operations

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 15 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & -2 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{pmatrix},$$

the matrix B is positive definite. The corresponding inner product is

$$\langle \vec{x}, \vec{y} \rangle = x_1y_1 + 5x_2y_2 + 15x_3y_3 + 2(x_1y_2 + x_2y_1) + 3(x_1y_3 + x_3y_1) + 4(x_2y_3 + x_3y_2).$$

Example 4.1.11. For

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 13 & 9 \\ 1 & 9 & 14 \end{pmatrix},$$

we gather together all the terms involving x and complete the square

$$\begin{aligned} \vec{x}^T A\vec{x} &= x^2 + 13y^2 + 14z^2 + 6xy + 2xz + 18yz \\ &= [x^2 + 2x(3y + z) + (3y + z)^2] + 13y^2 + 14z^2 + 18yz - (3y + z)^2 \\ &= (x + 3y + z)^2 + 4y^2 + 13z^2 + 12yz. \end{aligned}$$

The remaining terms involve only y and z. Gathering all the terms involving y and completing the square, we get $4y^2 + 13z^2 + 12yz = (2y + 3z)^2 + 4z^2$ and

$$\vec{x}^T A\vec{x} = (x + 3y + z)^2 + (2y + 3z)^2 + (2z)^2 = u^2 + v^2 + w^2,$$

for

$$\vec{y} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} x + 3y + z \\ 2y + 3z \\ 2z \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

or

$$\vec{u} = P\vec{x}, \quad P = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix}.$$

In particular, the matrix is positive definite.

The process of completing the square corresponds to the row operations

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 13 & 9 \\ 1 & 9 & 14 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 1 \\ 0 & 4 & 6 \\ 0 & 6 & 13 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 1 \\ 0 & 4 & 6 \\ 0 & 0 & 4 \end{pmatrix}.$$

The result of the completing the square can also be interpreted as

$$\vec{x}^T A\vec{x} = \vec{u}^T\vec{u} = (P\vec{x})^T(P\vec{x}) = \vec{x}^T P^T P\vec{x}.$$

By Exercise 4.29, this means $A = P^T P$.

If P is an invertible matrix, and $\vec{y} = P\vec{x}$, then completing the square means

$$\vec{x}^T A\vec{x} = b_1 y_1^2 + b_2 y_2^2 + \cdots + b_n y_n^2 = \vec{y}^T D\vec{y} = \vec{x}^T P^T DP\vec{x},$$

where D is diagonal

$$D = \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & b_n \end{pmatrix}.$$

By Exercise 4.29, the equality $\vec{x}^T A\vec{x} = \vec{x}^T P^T DP\vec{x}$ means $A = P^T DP$.

Exercise 4.30. Prove that all the diagonal terms in a positive definite matrix must be positive.

Exercise 4.31. For any $n \times n$ matrix A and $1 \le i_1 < i_2 < \cdots < i_k \le n$, let $A(i_1, i_2, \dots, i_k)$ be the $k \times k$ submatrix obtained by selecting the i_p-rows and i_q-columns, $1 \le p, q \le k$. If A is positive definite, prove that $A(i_1, i_2, \dots, i_k)$ is also positive definite. This generalizes Exercise 4.30.

Exercise 4.32. Determine whether the matrix is positive definite.

1. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$.
2. $\begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & -2 \\ 1 & -2 & 3 \end{pmatrix}$.
3. $\begin{pmatrix} 1 & -1 & -1 \\ -1 & 2 & -2 \\ -1 & -2 & 3 \end{pmatrix}$.

Exercise 4.33. Find the condition for $\begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix}$ and $\begin{pmatrix} a & 1 & 0 \\ 1 & a & 1 \\ 0 & 1 & a \end{pmatrix}$ to be positive definite, and generalise to $n \times n$ matrix.

4.2 Orthogonality

In Section 3.3, we learned that the essence of linear algebra is not individual vectors, but subspaces. The essence of span is sum of subspace, and the essence of linear independence is that the sum is direct. Similarly, the essence of orthogonal vectors is orthogonal subspaces.

4.2.1 Orthogonal Sum

Two vectors are orthogonal if their inner product is 0. Therefore we may define the orthogonality of two subspaces.

Definition 4.2.1. Two spaces H and H' are *orthogonal* and denoted $H \perp H'$, if $\langle \vec{u}, \vec{v} \rangle = 0$ for all $\vec{u} \in H$ and $\vec{v} \in H'$.

Clearly, $\vec{u} \perp \vec{v}$ if and only if $\mathbb{R}\vec{u} \perp \mathbb{R}\vec{v}$. We note that $\vec{u}$ and $\vec{v}$ are linearly dependent if the angle between them is 0 (or π). Although the two vectors become linearly independent when the angle is slightly away from 0, we still feel they are almost dependent. In fact, we feel they are more and more independent when the angle gets bigger and bigger. We feel the greatest independence when the angle is $\frac{1}{2}\pi$. This motivates the following result.

Theorem 4.2.2. *If subspaces $H_1, H_2, \dots, H_n$ are pairwise orthogonal, then $H_1 + H_2 + \cdots + H_n$ is a direct sum.*

Proof. Suppose $\vec{h}_i \in H_i$ satisfies $\vec{h}_1 + \vec{h}_2 + \cdots + \vec{h}_n = \vec{0}$. Then by the pairwise orthogonality, we have

$$0 = \langle \vec{h}_i, \vec{h}_1 + \vec{h}_2 + \cdots + \vec{h}_n \rangle = \langle \vec{h}_i, \vec{h}_1 \rangle + \langle \vec{h}_i, \vec{h}_2 \rangle + \cdots + \langle \vec{h}_i, \vec{h}_n \rangle = \langle \vec{h}_i, \vec{h}_i \rangle.$$

By the positivity of the inner product, this implies $\vec{h}_i = \vec{0}$. □

To emphasis the orthogonality between subspaces, we express the *orthogonal (direct) sum* as $H_1 \perp H_2 \perp \cdots \perp H_k$. Similarly, if

$$\begin{aligned} V &= H_1 \perp H_2 \perp \cdots \perp H_k, \\ W &= H_1' \perp H_2' \perp \cdots \perp H_k', \end{aligned}$$

and $L \colon V \to W$ satisfies $L(H_i) \subset H_i'$, then we have linear transformations $L_i \colon H_i \to H_i'$ and denote the direct sum of linear transformations (see Example 3.3.9) as an *orthogonal sum*

$$L = L_1 \perp L_2 \perp \cdots \perp L_n \colon H_1 \perp H_2 \perp \cdots \perp H_k \to H_1' \perp H_2' \perp \cdots \perp H_k'.$$

Example 4.2.1. We have the direct sum $P_n = P_n^{\text{even}} \oplus P_n^{\text{odd}}$ in Example 3.3.1. Moreover, the two subspaces are orthogonal with respect to the inner product in Exercise 4.21. Therefore we have $P_n = P_n^{\text{even}} \perp P_n^{\text{odd}}$.

Exercise 4.34. What is the subspace orthogonal to itself?

Exercise 4.35. Prove that $H_1+H_2+\cdots+H_m \perp H_1'+H_2'+\cdots+H_n'$ if and only if $H_i \perp H_j'$ for all i and j. What does this tell you about $\mathbb{R}\vec{v}_1+\mathbb{R}\vec{v}_2+\cdots+\mathbb{R}\vec{v}_m \perp \mathbb{R}\vec{w}_1+\mathbb{R}\vec{w}_2+\cdots+\mathbb{R}\vec{w}_n$?

Exercise 4.36. Show that $H_1 + H_2 + H_3 + H_4 + H_5$ is an orthogonal sum if and only if $H_1 + H_2 + H_3$, $H_4 + H_5$, $(H_1 + H_2 + H_3) + (H_4 + H_5)$ are orthogonal sums. In general, extend Proposition 3.3.2.

In case $H_i = \mathbb{R}\vec{v}_i$, the pairwise orthogonal property in Theorem 4.2.2 means $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ are pairwise orthogonal, i.e., $\vec{v}_i \cdot \vec{v}_j = 0$ whenever $i \neq j$. We call

$\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ an *orthogonal set*. Then Theorem 4.2.2 says that an orthogonal set of nonzero vectors is linearly independent.

If α is an orthogonal set of k nonzero vectors, and $k = \dim V$, then α is a basis of V, called an *orthogonal basis*.

If all vectors in an orthogonal set have unit length, i.e., we have

$$\vec{v}_i \cdot \vec{v}_j = \delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j, \end{cases}$$

then we have an *orthonormal set*. If the number of vectors in an orthonormal set is $\dim V$, then we have an *orthonormal basis*.

An orthogonal set of nonzero vectors can be changed to an orthonormal set by dividing the vector lengths.

Example 4.2.2. The vectors $\vec{v}_1 = (2, 2, -1)$ and $\vec{v}_2 = (2, -1, 2)$ are orthogonal (with respect to the dot product). To get an orthogonal basis of $\mathbb{R}^3$, we need to add one vector $\vec{v}_3 = (x, y, z)$ satisfying

$$\vec{v}_3 \cdot \vec{v}_1 = 2x + 2y - z = 0, \quad \vec{v}_3 \cdot \vec{v}_2 = 2x - y + 2z = 0.$$

The solution is $y = z = -2x$. Taking $x = -1$, or $\vec{v}_3 = (-1, 2, 2)$, we get an orthogonal basis $\{(2, 2, -1), (2, -1, 2), (-1, 2, 2)\}$. By dividing the length $\|\vec{v}_1\| = \|\vec{v}_2\| = \|\vec{v}_3\| = 3$, we get an orthonormal basis $\{\frac{1}{3}(2, 2, -1), \frac{1}{3}(2, -1, 2), \frac{1}{3}(-1, 2, 2)\}$.

Example 4.2.3. By the inner product in Example 4.1.3, we have

$$\langle t, t - a\rangle = \int_0^1 t(t - a)dt = \frac{1}{3} - \frac{1}{2}a.$$

Therefore t is orthogonal to $t - a$ if and only if $a = \frac{2}{3}$. By

$$\|t\| = \sqrt{\int_0^1 t^2 dt} = \frac{1}{\sqrt{3}}, \quad \|t\| = \sqrt{\int_0^1 \left(t - \frac{2}{3}\right)^2 dt} = \frac{1}{3},$$

we get an orthonormal basis $\{\sqrt{3}t, 3t - 2\}$ of P_1.

Exercise 4.37. For an orthogonal set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$, prove the *Pythagorean identity*

$$\|\vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_n\|^2 = \|\vec{v}_1\|^2 + \|\vec{v}_2\|^2 + \cdots + \|\vec{v}_n\|^2.$$

Exercise 4.38. Show that an orthonormal basis in $\mathbb{R}^2$ is either $\{(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\}$ or $\{(\cos\theta, \sin\theta), (\sin\theta, -\cos\theta)\}$.

Exercise 4.39. Find an orthonormal basis of $\mathbb{R}^n$ with the inner product in Exercise 4.18.

Exercise 4.40. For P_2 with the inner product in Example 4.1.3, find an orthogonal basis of P_2 of the form $a_0, b_0 + b_1 t, c_0 + c_1 t + c_2 t^2$. Then convert to an orthonormal basis.

4.2.2 Orthogonal Complement

By Exercise 4.35, if H is orthogonal to H_1' and H_2', then H is orthogonal to $H_1'+H_2'$. This suggests there is a maximal subspace orthogonal to H. This maximal subspace is given below.

Definition 4.2.3. The *orthogonal complement* of a subspace H of an inner product space V is

$$H^\perp = \{\vec{v}\colon \langle \vec{v}, \vec{h}\rangle = 0 \text{ for all } \vec{h} \in H\}.$$

The following shows that $H^\perp$ is indeed a subspace

$$\begin{aligned}
\vec{v}, \vec{w} \in H^\perp \iff& \langle \vec{v}, \vec{h}\rangle = \langle \vec{w}, \vec{h}\rangle = 0 \text{ for all } \vec{h} \in H \\
\implies& \langle a\vec{v} + b\vec{w}, \vec{h}\rangle = a\langle \vec{v}, \vec{h}\rangle + b\langle \vec{w}, \vec{h}\rangle = 0 \text{ for all } \vec{h} \in H \\
\iff& a\vec{v} + b\vec{w} \in H^\perp.
\end{aligned}$$

Proposition 4.2.4. *The orthogonal complement has the following properties.*

1. $H \subset H'$ *implies* $H^\perp \supset H'^\perp$.
2. $(H_1 + H_2 + \cdots + H_n)^\perp = H_1^\perp \cap H_2^\perp \cap \cdots \cap H_n^\perp$.
3. $H \subset (H^\perp)^\perp$.
4. *If* $V = H \perp H'$, *then* $H' = H^\perp$ *and* $H = (H')^\perp$.

Proof. We only prove the fourth statement. The other properties are left as exercise.

Assume $V = H \perp H'$. By the definition of orthogonal subspaces, we have $H' \subset H^\perp$. Conversely, we express any $\vec{x} \in H^\perp$ as $\vec{x} = \vec{h} + \vec{h}'$ with $\vec{h} \in H$ and $\vec{h}' \in H'$. Then we have

$$0 = \langle \vec{x}, \vec{h}\rangle = \langle \vec{h}, \vec{h}\rangle + \langle \vec{h}', \vec{h}\rangle = \langle \vec{h}, \vec{h}\rangle.$$

Here the first equality is due to $\vec{x} \in H^\perp, \vec{h} \in H$, and the third equality is due to $\vec{h}' \in H^\perp, \vec{h} \in H$. The overall equality implies $\vec{h} = \vec{0}$. Therefore $\vec{x} = \vec{h}' \in H'$. This proves $H^\perp \subset H'$. □

Exercise 4.41. Prove the first three statements in Proposition 4.2.4.

The following can be obtained from Exercise 4.35 and gives a practical way of computing the orthogonal complement.

Proposition 4.2.5. *The orthogonal complement of* $\mathbb{R}\vec{v}_1 + \mathbb{R}\vec{v}_2 + \cdots + \mathbb{R}\vec{v}_k$ *is all the vector orthogonal to* $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$.

Consider an $m \times n$ matrix $A = (\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n)$, $\vec{v}_i \in \mathbb{R}^m$. By Proposition 4.2.5, the orthogonal complement of the column space Col$A$ consists of vectors $\vec{x} \in \mathbb{R}^m$ satisfying $\vec{v}_1 \cdot \vec{x} = \vec{v}_2 \cdot \vec{x} = \cdots = \vec{v}_n \cdot \vec{x} = 0$. By the formula in Example 4.1.1, this means $A^T\vec{x} = \vec{0}$, or the null space of A^T. Therefore we get

$$(\mathrm{Col}A)^\perp = \mathrm{Nul}A^T.$$

Taking transpose, we also get

$$(\mathrm{Row}A)^\perp = \mathrm{Nul}A.$$

Example 4.2.4. The orthogonal complement of the line $H = \mathbb{R}(1,1,1)$ consists of vectors $(x,y,z) \in \mathbb{R}^3$ satisfying $(1,1,1)\cdot(x,y,z) = x+y+z = 0$. This is the plane in Example 2.1.13. In general, the solutions of $a_1x_1 + a_2x_2 + \cdots + a_mx_m = 0$ is the hyperplane orthogonal to the line $\mathbb{R}(a_1, a_2, \dots, a_m)$.

For another example, Example 3.2.4 shows that the orthogonal complement of $\mathbb{R}(1,4,7,10) + \mathbb{R}(2,5,8,11) + \mathbb{R}(3,6,9,12)$ is $\mathbb{R}(1,-2,0,0) + \mathbb{R}(2,-3,0,1)$.

Example 4.2.5. We try to calculate the orthogonal complement of P_1 (span of 1 and t) in P_3 with respect to the inner product in Example 4.1.3. A polynomial $f = a_0 + a_1t + a_2t^2 + a_3t^3$ is in the orthogonal complement if and only if

$$\begin{aligned}
\langle 1, f\rangle &= \int_0^1 (a_0 + a_1t + a_2t^2 + a_3t^3)dt = a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 + \frac{1}{4}a_3 = 0,\\
\langle t, f\rangle &= \int_0^1 t(a_0 + a_1t + a_2t^2 + a_3t^3)dt = \frac{1}{2}a_0 + \frac{1}{3}a_1 + \frac{1}{4}a_2 + \frac{1}{5}a_3 = 0.
\end{aligned}$$

We find two linearly independent solutions $2 - 9t + 10t^3$ and $1 - 18t^2 + 20t^3$, which form a basis of the orthogonal complement.

Exercise 4.42. Find the orthogonal complement of the span of $(1,4,7,10)$, $(2,5,8,11)$, $(3,6,9,12)$ with respect to the inner product $\langle (x_1,x_2,x_3,x_4),(y_1,y_2,y_3,y_4)\rangle = x_1y_1 + 2x_2y_2 + 3x_3y_3 + 4x_4y_4$.

Exercise 4.43. Find the orthogonal complement of P_1 in P_3 respect to the inner products in Exercises 4.7, 4.20, 4.21.

Exercise 4.44. Find all vectors orthogonal to the given vectors.

1. $(1,4,7)$, $(2,5,8)$, $(3,6,9)$.
2. $(1,2,3)$, $(4,5,6)$, $(7,8,9)$.
3. $(1,2,3,4)$, $(2,3,4,1)$, $(3,4,1,2)$.
4. $(1,0,1,0)$, $(0,1,0,1)$, $(1,0,0,1)$.

Exercise 4.45. Redo Exercise 4.44 with respect to the inner product in Exercise 4.18.

Exercise 4.46. Find all polynomials of degree 2 orthogonal to the given functions, with respect to the inner product in Example 4.1.3.

1. $1, t$. 2. $1, t, 1+t$. 3. $\sin t, \cos t$. 4. $1, t, t^2$.

Exercise 4.47. Redo Exercise 4.46 with respect to the inner product in Exercises 4.20, 4.21.

4.2.3 Orthogonal Projection

In Section 3.3.2, a direct sum $V = H \oplus H'$ is associated with a projection

$$P(\vec{h} + \vec{h}') = \vec{h}, \quad \vec{h} \in H,\ \vec{h}' \in H'.$$

The projection depends on the choice of the direct summand H' of H in V.

If $V = H \perp H'$, then the projection is an orthogonal projection. We will prove that, if H is finite dimensional, then $V = H \perp H^{\perp}$, and H' must be $H^{\perp}$. Therefore the orthogonal projection depends only on H, and we denote it by $\text{proj}_H\vec{x}$. Since we have not yet proved $V = H \perp H^{\perp}$, we define the orthogonal projection without using the result.

Definition 4.2.6. The *orthogonal projection* of $\vec{x} \in V$ onto a subspace $H \subset V$ is the vector $\vec{h} \in H$ satisfying $\vec{x} - \vec{h} \perp H$.

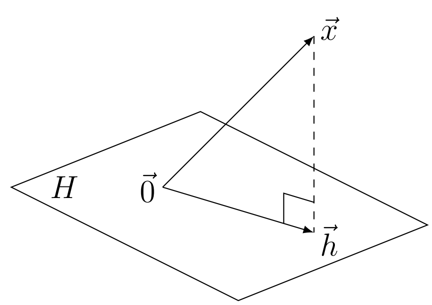

Figure 4.2.1: Orthogonal projection.

For the uniqueness of $\vec{h}$ (i.e., $\text{proj}_H\vec{x}$ is not ambiguous), we note that $\vec{x} - \vec{h} \perp H$ and $\vec{x} - \vec{h}' \perp H$ imply $\vec{h}' - \vec{h} = (\vec{x} - \vec{h}) - (\vec{x} - \vec{h}') \perp H$. Then by $\vec{h}' - \vec{h} \in H$, we get $\vec{h}' - \vec{h} = \vec{0}$.

Proposition 4.2.7. *Orthogonal projection onto H exists if and only if $V = H \perp H^{\perp}$. Moreover, we have $(H^{\perp})^{\perp} = H$.*

The orthogonal sum $V = H \perp H^{\perp}$ implies that the orthogonal projection is the projection associated to a direct sum, and is therefore a linear transformation.

Proof. Suppose the orthogonal projection onto H exists. Let the orthogonal projection of $\vec{x} \in V$ be $\vec{h} \in H$. Then $\vec{h}' = \vec{x} - \vec{h} \in H^{\perp}$, and we have $\vec{x} = \vec{h} + \vec{h}'$ with

$\vec{h} \in H$ and $\vec{h}' \in H^\perp$. This proves $V = H + H^\perp$. Since the sum $H + H^\perp$ is always an orthogonal sum, we get $V = H \perp H^\perp$.

Suppose $V = H \perp H^\perp$. Then the projection $P\colon V \to H$ assocated to the direct sum satisfies $\vec{x} - P(\vec{x}) \in H^\perp$. By definition, $P(\vec{x})$ is the orthogonal projection onto H. On the other hand, we may also apply the fourth statement in Proposition 4.2.4 to $H' = H^\perp$ to get $(H^\perp)^\perp = H$.

It remains to prove that $(H^\perp)^\perp = H$ implies $V = H \perp H^\perp$. □

The following gives the formula of the orthogonal projection in case H has a finite orthogonal basis.

Proposition 4.2.8. *If $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is an orthogonal basis of a subspace $H \subset V$, then the orthogonal projection of $\vec{x}$ on H is*

$$\text{proj}_H \vec{x} = \frac{\vec{x} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 + \frac{\vec{x} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 + \cdots + \frac{\vec{x} \cdot \vec{v}_k}{\vec{v}_k \cdot \vec{v}_k}\vec{v}_k.$$

If the basis is orthonormal, then

$$\text{proj}_H \vec{x} = (\vec{x} \cdot \vec{v}_1)\vec{v}_1 + (\vec{x} \cdot \vec{v}_2)\vec{v}_2 + \cdots + (\vec{x} \cdot \vec{v}_k)\vec{v}_k.$$

In Section 4.2.4, we will prove that any finite dimensional subspace has an orthogonal basis. Then the formula in the proposition shows the existence of orthogonal projection.

We also note that, if $\vec{x} \in H = \text{Span}\alpha$, then $\vec{x} = \text{proj}_H \vec{x}$, and we get

$$\vec{x} = \frac{\vec{x} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 + \frac{\vec{x} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 + \cdots + \frac{\vec{x} \cdot \vec{v}_k}{\vec{v}_k \cdot \vec{v}_k}\vec{v}_k.$$

Proof. Let $\vec{h}$ be the formula in the proposition. We need to verify $\vec{x} - \vec{h} \perp H$. By Proposition 4.2.5 (and Exercise 4.35), we only need to show $(\vec{x} - \vec{h}) \cdot \vec{v}_i = 0$. The following proves $\vec{h} \cdot \vec{v}_i = \vec{x} \cdot \vec{v}_i$.

$$\begin{aligned}
&\left(\frac{\vec{x} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 + \frac{\vec{x} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 + \cdots + \frac{\vec{x} \cdot \vec{v}_k}{\vec{v}_k \cdot \vec{v}_k}\vec{v}_k\right) \cdot \vec{v}_i \\
&= \frac{\vec{x} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 \cdot \vec{v}_i + \frac{\vec{x} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 \cdot \vec{v}_i + \cdots + \frac{\vec{x} \cdot \vec{v}_k}{\vec{v}_k \cdot \vec{v}_k}\vec{v}_k \cdot \vec{v}_i \\
&= \frac{\vec{x} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i}\vec{v}_i \cdot \vec{v}_i = \vec{x} \cdot \vec{v}_i.
\end{aligned}$$

□

Example 4.2.6. In Example 2.1.13, we found the formula for the orthogonal projection onto the subspace $H \subset \mathbb{R}^3$ given by $x + y + z = 0$. Now we find the formula again by using Proposition 4.2.8. To get an orthogonal basis of H, we start with $\vec{v}_1 = (1, -1, 0) \in H$. Since $\dim H = 2$, we only need to find $\vec{v}_2 = (x, y, z) \in H$ satisfying $\vec{v}_1 \cdot \vec{v}_2 = 0$. This means

$$x + y + z = 0, \quad x - y = 0.$$

The solution is $x = y$, $z = -2y$, with y arbitrary. By choosing $y = 1$, we get an orthogonal basis $\vec{v}_1 = (1, -1, 0)$, $\vec{v}_2 = (1, 1, -2)$ of H. Then by Proposition 4.2.8, we get

$$\begin{aligned}\text{proj}_{\{x+y+z=0\}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \frac{x_1 - x_2}{1+1+0} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \frac{x_1 + x_2 - 2x_3}{1+1+4} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 2x_1 - x_2 - x_3 \\ -x_1 + 2x_2 - x_3 \\ -x_1 - x_2 + 2x_3 \end{pmatrix}.\end{aligned}$$

Example 4.2.7 (Fourier series)**.** The inner product in Example 4.3.3 is defined for all continuous (in fact square integrable on $[0, 2\pi]$ is enough) periodic functions on $\mathbb{R}$ of period 2π. For integers $m \neq n$, we have

$$\begin{aligned}\langle \cos mt, \cos nt \rangle &= \frac{1}{2\pi} \int_0^{2\pi} \cos mt \cos nt dt \\ &= \frac{1}{4\pi} \int_0^{2\pi} (\cos(m+n)t + \cos(m-n)t) dt \\ &= \frac{1}{4\pi} \left(\frac{\sin(m+n)t}{m+n} + \frac{\sin(m-n)t}{m-n} \right)_0^{\pi} = 0.\end{aligned}$$

We may similarly find $\langle \sin mt, \sin nt \rangle = 0$ for $m \neq n$ and $\langle \cos mt, \sin nt \rangle = 0$. Therefore the vectors $(1 = \cos 0t)$

$$1, \cos t, \sin t, \cos 2t, \sin 2t, \dots, \cos nt, \sin nt, \dots$$

form an orthogonal set.

We can also easily find the lengths $\|\cos nt\| = \|\sin nt\| = \frac{1}{\sqrt{2}}$ for $n \neq 0$ and $\|1\| = 1$. If we pretend Proposition 4.2.8 works for infinite sum, then for a periodic function $f(t)$ of period 2π, we may expect

$$f(t) = a_0 + a_1 \cos t + b_1 \sin t + a_2 \cos 2t + b_2 \sin 2t + \cdots + a_n \cos nt + b_n \sin nt + \cdots,$$

with

$$\begin{aligned}a_0 &= \langle f(t), 1 \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t) dt, \\ a_n &= 2\langle f(t), \cos nt \rangle = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos nt dt, \\ b_n &= 2\langle f(t), \sin nt \rangle = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin nt dt.\end{aligned}$$

This is the Fourier series.

4.2.4 Gram-Schmidt Process

The following gives an inductive process for turning a basis of a subspace to an orthogonal basis of the subspace. In particular, it implies the existence of orthogonal basis for finite dimensional subspace. By dividing the vector length, we may even get orthonormal basis.

Proposition 4.2.9. *Suppose $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a linearly independent set in an inner product space. Then there is an orthogonal set $\beta = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n\}$ of nonzero vectors satisfying*

$$\mathbb{R}\vec{w}_1 + \mathbb{R}\vec{w}_2 + \cdots + \mathbb{R}\vec{w}_i = \mathbb{R}\vec{v}_1 + \mathbb{R}\vec{v}_2 + \cdots + \mathbb{R}\vec{v}_i, \quad i = 1, 2, \dots, n.$$

In particular, any finite dimensional inner product space has an orthonormal basis.

Proof. We start with $\vec{w}_1 = \vec{v}_1$ and get $\mathbb{R}\vec{w}_1 = \mathbb{R}\vec{v}_1$. Then we inductively assume $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_i$ are constructed, and $\mathbb{R}\vec{w}_1 + \mathbb{R}\vec{w}_2 + \cdots + \mathbb{R}\vec{w}_i = \mathbb{R}\vec{v}_1 + \mathbb{R}\vec{v}_2 + \cdots + \mathbb{R}\vec{v}_i$ is satisfied. We denote the subspace by H_i and take

$$\vec{w}_{i+1} = \vec{v}_{i+1} - \text{proj}_{H_i}\vec{v}_{i+1}.$$

Since α is linearly independent, we have $\vec{v}_{i+1} \notin H_i$. Therefore $\vec{w}_{i+1} \neq \vec{0}$. Moreover, by the definition of orthogonal projection, we get $\vec{w}_{i+1} \perp H_i$. This means $\vec{w}_{i+1} \perp \vec{w}_1, \vec{w}_{i+1} \perp \vec{w}_2, \dots, \vec{w}_{i+1} \perp \vec{w}_i$. Therefore we get an orthogonal basis at the end. □

Example 4.2.8. In Example 2.1.13, the subspace H of $\mathbb{R}^3$ given by the equation $x+y+z = 0$ has basis $\vec{v}_1 = (1, -1, 0)$, $\vec{v}_2 = (1, 0, -1)$. Then we derive an orthogonal basis of H

$$\begin{aligned}
\vec{w}_1 &= \vec{v}_1 = (1, -1, 0), \\
\vec{w}_2' &= \vec{v}_2 - \frac{\langle \vec{v}_2, \vec{w}_1 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle}\vec{w}_1 = (1, 0, -1) - \frac{1+0+0}{1+1+0}(1, -1, 0) = \frac{1}{2}(1, 1, -2), \\
\vec{w}_2 &= 2\vec{w}_2' = (1, 1, -2).
\end{aligned}$$

Here we simplify the choice of vectors by suitable scalar multiplication, which does not change orthogonal basis. The orthogonal basis we get is the same as the one used in Example 4.2.6.

Example 4.2.9. In Example 1.3.14, the vectors $\vec{v}_1 = (1, 2, 3)$, $\vec{v}_2 = (4, 5, 6)$, $\vec{v}_3 = (7, 8, 10)$ form a basis of $\mathbb{R}^3$. We apply the Gram-Schmidt process to get an orthogonal basis of $\mathbb{R}^3$.

$$
\begin{aligned}
\vec{w}_1 &= \vec{v}_1 = (1,2,3),\\
\vec{w}_2' &= \vec{v}_2 - \frac{\langle \vec{v}_2, \vec{w}_1\rangle}{\langle \vec{w}_1, \vec{w}_1\rangle}\vec{w}_1 = (4,5,6) - \frac{4+10+18}{1+4+9}(1,2,3) = -\frac{3}{7}(4,1,-2),\\
\vec{w}_2 &= (4,1,-2),\\
\vec{w}_3' &= \vec{v}_3 - \frac{\langle \vec{v}_3, \vec{w}_1\rangle}{\langle \vec{w}_1, \vec{w}_1\rangle}\vec{w}_1 - \frac{\langle \vec{v}_3, \vec{w}_2\rangle}{\langle \vec{w}_2, \vec{w}_2\rangle}\vec{w}_2\\
&= (7,8,10) - \frac{7+16+30}{1+4+9}(1,2,3) - \frac{28+8-20}{16+1+4}(4,1,-2) = -\frac{1}{6}(1,-2,1),\\
\vec{w}_3 &= (1,-2,1).
\end{aligned}
$$

Example 4.2.10. The natural basis $\{1, t, t^2\}$ of P_2 is not orthogonal with respect to the inner product in Example 4.1.3. We improve the basis to become orthogonal

$$
\begin{aligned}
f_1 &= 1,\\
f_2 &= t - \frac{\int_0^1 t\cdot 1 dt}{\int_0^1 1^2 dt}1 = t - \frac{1}{2},\\
f_3 &= t^2 - \frac{\int_0^1 t^2\cdot 1 dt}{\int_0^1 1^2 dt}1 - \frac{\int_0^1 t^2\left(t-\frac{1}{2}\right)dt}{\int_0^1 \left(t-\frac{1}{2}\right)^2 dt}\left(t-\frac{1}{2}\right) = t^2 - t + \frac{1}{6}.
\end{aligned}
$$

By rescaling, we find that $1, 2t-1, 6t^2-6t+1$ is an orthogonal basis of P_2.

We may use the orthogonal basis $1, 2t-1$ of P_1 to calculate the orthogonal projection of P_3 to P_1

$$
\begin{aligned}
\mathrm{proj}_{P_1} t^2 &= \frac{\int_0^1 t^2 dt}{\int_0^1 1^2 dt}1 + \frac{\int_0^1 (1-2t)t^2 dt}{\int_0^1 (1-2t)^2 dt}(1-2t) = \frac{1}{3}1 - \frac{1}{2}(1-2t) = -\frac{1}{6} + t,\\
\mathrm{proj}_{P_1} t^3 &= \frac{\int_0^1 t^3 dt}{\int_0^1 1^2 dt}1 + \frac{\int_0^1 (1-2t)t^3 dt}{\int_0^1 (1-2t)^2 dt}(1-2t) = \frac{1}{4}1 - \frac{9}{20}(1-2t) = -\frac{1}{5} + \frac{9}{10}t.
\end{aligned}
$$

Combined with $\mathrm{proj}_{P_1} 1 = 1$ and $\mathrm{proj}_{P_1} t = t$, we get

$$
\begin{aligned}
\mathrm{proj}_{P_1}(a_0 + a_1 t + a_2 t^2 + a_3 t^3) &= a_0 + a_1 t + a_2\left(-\frac{1}{6} + t\right) + a_3\left(-\frac{1}{5} + \frac{9}{10}t\right)\\
&= \left(a_0 - \frac{1}{6}a_2 - \frac{1}{5}a_3\right) + \left(a_1 + a_2 + \frac{9}{10}a_3\right)t.
\end{aligned}
$$

Example 4.2.11. With respect to the inner product in Exercise 4.21, even and odd polynomials are always orthogonal. Therefore to find an orthogonal basis of P_3 with respect to this inner product, we may apply the Gram-Schmidt process to $1, t^2$ and

t, t^3 separately, and then simply combine the results together. Specifically, we have

$$t^2 - \frac{\int_{-1}^{1} t^2 \cdot 1dt}{\int_{-1}^{1} 1^2 dt} 1 = t^2 - \frac{\int_{0}^{1} t^2 \cdot 1dt}{\int_{0}^{1} 1^2 dt} 1 = t^2 - \frac{1}{3},$$
$$t^3 - \frac{\int_{-1}^{1} t^3 \cdot tdt}{\int_{-1}^{1} t^2 dt} t = t^3 - \frac{\int_{0}^{1} t^3 \cdot tdt}{\int_{0}^{1} t^2 dt} t = t^3 - \frac{3}{5} t.$$

Therefore $1, t, t^2 - \frac{1}{3}, t^3 - \frac{3}{5}t$ form an orthogonal basis of P_3. By

$$\int_{-1}^{1} 1^2 dt = 2, \qquad \int_{-1}^{1} t^2 dt = \frac{2}{3},$$
$$\int_{-1}^{1} \left(t^2 - \frac{1}{3}\right)^2 dt = \frac{8}{3^2 \cdot 5}, \qquad \int_{-1}^{1} \left(t^3 - \frac{3}{5}t\right)^2 dt = \frac{8}{5^2 \cdot 7},$$

we divide the edge lengths and get an orthonormal basis

$$\frac{1}{\sqrt{2}},\ \frac{\sqrt{3}}{\sqrt{2}} t,\ \frac{2\sqrt{2}}{\sqrt{5}}(3t^2 - 1),\ \frac{2\sqrt{2}}{\sqrt{7}}(5t^3 - 3t).$$

Exercise 4.48. Find an orthogonal basis of the subspace in Example 4.2.8 by starting with $\vec{v}_2$ and then use $\vec{v}_1$.

Exercise 4.49. Find an orthogonal basis of the subspace in Example 4.2.8 with respect to the inner product $\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = x_1y_1 + 2x_2y_2 + 3x_3y_3$. Then extend the orthogonal basis to an orthogonal basis of $\mathbb{R}^3$.

Exercise 4.50. Apply the Gram-Schmidt process to $1, t, t^2$ with respect to the inner products in Exercises 4.7 and 4.20.

Exercise 4.51. Use the orthogonal basis in Example 4.2.3 to calculate the orthogonal projection in Example 4.2.10.

Exercise 4.52. Find the orthogonal projection of general polynomial onto P_1 with respect to the inner product in Example 4.1.3. What about the inner products in Exercises 4.7, 4.20, 4.21?

4.2.5 Property of Orthogonal Projection

Proposition 4.2.9 implies that any finite dimensional subspace H of an inner product space V has orthogonal basis. Then Proposition 4.2.8 implies that the orthogonal projection onto H exists. Further, Proposition 4.2.7 shows that $(H^\perp)^\perp = H$.

In Section 4.2.2, for a matrix A, we have

$$(\mathrm{Col}A)^\perp = \mathrm{Nul}A^T, \quad (\mathrm{Row}A)^\perp = \mathrm{Nul}A.$$

Applying $(H^\perp)^\perp = H$, we get

$$(\text{Nul}A)^\perp = \text{Row}A, \quad (\text{Nul}A^T)^\perp = \text{Col}A.$$

We remark that the equality $(\text{Nul}A^T)^\perp = \text{Col}A$ means that $A\vec{x} = \vec{b}$ has solution if and only if $\vec{b}$ is orthogonal to all the solutions of the equation $A^T\vec{x} = \vec{0}$. Similarly, the equality $(\text{Row}A)^\perp = \text{Nul}A$ means that $A\vec{x} = \vec{0}$ if and only if $\vec{x}$ is orthogonal to all $\vec{b}$ such that $A^T\vec{x} = \vec{b}$ has solution. These are called the *complementarity principles*.

Example 4.2.12. By Example 4.2.4 and $(H^\perp)^\perp = H$, the orthogonal complement of $\mathbb{R}(1,-2,0,0) + \mathbb{R}(2,-3,0,1)$ is $\mathbb{R}(1,4,7,10) + \mathbb{R}(2,5,8,11) + \mathbb{R}(3,6,9,12)$.

Similarly, in Example 4.2.5, the orthogonal complement of $\mathbb{R}(2 - 9t + 10t^3) + \mathbb{R}(1 - 18t^2 + 20t^3)$ in P_3 is P_1.

By Section 3.3.2, the orthogonal sum $V = H \perp H^\perp$ implies (see Exercise 3.65)

$$\vec{x} = \text{proj}_H\vec{x} + \text{proj}_{H^\perp}\vec{x}.$$

In fact, if subspaces H_1, H_2 are orthogonal, then by Proposition 4.2.4, we have

$$V = H_1 \perp H_2 \perp (H_1 \perp H_2)^\perp = H_1 \perp H_2 \perp (H_1^\perp \cap H_2^\perp).$$

By Exercise 3.68, we get

$$\text{proj}_{H_1 \perp H_2} = \text{proj}_{H_1} + \text{proj}_{H_2}.$$

Example 4.2.13. We wish to calculate the matrix of the orthogonal projection to the subspace of $\mathbb{R}^n$ of dimension $n-1$

$$H = \{(x_1, x_2, \dots, x_n) \colon a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0\}.$$

It would be complicated to find an orthogonal basis of H, like what we did in Example 4.2.6. Instead, we take advantage of the fact that

$$H = (\mathbb{R}\vec{a})^\perp, \quad \vec{a} = (a_1, a_2, \dots, a_n).$$

Moreover, we may assume $\|\vec{a}\| = 1$ by dividing the length, then

$$\begin{aligned}
\text{proj}_{\{a_1x_1+a_2x_2+\cdots+a_nx_n=0\}}\vec{x} &= \vec{x} - \text{proj}_{\mathbb{R}\vec{a}}\vec{x} = \vec{x} - (a_1x_1 + a_2x_2 + \cdots + a_nx_n)\vec{a} \\
&= \begin{pmatrix} x_1 - a_1(a_1x_1 + a_2x_2 + \cdots + a_nx_n) \\ x_2 - a_2(a_1x_1 + a_2x_2 + \cdots + a_nx_n) \\ \vdots \\ x_n - a_n(a_1x_1 + a_2x_2 + \cdots + a_nx_n) \end{pmatrix} \\
&= \begin{pmatrix} 1-a_1^2 & -a_1a_2 & \cdots & -a_1a_n \\ -a_2a_1 & 1-a_2^2 & \cdots & -a_2a_n \\ \vdots & \vdots & & \vdots \\ -a_na_1 & -a_na_2 & \cdots & 1-a_n^2 \end{pmatrix}\vec{x}.
\end{aligned}$$

For $\vec{a} = \frac{1}{\sqrt{3}}(1,1,1)$, we get the same matrix as the earlier examples.

Example 4.2.14. Let

$$A = \begin{pmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{pmatrix}.$$

In Example 3.2.4, we get a basis $(1,-2,1,0),(2,-3,0,1)$ for the null space Nul$A$. The two vectors are not orthogonal. By

$$(2,-3,0,1) - \frac{(2,-3,0,1)\cdot(1,-2,1,0)}{(1,-2,1,0)\cdot(1,-2,1,0)}(1,-2,1,0) = \frac{1}{3}(2,-1,-4,3),$$

we get an orthogonal basis $(1,-2,1,0),(2,-1,-4,3)$ of Nul$A$. Then

$$\begin{aligned}\text{proj}_{\text{Nul}A}\vec{x} &= \frac{x_1-2x_2+x_3}{1+4+1}\begin{pmatrix}1\\-2\\1\\0\end{pmatrix} + \frac{2x_1-x_2-4x_3+3x_4}{4+1+16+9}\begin{pmatrix}2\\-1\\-4\\3\end{pmatrix} \\ &= \frac{1}{10}\begin{pmatrix} 3 & 4 & -1 & 2 \\ -4 & 7 & -2 & -1 \\ -1 & -2 & 7 & -4 \\ 2 & -1 & -4 & 3 \end{pmatrix}\vec{x}.\end{aligned}$$

By Row$A = (\text{Nul}A)^{\perp}$, we also get

$$\text{proj}_{\text{Row}A}\vec{x} = \vec{x} - \text{proj}_{\text{Nul}A}\vec{x} = \frac{1}{10}\begin{pmatrix} 7 & -4 & 1 & -2 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -2 & 1 & 4 & 7 \end{pmatrix}\vec{x}.$$

Example 4.2.15. In Example 4.2.1, we have orthogonal sum decomposition

$$P_n = P_n^{\text{even}} \oplus P_n^{\text{odd}}$$

with respect to the inner product in Exercise 4.21. By Example 4.2.11, we have further orthogonal sum decompositions

$$P_3^{\text{even}} = \mathbb{R}1 \oplus \mathbb{R}(t^2 - \tfrac{1}{3}), \quad P_3^{\text{odd}} = \mathbb{R}t \oplus \mathbb{R}(t^3 - \tfrac{3}{5}t).$$

This gives two orthogonal projections

$$\begin{aligned} P_{\mathbb{R}1}^{\text{even}}(a_0 + a_2t^2) &= P_{\mathbb{R}1}^{\text{even}}((a_0 + \tfrac{1}{3}a_2) + a_2(t^2 - \tfrac{1}{3})) = a_0 + \tfrac{1}{3}a_2 \colon P_3^{\text{even}} \to \mathbb{R}1 \\ P_{\mathbb{R}t}^{\text{odd}}(a_1t + a_3t^3) &= P_{\mathbb{R}t}^{\text{odd}}((a_1 + \tfrac{3}{5}a_3)t + a_2(t^3 - \tfrac{3}{5}t)) = (a_1 + \tfrac{3}{5}a_3)t \colon P_3^{\text{odd}} \to \mathbb{R}t. \end{aligned}$$

Then the orthogonal projection to $P_1 = \mathbb{R}1 \oplus \mathbb{R}t$ is

$$\begin{aligned} \text{proj}_{\mathbb{R}1\oplus\mathbb{R}t}(a_0 + a_1t + a_2t^2 + a_3t^3) &= P_{\mathbb{R}1}^{\text{even}}(a_0 + a_2t^2) + P_{\mathbb{R}t}^{\text{odd}}(a_1t + a_3t^3) \\ &= a_0 + \tfrac{1}{3}a_2 + (a_1 + \tfrac{3}{5}a_3)t. \end{aligned}$$

The idea of the example is extended in Exercise 4.53.

Exercise 4.53. Suppose $V = H_1 \perp H_2 \perp \cdots \perp H_k$ is an orthogonal sum. Suppose $H_i' \subset H_i$ is a subspace, and $P_i \colon H_i \to H_i'$ are the orthogonal projections inside subspaces. Prove that

$$\mathrm{proj}_{H_1'\perp H_2'\perp\cdots\perp H_k'} = P_1 \oplus P_2 \oplus \cdots \oplus P_k.$$

Exercise 4.54. Suppose $H_1, H_2, \ldots, H_k$ are pairwise orthogonal subspaces. Prove that

$$\mathrm{proj}_{H_1\perp H_2\perp\cdots\perp H_k} = \mathrm{proj}_{H_1} + \mathrm{proj}_{H_2} + \cdots + \mathrm{proj}_{H_k}.$$

Then use the orthogonal projection to a line

$$\mathrm{proj}_{\mathbb{R}\vec{u}} = \frac{\langle \vec{v}, \vec{u}\rangle}{\langle \vec{u}, \vec{u}\rangle}\vec{u}$$

to derive the formula in Proposition 4.2.8.

Exercise 4.55. Prove that $I = \mathrm{proj}_H + \mathrm{proj}_{H'}$ if and only if H' is the orthogonal complement of H.

Exercise 4.56. Find the orthogonal projection the subspace $x+y+z=0$ in $\mathbb{R}^3$ with respect to the inner product $\langle (x_1,x_2,x_3),(y_1,y_2,y_3)\rangle = x_1y_1 + 2x_2y_2 + 3x_3y_3$.

Exercise 4.57. Directly find the orthogonal projection to the row space in Example 4.2.14.

4.3 Adjoint

The inner product gives a dual pairing of an inner product space with itself, making the space self-dual. Then the dual linear transformation can be interpreted as linear transformations of the original vector spaces. This is the adjoint. We may use the adjoint to describe isometric linear transformations, which can also be described by orthonormal basis.

4.3.1 Adjoint

The inner product fits into Definition 2.4.6 of the dual pairing. The symmetric property of inner product implies that the two linear transformations in the definition are the same.

Proposition 4.3.1. *Suppose V is a finite dimensional inner product space. Then the inner product induces an isomorphism*

$$\vec{v} \mapsto \langle \cdot, \vec{v}\rangle \colon V \cong V^*.$$

Proof. By $\dim V^* = \dim V$, to show that $\vec{v} \mapsto \langle \cdot, \vec{v}\rangle$ is an isomorphism, it is sufficient to argue the one-to-one property, or the triviality of the kernel. A vector $\vec{v}$ is in this kernel means that $\langle \vec{x}, \vec{v}\rangle = 0$ for all $\vec{x}$. By taking $\vec{x} = \vec{v}$ and applying the positivity property of the inner product, we get $\vec{v} = \vec{0}$. □

A linear transformation $L: V \to W$ between vector spaces has the dual transformation $L^*: W^* \to V^*$. If V and W are finite dimensional inner product spaces, then we may use Proposition 4.3.1 to identify L^* with a linear transformation from W to V, which we still denote by L^*.

$$
\begin{array}{ccccccc}
W^* & \xrightarrow{L^*} & V^* & \qquad & \langle \cdot, \vec{w}\rangle_W & \xrightarrow{L^*} & \langle \cdot, L^*(\vec{w})\rangle_V \\
{\scriptstyle \langle\,,\rangle_W}\uparrow\cong & & \cong\uparrow{\scriptstyle \langle\,,\rangle_V} & & \uparrow & & \uparrow \\
W & \xrightarrow{L^*} & V & & \vec{w} & \xrightarrow{L^*} & L^*(\vec{w})
\end{array}
$$

If we start with $\vec{w} \in W$, then the definition means $L^*(\langle \cdot, \vec{w}\rangle_W) = \langle \cdot, L^*(\vec{w})\rangle_V$. Applying the equality to $\cdot = \vec{v} \in V$, we get the following definition.

Definition 4.3.2. Suppose $L: V \to W$ is a linear transformation between two inner product spaces. Then the *adjoint* of L is the linear transformation $L^*: W \to V$ satisfying

$$\langle L(\vec{v}), \vec{w}\rangle = \langle \vec{v}, L^*(\vec{w})\rangle \quad \text{for all } \vec{v} \in V,\ \vec{w} \in W.$$

Since the proof of Proposition 4.3.1 makes use of finite dimension, we know the adjoint exists for finite dimensional inner product spaces.

Example 4.3.1. In Example 4.1.1, the dot product on the Euclidean space is $\vec{x}\cdot\vec{y} = \vec{x}^T\vec{y}$. Then for $L(\vec{v}) = A\vec{v}$, we have

$$L(\vec{v})\cdot\vec{w} = (A\vec{v})^T\vec{w} = \vec{v}^T A^T\vec{w} = \vec{v}\cdot A^T\vec{w}.$$

Therefore $L^*(\vec{w}) = A^T\vec{w}$.

Example 4.3.2. Consider the vector space of polynomials with inner product in Example 4.1.3. The adjoint $D^*: P_{n-1} \to P_n$ of the derivative linear transformation $D(f) = f': P_n \to P_{n-1}$ is characterised by

$$\int_0^1 t^p D^*(t^q)dt = \int_0^1 D(t^p)t^q dt = \int_0^1 pt^{p-1}t^q dt = \frac{p}{p+q}, \quad 0 \le p \le n,\ 0 \le q \le n-1.$$

For fixed q, let $D^*(t^q) = x_0 + x_1 t + \cdots + x_n t^n$. Then we get a system of linear equations

$$\frac{1}{p+1}x_0 + \frac{1}{p+2}x_1 + \cdots + \frac{1}{p+n+1}x_n = \frac{p}{p+q}, \quad 0 \le p \le n.$$

The solution is quite non-trivial. For $n = 2$, we have

$$D^*(t^q) = \frac{2}{(1+q)(2+q)}(11 - 2q + 6(17+q)t - 90t^2), \quad q = 0, 1.$$

Example 4.3.3. Let V be the vector space of all smooth periodic functions on $\mathbb{R}$ of period 2π, with inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(t)g(t)dt.$$

The derivative operator $D(f) = f' \colon V \to V$ takes periodic functions to periodic functions. By the integration by parts and period 2π, we have

$$\begin{aligned}\langle D(f), g\rangle &= \frac{1}{2\pi}\int_0^{2\pi} f'(t)g(t)dt = f(2\pi)g(2\pi) - f(0)g(0) - \frac{1}{2\pi}\int_0^{2\pi} f(t)g'(t)dt \\ &= -\frac{1}{2\pi}\int_0^{2\pi} f(t)g'(t)dt = -\langle f, D(g)\rangle.\end{aligned}$$

This implies $D^* = -D$.

The same argument can be applied to the vector space of all smooth functions f on $\mathbb{R}$ satisfying $\lim_{t\to\infty} f^{(n)}(t) = 0$ for all $n \geq 0$, with inner product

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(t)g(t)dt.$$

We still get $D^* = -D$.

Exercise 4.58. Prove that a linear operator $L\colon V \to V$ satisfies

$$\langle L(\vec{u}), \vec{v}\rangle + \langle L(\vec{v}), \vec{u}\rangle = \langle L(\vec{u}+\vec{v}), \vec{u}+\vec{v}\rangle - \langle L(\vec{u}), \vec{u}\rangle - \langle L(\vec{v}), \vec{v}\rangle.$$

Then prove that $\langle L(\vec{v}), \vec{v}\rangle = 0$ for all $\vec{v}$ if and only if $L + L^* = 0$.

Exercise 4.59. Calculate the adjoint of the derivative linear transformation $D(f) = f' \colon P_n \to P_{n-1}$ with respect to the inner products in Exercises 4.7, 4.20, 4.21.

Exercise 4.60. Prove that $(L_1 \perp L_2 \perp \cdots \perp L_k)^* = L_1^* \perp L_2^* \perp \cdots \perp L_k^*$. What if the subspaces are not orthogonal?

Since the adjoint is only the "translation" of the dual via the inner product, properties of the dual can be translated into properties of the adjoint

$$I^* = I, \quad (L+K)^* = L^* + K^*, \quad (aL)^* = aL^*, \quad (L\circ K)^* = K^*\circ L^*, \quad (L^*)^* = L.$$

All properties can be directly verified by definition. See Exercise 4.61.

Example 4.3.1 and $(\text{Col}A)^\perp = \text{Nul}A^T$ suggest

$$(\text{Ran}L)^\perp = \text{Ker}L^*.$$

The following is a direct argument for the equality

$$\begin{aligned}
\vec{x} \in (\mathrm{Ran}L)^{\perp} &\iff \langle \vec{w}, \vec{x} \rangle = 0 \text{ for all } \vec{w} \in \mathrm{Ran}L \subset W \\
&\iff \langle L(\vec{v}), \vec{x} \rangle = 0 \text{ for all } \vec{v} \in V \\
&\iff \langle \vec{v}, L^*(\vec{x}) \rangle = 0 \text{ for all } \vec{v} \in V \\
&\iff L^*(\vec{x}) = \vec{0} \text{ for all } \vec{v} \in V \\
&\iff \vec{x} \in \mathrm{Ker}L^*.
\end{aligned}$$

Substituting L^* in place of L, we get $(\mathrm{Ran}L^*)^{\perp} = \mathrm{Ker}L$. Then we may use $(H^{\perp})^{\perp} = H$ to get

$$(\mathrm{Ran}L)^{\perp} = \mathrm{Nul}L^*, \quad (\mathrm{Ran}L^*)^{\perp} = \mathrm{Nul}L, \quad (\mathrm{Ker}L)^{\perp} = \mathrm{Ran}L^*, \quad (\mathrm{Ker}L^*)^{\perp} = \mathrm{Ran}L.$$

Exercise 4.61. Use the definition of adjoint to directly prove its properties.

Exercise 4.62. Prove that $\mathrm{Ker}L = \mathrm{Ker}L^*L$ and $\mathrm{rank}L = \mathrm{rank}L^*L$.

Exercise 4.63. Prove that the following are equivalent.

1. L is one-to-one.
2. L^* is onto.
3. L^*L is invertible.

Exercise 4.64. Prove that an operator P on an inner product space is the orthogonal projection to a subspace if and only if $P^2 = P = P^*$.

Exercise 4.65. Prove that there is a one-to-one correspondence between the decomposition of an inner product space into an orthogonal sum of subspaces and collection of linear operators P_i satisfying

$$P_1 + P_2 + \cdots + P_k = I, \quad P_iP_j = O \text{ for } i \neq j, \quad P_i^* = P_i.$$

This is the orthogonal sum analogue of Proposition 3.3.5.

4.3.2 Adjoint Basis

Let $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be a basis of V. Then we have the dual basis α^* of the dual space V^*. Using the isomorphism in Proposition 4.3.1, we may identify α^* with a basis $\beta = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n\}$ of V. This means

$$\langle \cdot, \beta \rangle = \{\langle \cdot, \vec{w}_1 \rangle, \langle \cdot, \vec{w}_2 \rangle, \dots, \langle \cdot, \vec{w}_n \rangle\} = \alpha^* = \{\vec{v}_1^*, \vec{v}_2^*, \dots, \vec{v}_n^*\}.$$

By the definition of dual basis in V^* and applying the above to $\cdot = \vec{v}_i$, this means

$$\langle \vec{v}_i, \vec{w}_j \rangle = \vec{v}_j^*(\vec{v}_i) = \delta_{ij}.$$

Therefore the dual basis is the specialisation of the dual basis in Section 2.4.4 when the inner product is used as the dual pairing.

We call β the *adjoint basis* of α with respect to the inner product, and even denote $\beta = \alpha^*$, $\vec{w}_j = \vec{v}_j^*$. However, we need to be clear that the same notation is used for two meanings:

1. The dual basis of α is a basis of V^*. The concept is independent of the inner product.

2. The adjoint basis of α (with respect to an inner product) is a basis of V. The concept depends on the inner product.

The two meanings are related by the isomorphism in Proposition 4.3.1.

In the second sense, a natural question is whether the adjoint β is the same as the original α. In other words, whether α is a *self-adjoint basis* with respect to the inner product. Using the characterisation $\langle \vec{v}_i, \vec{w}_j \rangle = \delta_{ij}$, we get the following.

Proposition 4.3.3. *A basis α of an inner product space is self-adjoint, in the sense that $\alpha^* = \langle \cdot, \alpha \rangle$, if and only if α is an orthonormal basis.*

Let α and β be bases of V and W. Then Proposition 2.4.2 gives the equality $[L^*]_{\alpha^*\beta^*} = [L]_{\beta\alpha}^T$ for the dual linear transformation and dual basis. Since the dual bases of V^*, W^* are translated into the adjoint basis of V, W under the isomorphism in Proposition 4.3.1, and both are still denoted α^*, β^*, the equality is also true for the adjoint linear transformation L^* and adjoint bases α^*, β^*. In particular, by Proposition 4.3.3, we have the following.

Proposition 4.3.4. *Suppose $L\colon V \to W$ is a linear transformation of inner product spaces, and $L^*\colon W \to V$ is the adjoint. If α and β are orthonormal bases, then $[L^*]_{\alpha\beta} = [L]_{\beta\alpha}^T$.*

Example 4.3.4. For the basis of $\mathbb{R}^3$ in Examples 1.3.17 and 2.1.13

$$\vec{v}_1 = (1, -1, 0), \quad \vec{v}_2 = (1, 0, -1), \quad \vec{v}_3 = (1, 1, 1),$$

we would like to find the adjoint basis $\vec{w}_1, \vec{w}_2, \vec{w}_3$ of $\mathbb{R}^3$ with respect to the dot product. Let $X = (\vec{v}_1\ \vec{v}_2\ \vec{v}_3)$ and $Y = (\vec{w}_1\ \vec{w}_2\ \vec{w}_3)$, then the condition $\vec{v}_i \cdot \vec{w}_j = \delta_{ij}$ means $X^T Y = I$. By Example 2.2.18, we have

$$Y = (X^{-1})^T = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ -2 & 1 & 1 \\ 1 & -2 & 1 \end{pmatrix}.$$

This gives

$$\vec{w}_1 = \tfrac{1}{3}(1, -2, 1), \quad \vec{w}_2 = \tfrac{1}{3}(1, 1, -2), \quad \vec{w}_3 = \tfrac{1}{3}(1, 1, 1).$$

Example 4.3.5. For the basis $\alpha = \{1, t, t^2\}$ of P_2, we would like to find the adjoint basis $\alpha^* = \{p_0(t), p_1(t), p_2(t)\}$ with respect to the inner product in Example 4.1.3. Note that for $p(t) = a_0 + a_1 t + a_2 t^2$, we have

$$\langle \alpha, p(t) \rangle = \begin{pmatrix} \int_0^1 p(t)dt \\ \int_0^1 tp(t)dt \\ \int_0^1 t^2 p(t)dt \end{pmatrix} = \begin{pmatrix} a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 \\ \frac{1}{2}a_0 + \frac{1}{3}a_1 + \frac{1}{4}a_2 \\ \frac{1}{3}a_0 + \frac{1}{4}a_1 + \frac{1}{5}a_2 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix} [p(t)]_{\{1,t,t^2\}}.$$

Then the adjoint basis means $\langle \alpha, p_0(t) \rangle = \vec{e}_1$, $\langle \alpha, p_1(t) \rangle = \vec{e}_2$, $\langle \alpha, p_2(t) \rangle = \vec{e}_3$. This is the same as

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix} ([p_0(t)]\ [p_1(t)]\ [p_2(t)]) = I.$$

Therefore

$$([p_0(t)]\ [p_1(t)]\ [p_2(t)]) = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}^{-1} = \begin{pmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix},$$

and we get the adjoint basis

$$p_0(t) = 9 - 36t + 30t^2, \quad p_1(t) = -36 + 192t - 180t^2, \quad p_2(t) = 30 - 180t + 180t^2.$$

Exercise 4.66. Find the adjoint basis of (a, b), (c, d) with respect to the dot product on $\mathbb{R}^2$.

Exercise 4.67. Find the adjoint basis of $(1, 2, 3)$, $(4, 5, 6)$, $(7, 8, 10)$ with respect to the dot product on $\mathbb{R}^3$.

Exercise 4.68. If the inner product on $\mathbb{R}^3$ is given by $\langle \vec{x}, \vec{y} \rangle = x_1y_1 + 2x_2y_2 + 3x_3y_3$, what would be the adjoint basis in Example 4.3.4?

Exercise 4.69. If the inner product on P_2 is given by Exercise 4.21, what would be the adjoint basis in Example 4.3.5?

Exercise 4.70. Suppose $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a basis of $\mathbb{R}^n$ and A is positive definite. What is the adjoint basis of α with respect to the inner product $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T A \vec{y}$?

Exercise 4.71. Prove that if β is the adjoint basis of α with respect to an inner product, then α is the adjoint basis of β with respect to the same inner product.

Exercise 4.72. The adjoint basis can be introduced in another way. For a basis α of V, we have a basis $\langle \cdot, \alpha \rangle$ of V^*. Then the basis $\langle \cdot, \alpha \rangle$ corresponds to a basis β of V under the isomorphism in Proposition 4.3.1. Prove that β is also the adjoint basis of α.

Exercise 4.73. Prove that $\text{rank} L = \text{rank} L^*$.

4.3.3 Isometry

A linear transformations preserves addition and scalar multiplication. Naturally, we wish a linear transformation between inner product spaces to additionally preserve the inner product.

Definition 4.3.5. A linear transformation $L\colon V \to W$ between inner product spaces is an *isometry* if it preserves the inner product

$$\langle L(\vec{u}), L(\vec{v})\rangle_W = \langle \vec{u}, \vec{v}\rangle_V \quad \text{for all } \vec{u}, \vec{v} \in V.$$

If the isometry is also invertible, then it is an *isometric isomorphism*. In case $V = W$, the isometric isomorphism is also called an *orthogonal operator*.

An isometry preserves all the concepts defined by the inner product, such as the length, the angle, the orthogonality, the area, and the *distance*

$$\|L(\vec{u}) - L(\vec{v})\| = \|L(\vec{u} - \vec{v})\| = \|\vec{u} - \vec{v}\|.$$

Conversely, a linear transformation preserving the length (or distance) must be an isometry. See Exercise 4.76.

A linear transformation L between finite dimensional inner product spaces is an isometry if

$$\langle \vec{u}, \vec{v}\rangle = \langle L(\vec{u}), L(\vec{v})\rangle = \langle \vec{u}, L^*L(\vec{v})\rangle.$$

By Exercise 4.4, this means that $L^*L = I$ is the identity.

An isometry is always one-to-one by the following argument

$$\vec{u} \neq \vec{v} \implies \|\vec{u} - \vec{v}\| \neq 0 \implies \|L(\vec{u}) - L(\vec{v})\| \neq 0 \implies L(\vec{u}) \neq L(\vec{v}).$$

By Theorem 2.2.6, for finite dimensional spaces, an isometry is an isomorphism if and only if $\dim V = \dim W$. In this case, by $L^*L = I$, we get $L^{-1} = L^*$.

The following is the isometric version of Proposition 2.1.3.

Proposition 4.3.6. *Suppose $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ span a vector space V, and $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n$ are vectors in W. Then $L(\vec{v}_i) = \vec{w}_i$ (see (2.1.1)) gives a well defined isometry $L\colon V \to W$ if and only if*

$$\langle \vec{v}_i, \vec{v}_j\rangle = \langle \vec{w}_i, \vec{w}_j\rangle \quad \text{for all } i, j.$$

Proof. By the definition, the equality $\langle \vec{v}_i, \vec{v}_j\rangle = \langle \vec{w}_i, \vec{w}_j\rangle$ is satisfied if L is an isometry. Conversely, suppose the equality is satisfied, then using the bilinearity of the inner product, we have

$$\begin{aligned}
&\langle x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n, y_1\vec{v}_1 + y_2\vec{v}_2 + \cdots + y_n\vec{v}_n\rangle \\
=&\sum_{ij} x_iy_j\langle \vec{v}_i, \vec{v}_j\rangle = \sum_{ij} x_iy_j\langle \vec{w}_i, \vec{w}_j\rangle \\
=&\langle x_1\vec{w}_1 + x_2\vec{w}_2 + \cdots + x_n\vec{w}_n, y_1\vec{w}_1 + y_2\vec{w}_2 + \cdots + y_n\vec{w}_n\rangle.
\end{aligned}$$

By taking $x_i = y_i$, this implies

$$\|x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n\| = \|x_1\vec{w}_1 + x_2\vec{w}_2 + \cdots + x_n\vec{w}_n\|.$$

By $\vec{x} = \vec{0} \iff \|\vec{x}\| = 0$, the condition in Proposition 2.1.3 for L being well defined is satisfied. Moreover, the equality above means $\langle\vec{x}, \vec{y}\rangle = \langle L(\vec{x}), L(\vec{y})\rangle$, i.e., L is an isometry. $\square$

Applying Proposition 4.3.6 to the α-coordinate of a finite dimensional vector space, we find that $[\cdot]_\alpha \colon V \to \mathbb{R}^n$ is an isometric isomorphism between V and the Euclidean space (with dot product) if and only if α is an orthonormal basis. Since Proposition 4.2.9 says that any finite dimensional inner product space has an orthonormal basis, we get the following.

Theorem 4.3.7. *Any finite dimensional inner product space is isometrically isomorphic to a Euclidean space with the dot product.*

Example 4.3.6. The inner product in Exercise 4.18 is

$$\langle\vec{x}, \vec{y}\rangle = x_1y_1 + 2x_2y_2 + \cdots + nx_ny_n = x_1y_1 + (\sqrt{2}x_2)(\sqrt{2}y_2) + \cdots + (\sqrt{n}x_n)(\sqrt{n}y_n).$$

This shows that $L(x_1, x_2, \dots, x_n) = (x_1, \sqrt{2}x_2, \dots, \sqrt{n}x_n)$ is an isometric isomorphism from this inner product to the dot product.

Example 4.3.7. In Example 4.2.3, we get an orthonormal basis $\{\sqrt{3}t, 3t - 2\}$ of P_1 with respect to the inner product in Example 4.1.3. Then

$$L(x_1, x_2) = x_1(\sqrt{3}t) + x_2(3t - 2) = -2x_2 + (\sqrt{3}x_1 + 3x_2)t \colon \mathbb{R}^2 \to P_1$$

is an isometric isomorphism.

Example 4.3.8. In Exercise 4.76, we will show that a linear transformation is an isometry if and only if it preserves length. For example, the length in P_1 with respect to the inner product in Example 4.1.3 is given by

$$\|a_0 + a_1t\|^2 = \int_0^1 (a_0 + a_1t)^2 dt = a_0^2 + a_0a_1 + \frac{1}{3}a_1^2 = \left(a_0 + \frac{1}{2}a_1\right)^2 + \left(\frac{1}{2\sqrt{3}}a_1\right)^2.$$

Since the right side is the square of the Euclidean length of $(a_0 + \frac{1}{2}a_1, \frac{1}{2\sqrt{3}}a_1) \in \mathbb{R}^2$, we find that

$$L(a_0 + a_1t) = (a_0 + \tfrac{1}{2}a_1, \tfrac{1}{2\sqrt{3}}a_1) \colon P_1 \to \mathbb{R}^2$$

is an isometric isomorphism between P_1 with the inner product in Example 4.1.3 and $\mathbb{R}^2$ with the dot product.

Example 4.3.9. The map $t \to 2t-1\colon [0,1] \to [-1,1]$ is a linear change of variable between two intervals. We have

$$\int_{-1}^{1} f(t)g(t)dt = \int_{0}^{1} f(2t-1)g(2t-1)d(2t-1) = 2\int_{0}^{1} f(2t-1)g(2t-1)dt.$$

This implies that $L(f(t)) = \sqrt{2}f(2t-1)\colon P_n \to P_n$ is an isometric isomorphism between the inner product in Exercise 4.21 and the inner product in Example 4.1.3. We may apply the isometric isomorphism to the orthonormal basis in Example 4.2.11 to get an orthonormal basis of P_3 with respect to the inner product in Example 4.1.3

$$\begin{aligned}
L\left(\tfrac{1}{\sqrt{2}}\right) &= \sqrt{2}\tfrac{1}{\sqrt{2}} = 1,\\
L\left(\tfrac{\sqrt{3}}{\sqrt{2}}t\right) &= \sqrt{2}\tfrac{\sqrt{3}}{\sqrt{2}}(2t-1) = \sqrt{3}(2t-1),\\
L\left(\tfrac{2\sqrt{2}}{\sqrt{5}}(3t^2-1)\right) &= \sqrt{2}\tfrac{2\sqrt{2}}{\sqrt{5}}(3(2t-1)^2-1) = \tfrac{8}{\sqrt{5}}(6t^2-6t+1),\\
L\left(\tfrac{2\sqrt{2}}{\sqrt{7}}(5t^3-3t)\right) &= \sqrt{2}\tfrac{2\sqrt{2}}{\sqrt{7}}(5(2t-1)^3-3(2t-1)) = \tfrac{8}{\sqrt{7}}(2t-1)(10t^2-10t+1).
\end{aligned}$$

Exercise 4.74. Prove that a composition of isometries is an isometry.

Exercise 4.75. Prove that the inverse of an isometric isomorphism is also an isometric isomorphism.

Exercise 4.76. Use the polarisation identity in Exercise 4.12 to prove that a linear transformation L is an isometry if and only if it preserves the length: $\|L(\vec{v})\| = \|\vec{v}\|$.

Exercise 4.77. Use Exercise 4.38 to prove that an orthogonal operator on $\mathbb{R}^2$ is either a rotation or a reflection. What about an orthogonal operator on $\mathbb{R}^1$?

Exercise 4.78. In Example 4.1.2, we know that $\langle (x_1,x_2),(y_1,y_2)\rangle = x_1y_1 + 2x_1y_2 + 2x_2y_1 + ax_2y_2$ is an inner product on $\mathbb{R}^2$ for $a > 4$. Find an isometric isomorphism between $\mathbb{R}^2$ with this inner product and $\mathbb{R}^2$.

Exercise 4.79. Find an isometric isomorphism between $\mathbb{R}^3$ and P_2 with the inner product in Example 4.1.3, by two ways.

1. Similar to Example 4.3.8, calculate the length of $a_0 + a_1t + a_2t^2$, and then complete the square.

2. Similar to Example 4.3.9, find an orthogonal set of the form $1, t, a+t^2$ with respect to the inner product in Exercise 4.21, divide the length, and then translate to the inner product in Example 4.1.3.

Exercise 4.80. Prove that a linear transformation $L\colon V \to W$ is an isometry if and only if L takes an orthonormal bases of V to an orthonormal set of W.

Exercise 4.81. For an orthonormal basis $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$, prove *Parsival's identity*

$$\langle \vec{x}, \vec{y}\rangle = \langle \vec{x}, \vec{v}_1\rangle\langle \vec{v}_1, \vec{y}\rangle + \langle \vec{x}, \vec{v}_2\rangle\langle \vec{v}_2, \vec{y}\rangle + \cdots + \langle \vec{x}, \vec{v}_n\rangle\langle \vec{v}_n, \vec{y}\rangle.$$

Exercise 4.82. Suppose $L\colon V \to W$ is an isometric isomorphism, and β is the adjoint basis of a basis α of V. Explain that $L(\beta)$ is the adjoint basis of $L(\alpha)$. Then use the isometric isomorphism in Example 4.3.9 to relate Example 4.3.5 and Exercise 4.69.

Exercise 4.83. A linear transformation is *conformal* if it preserves the angle. Prove that a linear transformation is conformal if and only if it is a scalar multiple of an isometry.

Let α and β be orthonormal bases of V and W. Then the isometric isomorphisms $[\cdot]_\alpha\colon V \cong \mathbb{R}^n$ and $[\cdot]_\beta\colon W \cong \mathbb{R}^m$ translates an isometry $L\colon V \to W$ to an isometry $L_{\beta\alpha}\colon \mathbb{R}^n \to \mathbb{R}^m$. The columns of the matrix $Q = [L]_{\beta\alpha} = [L_{\beta\alpha}]$ is the isometric image of the standard basis of $\mathbb{R}^n$, and is therefore an orthonormal set in $\mathbb{R}^m$. The property means $Q^TQ = I$, which is consistent with $L^*L = I$ interpretation of isometry. In case $m = n$, L becomes an isomorphism, and Q is described by the following concept.

Definition 4.3.8. An *orthogonal* matrix is a square matrix Q satisfying $Q^TQ = I$.

Orthogonal matrices are exactly the matrices of isometric isomorphisms with respect to orthonormal bases. We have $Q^{-1} = Q^T$, corresponding to $L^{-1} = L^*$. Note that $QQ^T = I$ means that the rows of Q also form an orthonormal basis.

Example 4.3.10. The orthogonal basis of $\mathbb{R}^3$ in Example 4.2.2 gives an orthogonal matrix

$$Q = \frac{1}{3}\begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix}, \quad Q^TQ = I.$$

The inverse of the matrix is simply the transpose

$$Q^{-1} = Q^T = \frac{1}{3}\begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix}.$$

Exercise 4.84. Find all the 2×2 orthogonal matrices.

Exercise 4.85. Find an orthogonal matrix such that the first two columns are parallel to $(1, -1, 0)$, $(1, a, 1)$. Then find the inverse of the orthogonal matrix.

Exercise 4.86. Find an orthogonal matrix such that the first three columns are parallel to $(1, 0, 1, 0)$, $(0, 1, 0, 1)$, $(1, -1, 1, -1)$. Then find the inverse of the orthogonal matrix.

Exercise 4.87. Prove that the transpose, inverse and multiplication of orthogonal matrices are orthogonal matrices.

Exercise 4.88. Suppose α is an orthonormal basis. Prove that another basis β is orthonormal if and only if $[I]_{\beta\alpha}$ is an orthogonal matrix.

4.3.4 QR-Decomposition

In Proposition 4.2.9, we apply the Gram-Schmidt process to a basis α to get an orthogonal basis β. The span property means that the two bases are related in "triangular" way, with $a_{ii} \neq 0$

$$\begin{aligned}
\vec{v}_1 &= a_{11}\vec{w}_1, \\
\vec{v}_2 &= a_{12}\vec{w}_1 + a_{22}\vec{w}_2, \\
&\vdots \\
\vec{v}_n &= a_{1n}\vec{w}_1 + a_{2n}\vec{w}_2 + \cdots + a_{nn}\vec{w}_n.
\end{aligned}$$

The vectors $\vec{w}_i$ can also be expressed in $\vec{v}_i$ in similar triangular way. The relation can be rephrased in the matrix form

$$A = QR, \quad A = (\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n),\ Q = (\vec{w}_1\ \vec{w}_2\ \cdots\ \vec{w}_n),\ R = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}.$$

If we make β to become orthonormal (by dividing the lengths), then the columns of Q are orthonormal. The expression $A = QR$, with $Q^TQ = I$ and R upper triangular, is called the *QR-decomposition* of A. Any $m \times n$ matrix A of rank n has QR-decomposition.

For the calculation, we first use the Gram-Schmidt process to get Q. Then $Q^TA = Q^TQR = IR = R$.

Example 4.3.11. After dividing the vector length, the QR-decomposition for the Gram-Schmidt process in Example 4.2.8 has

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}.$$

Then

$$R = Q^TA = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \frac{\sqrt{3}}{\sqrt{2}} \end{pmatrix},$$

and we get the QR-decomposition

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \frac{\sqrt{3}}{\sqrt{2}} \end{pmatrix}.$$

For the Gram-Schmidt process in Example 4.2.9, we have

$$A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{pmatrix}, \quad Q = \begin{pmatrix} \frac{1}{\sqrt{14}} & \frac{4}{\sqrt{21}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{14}} & \frac{1}{\sqrt{21}} & -\frac{2}{\sqrt{6}} \\ \frac{3}{\sqrt{14}} & -\frac{2}{\sqrt{21}} & \frac{1}{\sqrt{6}} \end{pmatrix}.$$

Therefore

$$R = Q^T A = \begin{pmatrix} \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}} \\ \frac{4}{\sqrt{21}} & \frac{1}{\sqrt{21}} & -\frac{2}{\sqrt{21}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{pmatrix} = \begin{pmatrix} \sqrt{14} & \frac{32}{\sqrt{14}} & \frac{53}{\sqrt{14}} \\ 0 & \frac{9}{\sqrt{21}} & -\frac{12}{\sqrt{21}} \\ 0 & 0 & \frac{1}{\sqrt{6}} \end{pmatrix}.$$

Exercise 4.89. In Example 2.1.13, the basis in Example 4.2.8 is extended to a basis of $\mathbb{R}^3$, and we get the invertible matrix $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$ in Example 1.3.17. Find the QR-decomposition of this matrix.

Exercise 4.90. Use Example 4.2.9 to find the QR-decomposition for $\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$ and $\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$. Then make a general observation.

The QR-decomposition is related to the least square solution. Here is the problem: For a system of linear equations $A\vec{x} = \vec{b}$ to have solution, we need $\vec{b} \in \mathrm{Col}A$. If $\vec{b} \notin \mathrm{Col}A$, then we may settle with the best approximate solution, in the sense that the Euclidean distance $\|A\vec{x} - \vec{b}\|$ is the smallest possible.

Let $\vec{h} = \mathrm{proj}_{\mathrm{Col}A}\vec{b}$. Then $A\vec{x}, \vec{h} \in \mathrm{Col}A$, so that $A\vec{x} - \vec{h} \in \mathrm{Col}A$ and $\vec{b} - \vec{h} \in (\mathrm{Col}A)^\perp$ are orthogonal. Then we have

$$\|A\vec{x} - \vec{b}\|^2 = \|A\vec{x} - \vec{h}\|^2 + \|\vec{b} - \vec{h}\|^2 \geq \|\vec{b} - \vec{h}\|^2.$$

This shows that the best approximate solution is the solution of $A\vec{x} = \vec{h}$. This is characterised by $A\vec{x} - \vec{b} \perp \mathrm{Col}A$. In other words, for any $\vec{y}$, we have

$$0 = (A\vec{x} - \vec{b}) \cdot A\vec{y} = A^T(A\vec{x} - \vec{b}) \cdot \vec{y}.$$

Since this should hold for all $\vec{y}$, we get $A^T A\vec{x} = A^T\vec{b}$.

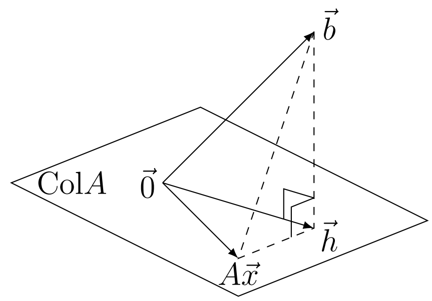

Figure 4.3.1: Least square method.

We conclude that the least square solution of $A\vec{x} = \vec{b}$ is the same as the solution of $A^T A\vec{x} = A^T\vec{b}$. In general, the solution is not unique. In fact, by Exercise 4.62, we have Nul$A^T A$ = Nul$A$. In case the solution of $A\vec{x} = \vec{0}$ is unique, i.e., the columns of A are linearly independent, then the square matrix $A^T A$ is invertible, and we get the unique least square solution

$$\vec{x} = (A^T A)^{-1} A^T \vec{b}.$$

Note that the linear independence of the columns of A implies $A = QR$. Then $A^T A = R^T Q^T QR = R^T R$, and the solution becomes

$$\vec{x} = (R^T R)^{-1} R^T Q^T \vec{b} = R^{-1} Q^T \vec{b}.$$

Since R is upper triangular, it is very easy to calculate R^{-1}. Therefore $R^{-1}Q^T\vec{b}$ is easier than $(A^T A)^{-1} A^T \vec{b}$.

Chapter 5

Determinant

Determinant first appeared as the numerical criterion that "determines" whether a system of linear equations has a unique solution. More properties of the determinant were discovered later, especially its relation to geometry. We will define the determinant by axioms, derive the calculation technique from the axioms, and then discuss the geometric meaning.

5.1 Algebra

Definition 5.1.1. The *determinant* of $n \times n$ matrices A is the function $\det A$ satisfying the following properties.

1. Multilinear: The function is linear in each column vector
$$\det(\cdots\ a\vec{u}+b\vec{v}\ \cdots) = a\det(\cdots\ \vec{u}\ \cdots) + b\det(\cdots\ \vec{v}\ \cdots).$$
2. Alternating: Exchanging two columns introduces a negative sign
$$\det(\cdots\ \vec{v}\ \cdots\ \vec{u}\ \cdots) = -\det(\cdots\ \vec{u}\ \cdots\ \vec{v}\ \cdots).$$
3. Normal: The determinant of the identity matrix is 1
$$\det I = \det(\vec{e}_1\ \vec{e}_2\ \cdots\ \vec{e}_n) = 1.$$

For a multilinear function D, taking $\vec{u}=\vec{v}$ in the alternating property gives
$$D(\cdots\ \vec{u}\ \cdots\ \vec{u}\ \cdots) = 0.$$
Conversely, if D satisfies the equality above, then
$$\begin{aligned}
0 &= D(\cdots\ \vec{u}+\vec{v}\ \cdots\ \vec{u}+\vec{v}\ \cdots) \\
&= D(\cdots\ \vec{u}\ \cdots\ \vec{u}\ \cdots) + D(\cdots\ \vec{u}\ \cdots\ \vec{v}\ \cdots) \\
&\quad + D(\cdots\ \vec{v}\ \cdots\ \vec{u}\ \cdots) + D(\cdots\ \vec{v}\ \cdots\ \vec{v}\ \cdots) \\
&= D(\cdots\ \vec{u}\ \cdots\ \vec{v}\ \cdots) + D(\cdots\ \vec{v}\ \cdots\ \vec{u}\ \cdots).
\end{aligned}$$
We recover the alternating property.

5.1.1 Multilinear and Alternating Function

A multilinear and alternating function D of 2×2 matrices is given by

$$\begin{aligned}
D\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} &= D(a_{11}\vec{e}_1 + a_{21}\vec{e}_2 \quad a_{12}\vec{e}_1 + a_{22}\vec{e}_2) \\
&= D(\vec{e}_1\ \vec{e}_1)a_{11}a_{12} + D(\vec{e}_1\ \vec{e}_2)a_{11}a_{22} + D(\vec{e}_2\ \vec{e}_1)a_{21}a_{12} + D(\vec{e}_2\ \vec{e}_2)a_{21}a_{22} \\
&= 0a_{11}a_{12} + D(\vec{e}_1\ \vec{e}_2)a_{11}a_{22} - D(\vec{e}_1\ \vec{e}_2)a_{21}a_{12} + 0a_{21}a_{22} \\
&= D(\vec{e}_1\ \vec{e}_2)(a_{11}a_{22} - a_{21}a_{12}).
\end{aligned}$$

A multilinear and alternating function D of 3×3 matrices is given by

$$\begin{aligned}
&D(a_{11}\vec{e}_1 + a_{21}\vec{e}_2 + a_{31}\vec{e}_3 \quad a_{12}\vec{e}_1 + a_{22}\vec{e}_2 + a_{32}\vec{e}_3 \quad a_{13}\vec{e}_1 + a_{23}\vec{e}_2 + a_{33}\vec{e}_3) \\
&= D(\vec{e}_1\ \vec{e}_2\ \vec{e}_3)a_{11}a_{22}a_{33} + D(\vec{e}_2\ \vec{e}_3\ \vec{e}_1)a_{21}a_{32}a_{13} + D(\vec{e}_3\ \vec{e}_1\ \vec{e}_2)a_{31}a_{12}a_{23} \\
&\quad + D(\vec{e}_1\ \vec{e}_3\ \vec{e}_2)a_{11}a_{32}a_{23} + D(\vec{e}_3\ \vec{e}_2\ \vec{e}_1)a_{31}a_{22}a_{13} + D(\vec{e}_2\ \vec{e}_1\ \vec{e}_3)a_{21}a_{12}a_{33} \\
&= D(\vec{e}_1\ \vec{e}_2\ \vec{e}_3)(a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{11}a_{32}a_{23} - a_{31}a_{22}a_{13} - a_{21}a_{12}a_{33}).
\end{aligned}$$

Here in the first equality, we use the alternating property that $D(\vec{e}_i\ \vec{e}_j\ \vec{e}_k) = 0$ whenever two of i, j, k are equal. In the second equality, we exchange (i.e., switching two indices) distinct i, j, k (which must be a rearrangement of $1, 2, 3$) to get the usual order $1, 2, 3$. For example, we have

$$D(\vec{e}_3\ \vec{e}_1\ \vec{e}_2) = -D(\vec{e}_1\ \vec{e}_3\ \vec{e}_2) = D(\vec{e}_1\ \vec{e}_2\ \vec{e}_3).$$

In general, a multilinear and alternating function D of $n \times n$ matrices $A = (x_{ij})$ is

$$D(A) = D(\vec{e}_1\ \vec{e}_2\ \cdots\ \vec{e}_n) \sum \pm a_{i_1 1}a_{i_2 2}\cdots a_{i_n n},$$

where the sum runs over all the rearrangements (or *permutations*) $(i_1, i_2, \dots, i_n)$ of $(1, 2, \dots, n)$. Moreover, the $\pm$ sign, usually denoted by $\text{sign}(i_1, i_2, \dots, i_n)$, is determined in the following way.

- If it takes even number of exchanges to transform $(i_1, i_2, \dots, i_n)$ to $(1, 2, \dots, n)$, then $\text{sign}(i_1, i_2, \dots, i_n) = 1$.
- If it takes odd number of exchanges to transform $(i_1, i_2, \dots, i_n)$ to $(1, 2, \dots, n)$, then $\text{sign}(i_1, i_2, \dots, i_n) = -1$.

The argument above can be carried out on any vector space, by replacing the standard basis of $\mathbb{R}^n$ by an (ordered) basis of V.

Theorem 5.1.2. *Multilinear and alternating functions of n vectors in a vector space of dimension n are unique up to multiplying constants. Specifically, in terms of the coordinates with respect to a basis α, such a function D is given by*

$$D(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n) = c \sum_{(i_1, i_2, \dots, i_n)} \text{sign}(i_1, i_2, \dots, i_n) a_{i_1 1}a_{i_2 2}\cdots a_{i_n n},$$

where

$$[\vec{v}_j]_\alpha = (a_{1j}, a_{2j}, \dots, a_{nj}).$$

In case the constant $c = D(\alpha) = 1$, the formula in the theorem is the determinant.

Exercise 5.1. What is the determinant of a 1×1 matrix?

Exercise 5.2. How many terms are in the explicit formula for the determinant of $n \times n$ matrices?

Exercise 5.3. Show that any alternating and bilinear function on 2×3 matrices is zero. Can you generalise this observation?

Exercise 5.4. Find explicit formula for an alternating and bilinear function on 3×2 matrices.

Theorem 5.1.2 is a very useful tool for deriving properties of the determinant. The following is a typical example.

Proposition 5.1.3. $\det AB = \det A \det B$.

Proof. For fixed A, we consider the function $D(B) = \det AB$. Since AB is obtained by multiplying A to the columns of B, the function $D(B)$ is multilinear and alternating in the columns of B. By Theorem 5.1.2, we get $D(B) = c \det B$. To determine the constant c, we let B to be the identity matrix and get $\det A = D(I) = c \det I = c$. Therefore $\det AB = D(B) = c \det B = \det A \det B$. □

Exercise 5.5. Prove that $\det A^{-1} = \frac{1}{\det A}$. More generally, we have $\det A^n = (\det A)^n$ for any integer n.

Exercise 5.6. Use the explicit formula for the determinant to verify $\det AB = \det A \det B$ for 2×2 matrices.

5.1.2 Column Operation

The explicit formula in Theorem 5.1.2 is too complicated to be a practical way of calculating the determinant. Since matrices can be simplified by row and column operations, it is useful to know how the determinant is changed by the operations.

The alternating property means that the column operation $C_i \leftrightarrow C_j$ introduces a negative sign

$$\det(\cdots\ \vec{v}\ \cdots\ \vec{u}\ \cdots) = -\det(\cdots\ \vec{u}\ \cdots\ \vec{v}\ \cdots).$$

The multilinear property implies that the column operation $c\,C_i$ multiplies the determinant by the scalar

$$\det(\cdots\ c\vec{u}\ \cdots) = c\det(\cdots\ \vec{u}\ \cdots).$$

Combining the multilinear and alternating properties, the column operation $C_i + c\,C_j$ preserves the determinant

$$\begin{aligned}\det(\cdots\ \vec{u}+c\vec{v}\ \cdots\ \vec{v}\ \cdots) &= \det(\cdots\ \vec{u}\ \cdots\ \vec{v}\ \cdots) + c\det(\cdots\ \vec{v}\ \cdots\ \vec{v}\ \cdots)\\ &= \det(\cdots\ \vec{u}\ \cdots\ \vec{v}\ \cdots) + c\cdot 0\\ &= \det(\cdots\ \vec{u}\ \cdots\ \vec{v}\ \cdots).\end{aligned}$$

Example 5.1.1. By column operations, we have

$$\begin{aligned}\det\begin{pmatrix}1&4&7\\2&5&8\\3&6&a\end{pmatrix} &\overset{\substack{C_2-4C_1\\C_3-7C_1}}{=\!=\!=} \det\begin{pmatrix}1&0&0\\2&-3&-6\\3&-6&a-21\end{pmatrix} \overset{\substack{C_3-2C_2\\3C_2}}{=\!=\!=} -3\det\begin{pmatrix}1&0&0\\2&1&0\\3&2&a-9\end{pmatrix}\\ &\overset{(a-9)C_3}{=\!=\!=} -3(a-9)\det\begin{pmatrix}1&0&0\\2&1&0\\3&2&1\end{pmatrix} \overset{\substack{C_2-2C_3\\C_1-2C_2\\C_1-3C_3}}{=\!=\!=} -3(a-9)\det\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\\ &= -3(a-9).\end{aligned}$$

The column operations can simplify a matrix to a column echelon form, which is a *lower triangular matrix*. By column operations, the determinant of a lower triangular matrix is the product of diagonal entries

$$\begin{aligned}\det\begin{pmatrix}a_1&0&\cdots&0\\ *&a_2&\cdots&0\\ \vdots&\vdots&&\vdots\\ *&*&\cdots&a_n\end{pmatrix} &= a_1a_2\cdots a_n\det\begin{pmatrix}1&0&\cdots&0\\ *&1&\cdots&0\\ \vdots&\vdots&&\vdots\\ *&*&\cdots&1\end{pmatrix}\\ &= a_1a_2\cdots a_n\det\begin{pmatrix}1&0&\cdots&0\\ 0&1&\cdots&0\\ \vdots&\vdots&&\vdots\\ 0&0&\cdots&1\end{pmatrix} = a_1a_2\cdots a_n.\end{aligned}$$

The argument above assumes all $a_i \neq 0$. If some $a_i = 0$, then the last column in the column echelon form is the zero vector $\vec{0}$. By the linearity of the determinant in the last column, the determinant is 0. This shows that the equality always holds. We also note that the argument for lower triangular matrices also applies to *upper triangular matrices*

$$\det\begin{pmatrix}a_1&*&\cdots&*\\ 0&a_2&\cdots&*\\ \vdots&\vdots&&\vdots\\ 0&0&\cdots&a_n\end{pmatrix} = a_1a_2\cdots a_n = \det\begin{pmatrix}a_1&0&\cdots&0\\ *&a_2&\cdots&0\\ \vdots&\vdots&&\vdots\\ *&*&\cdots&a_n\end{pmatrix}.$$

Example 5.1.2. We subtract the first column from the later columns and get

$$\begin{aligned}
&\det\begin{pmatrix}
a_1 & a_1 & a_1 & \cdots & a_1 & a_1\\
a_2 & b_2 & a_2 & \cdots & a_2 & a_2\\
a_3 & * & b_3 & \cdots & a_3 & a_3\\
\vdots & \vdots & \vdots & & \vdots & \vdots\\
a_{n-1} & * & * & \cdots & b_{n-1} & a_{n-1}\\
a_n & * & * & \cdots & * & b_n
\end{pmatrix}\\
&= \det\begin{pmatrix}
a_1 & 0 & 0 & \cdots & 0 & 0\\
a_2 & b_2-a_2 & 0 & \cdots & 0 & 0\\
a_3 & * & b_3-a_3 & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & & \vdots & \vdots\\
a_{n-1} & * & * & \cdots & b_{n-1}-a_{n-1} & 0\\
a_n & * & * & \cdots & * & b_n-a_n
\end{pmatrix}\\
&= a_1(b_2-a_2)(b_3-a_3)\dots(b_n-a_n).
\end{aligned}$$

A square matrix is invertible if and only if all the diagonal entries in the column echelon form are pivots, i.e., all $a_i \neq 0$. This proves the following.

Theorem 5.1.4. *A square matrix is invertible if and only if its determinant is nonzero.*

In terms of a system of linear equations with equal number of equations and variables, this means that whether $A\vec{x} = \vec{b}$ has unique solution is "determined" by $\det A \neq 0$.

5.1.3 Row Operation

To find out the effect of row operations on the determinant, we use the idea for the proof of Proposition 5.1.3. The key is that a row operation preserves the multilinear and alternating property. In other words, suppose $A \mapsto \tilde{A}$ is a row operation, then $\det \tilde{A}$ is still a multilinear and alternating function of A.

By Theorem 5.1.2, therefore, we have $\det \tilde{A} = a \det A$ for a constant a. The constant a can be calculated from the special case that $A = I$ is the identity matrix. In Section 2.1.5, we see that $\tilde{I} = T_{ij}, D_i(c), E_{ij}(c)$ respectively for the three row operations. Then we may use column operations to get $a = \det \tilde{I} = -1, c, 1$ respectively for the three operations. We conclude that the effect of row operations on the determinant is the same as the effect of column operations. By the same idea of proving $\text{rank} A^T = \text{rank} A$, we get the following result.

Proposition 5.1.5. $\det A^T = \det A$.

A consequence of the proposition is that the determinant is also multilinear and alternating in the row vectors.

Proof. A column operation $A \mapsto B$ is equivalent to a row operation $A^T \mapsto B^T$. The discussion above shows that $\det B = a \det A$ if and only if $\det B^T = a \det A^T$. Therefore $\det A = \det A^T$ if and only if $\det B = \det B^T$.

If A is invertible, then we may apply a sequence of column operations to reduce A to I. The transpose of these column operations is a sequence of row operations that reduces A^T to $I^T = I$. Since $\det I = \det I^T$, by what we proved above, we get $\det A = \det A^T$.

If A is not invertible, then A^T is also not invertible. By Theorem 5.1.4, we get $\det A^T = 0 = \det A$. □

Example 5.1.3. We calculate the determinant in Example 5.1.1 by mixing row and column operations

$$\det\begin{pmatrix}1 & 4 & 7\\ 2 & 5 & 8\\ 3 & 6 & a\end{pmatrix} \xlongequal[R_2-R_1]{R_3-R_2} \det\begin{pmatrix}1 & 4 & 7\\ 1 & 1 & 1\\ 1 & 1 & a-8\end{pmatrix} \xlongequal[C_2-C_1]{C_3-C_2} \det\begin{pmatrix}1 & 3 & 3\\ 1 & 0 & 0\\ 1 & 0 & a-9\end{pmatrix}$$
$$\xlongequal[R_2\leftrightarrow R_3]{C_1\leftrightarrow C_2,\ C_2\leftrightarrow C_3} -\det\begin{pmatrix}3 & 3 & 1\\ 0 & a-9 & 1\\ 0 & 0 & 1\end{pmatrix} = -3\cdot(a-9)\cdot 1 = -3(a-9).$$

Note that the negative sign after the third equality is due to odd number of exchanges.

Exercise 5.7. Prove that any orthogonal matrix has determinant ± 1.

Exercise 5.8. Prove that the determinant is the unique function that is multilinear and alternating on the row vectors, and satisfies $\det I = 1$.

Exercise 5.9. Suppose A and B are square matrices. Suppose O is the zero matrix. Use the multilinear and alternating property on columns of A and rows of B to prove

$$\det\begin{pmatrix}A & *\\ O & B\end{pmatrix} = \det A \det B = \det\begin{pmatrix}A & O\\ * & B\end{pmatrix}.$$

Exercise 5.10. Suppose A is an $m\times n$ matrix. Use Exercise 3.38 to prove that, if $\mathrm{rank} A = n$, then there is an $n\times n$ submatrix B inside A, such that $\det B \neq 0$. Similarly, if $\mathrm{rank} A = m$, then there is an $m\times m$ submatrix B inside A, such that $\det B \neq 0$.

Exercise 5.11. Suppose A is an $m \times n$ matrix. Prove that if columns of A are linearly dependent, then for any $n \times n$ submatrix B, we have $\det B = 0$. Similarly, if rows of A are linearly dependent, then any $m \times m$ submatrix inside A has vanishing determinant.

Exercise 5.12. Let $r = \mathrm{rank} A$.

1. Prove that there is an $r \times r$ submatrix B of A, such that $\det B \neq 0$.

2. Prove that if $s > r$, then any $s \times s$ submatrix of A has vanishing determinant.

We conclude that the rank r is the biggest number such that there is an $r \times r$ submatrix with non-vanishing determinent.

5.1.4 Cofactor Expansion

The first column of an $n \times n$ matrix $A = (a_{ij}) = (\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n)$ is

$$\vec{v}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} = a_{11}\vec{e}_1 + a_{21}\vec{e}_2 + \cdots + a_{n1}\vec{e}_n.$$

By the linearity of $\det A$ in the first column, we get

$$\det A = a_{11}D_1(A) + a_{21}D_2(A) + \cdots + a_{n1}D_n(A),$$

where

$$\begin{aligned} D_i(A) &= \det(\vec{e}_i\ \vec{v}_2\ \cdots\ \vec{v}_n) \\ &= \det \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{(i-1)2} & \cdots & a_{(i-1)n} \\ 1 & a_{i2} & \cdots & a_{in} \\ 0 & a_{(i+1)2} & \cdots & a_{(i+1)n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \det \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{(i-1)2} & \cdots & a_{(i-1)n} \\ 1 & 0 & \cdots & 0 \\ 0 & a_{(i+1)2} & \cdots & a_{(i+1)n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{pmatrix} \\ &= (-1)^{i+1} \det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{(i-1)2} & \cdots & a_{(i-1)n} \\ 0 & a_{(i+1)2} & \cdots & a_{(i+1)n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{pmatrix} \\ &= (-1)^{i-1} \det \begin{pmatrix} 1 & O \\ O & A_{i1} \end{pmatrix} = (-1)^{i-1} \det A_{i1}. \end{aligned}$$

Here the third equality is by the third type column operations, the fourth equality is by the first type row operations. The $(n-1) \times (n-1)$ matrix A_{i1} is obtained by deleting the i-th row and the 1st column of A. The last equality is due to the fact that $\det \begin{pmatrix} 1 & O \\ O & A_{i1} \end{pmatrix}$ is multilinear and alternating in columns of A_{i1}. We conclude the *cofactor expansion* formula

$$\det A = a_{11} \det A_{11} - a_{21} \det A_{21} + \cdots + (-1)^{n-1} a_{n1} \det A_{n1}.$$

For a 3×3 matrix, this means

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{21} \det \begin{pmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{pmatrix} + a_{31} \det \begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix}.$$

We may carry out the same argument with respect to the i-th column instead of the first one. Let A_{ij} be the matrix obtained by deleting the i-th row and j-th column from A. Then we get the cofactor expansions along the i-th column

$$\det A = (-1)^{1-i} a_{1i} \det A_{1i} + (-1)^{2-i} a_{2i} \det A_{2i} + \cdots + (-1)^{n-i} a_{ni} \det A_{ni}.$$

By $\det A^T = \det A$, we also have the cofactor expansion along the i-th row

$$\det A = (-1)^{i-1} a_{i1} \det A_{i1} + (-1)^{i-2} a_{i2} \det A_{i2} + \cdots + (-1)^{i-n} a_{in} \det A_{in}.$$

The combination of the row operation, column operation, and cofactor expansion gives an effective way of calculating the determinant.

Example 5.1.4. Cofactor expansion is the most convenient along rows or columns with only one nonzero entry.

$$\begin{aligned} \det \begin{pmatrix} t-1 & 2 & 4 \\ 2 & t-4 & 2 \\ 4 & 2 & t-1 \end{pmatrix} &\overset{C_1 - C_3}{=\!=\!=} \det \begin{pmatrix} t-5 & 2 & 4 \\ 0 & t-4 & 2 \\ -t+5 & 2 & t-1 \end{pmatrix} \\ &\overset{R_3 + R_1}{=\!=\!=} \det \begin{pmatrix} t-5 & 2 & 4 \\ 0 & t-4 & 2 \\ 0 & 4 & t+3 \end{pmatrix} \\ &\overset{\text{cofactor } C_1}{=\!=\!=} (t-5) \det \begin{pmatrix} t-4 & 2 \\ 4 & t+3 \end{pmatrix} \\ &= (t-5)(t^2 - t - 20) = (t-5)^2(t+4). \end{aligned}$$

Example 5.1.5. We calculate the determinant of the 4×4 Vandermonde matrix in

Example 2.3.5

$$\det\begin{pmatrix}1 & t_0 & t_0^2 & t_0^3\\ 1 & t_1 & t_1^2 & t_1^3\\ 1 & t_2 & t_2^2 & t_2^3\\ 1 & t_3 & t_3^2 & t_3^3\end{pmatrix} \overset{\substack{C_4-t_0C_3\\ C_3-t_0C_2\\ C_2-t_0C_1}}{=\!=\!=} \det\begin{pmatrix}1 & 0 & 0 & 0\\ 1 & t_1-t_0 & t_1(t_1-t_0) & t_1^2(t_1-t_0)\\ 1 & t_2-t_0 & t_2(t_2-t_0) & t_2^2(t_2-t_0)\\ 1 & t_3-t_0 & t_3(t_3-t_0) & t_3^2(t_3-t_0)\end{pmatrix}$$
$$= \det\begin{pmatrix}t_1-t_0 & t_1(t_1-t_0) & t_1^2(t_1-t_0)\\ t_2-t_0 & t_2(t_2-t_0) & t_2^2(t_2-t_0)\\ t_3-t_0 & t_3(t_3-t_0) & t_3^2(t_3-t_0)\end{pmatrix}$$
$$\overset{\substack{(t_1-t_0)R_1\\ (t_2-t_0)R_2\\ (t_3-t_0)R_3}}{=\!=\!=} (t_1-t_0)(t_2-t_0)(t_3-t_0)\det\begin{pmatrix}1 & t_1 & t_1^2\\ 1 & t_2 & t_2^2\\ 1 & t_3 & t_3^2\end{pmatrix}.$$

The second equality uses the cofactor expansion along the first row. We find that the calculation is reduced to the determinant of a 3×3 Vandermonde matrix. In general, by induction, we have

$$\det\begin{pmatrix}1 & t_0 & t_0^2 & \cdots & t_0^n\\ 1 & t_1 & t_1^2 & \cdots & t_1^n\\ 1 & t_2 & t_2^2 & \cdots & t_2^n\\ \vdots & \vdots & \vdots & & \vdots\\ 1 & t_n & t_n^2 & \cdots & t_n^n\end{pmatrix} = \prod_{i<j}(t_j - t_i).$$

Exercise 5.13. Calculate the determinant.

1. $\begin{pmatrix}2 & 1 & -3\\ -1 & 2 & 1\\ 3 & -2 & 1\end{pmatrix}$.
2. $\begin{pmatrix}0 & 1 & 2 & 3\\ 1 & 2 & 3 & 0\\ 2 & 3 & 0 & 1\\ 3 & 0 & 1 & 2\end{pmatrix}$.
3. $\begin{pmatrix}2 & 0 & 0 & 1\\ 0 & 1 & 3 & 0\\ -2 & 0 & -1 & 2\\ 0 & 3 & 1 & 2\end{pmatrix}$.

Exercise 5.14. Calculate the determinant.

1. $\begin{pmatrix}t-3 & -1 & 3\\ 1 & t-5 & 3\\ 6 & -6 & t+2\end{pmatrix}$.
2. $\begin{pmatrix}t & 2 & 3\\ 1 & t-1 & 1\\ -2 & -2 & t-5\end{pmatrix}$.
3. $\begin{pmatrix}t-2 & 1 & 1 & -1\\ 1 & t-2 & -1 & 1\\ 1 & -1 & t-2 & 1\\ -1 & 1 & 1 & t-2\end{pmatrix}$.

Exercise 5.15. Calculate the determinant.

1. $\begin{pmatrix}a & b & c\\ b & c & a\\ c & a & b\end{pmatrix}$.
2. $\begin{pmatrix}1 & t_0 & t_0^2 & t_0^4\\ 1 & t_1 & t_1^2 & t_1^4\\ 1 & t_2 & t_2^2 & t_2^4\\ 1 & t_3 & t_3^2 & t_3^4\end{pmatrix}$.

3. $\begin{pmatrix} 1 & t_0^2 & t_0^3 & t_0^4 \\ 1 & t_1^2 & t_1^3 & t_1^4 \\ 1 & t_2^2 & t_2^3 & t_2^4 \\ 1 & t_3^2 & t_3^3 & t_3^4 \end{pmatrix}$.

Exercise 5.16. Calculate the determinant.

1. $\begin{pmatrix} t & 0 & \cdots & 0 & 0 & a_0 \\ -1 & t & \cdots & 0 & 0 & a_1 \\ 0 & -1 & \cdots & 0 & 0 & a_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & t & a_{n-2} \\ 0 & 0 & \cdots & 0 & -1 & t+a_{n-1} \end{pmatrix}$.

2. $\det \begin{pmatrix} b_1 & a_1 & a_1 & \cdots & a_1 & a_1 \\ a_2 & b_2 & a_2 & \cdots & a_2 & a_2 \\ a_3 & a_3 & b_3 & \cdots & a_3 & a_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_n & a_n & a_n & \cdots & a_n & b_n \end{pmatrix}$.

Exercise 5.17. The Vandermonde matrix comes from the evaluation of polynomials $f(t)$ of degree n at $n+1$ distinct points. If two points t_0, t_1 are merged to the same point t_0, then the evaluations $f(t_0)$ and $f(t_1)$ should be replaced by $f(t_0)$ and $f'(t_0)$. The idea leads to the "derivative" Vandermonde matrix such as

$$\begin{pmatrix} 1 & t_0 & t_0^2 & t_0^3 & \cdots & t_0^n \\ 0 & 1 & 2t_0 & 3t_0^2 & \cdots & nt_0^{n-1} \\ 1 & t_2 & t_2^2 & t_2^3 & \cdots & t_2^n \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & t_n & t_n^2 & t_n^3 & \cdots & t_n^n \end{pmatrix}.$$

Find the determinant of the matrix. What about the general case of evaluations at $t_1, t_2 \dots, t_k$, with multiplicities $m_1, m_2, \dots, m_k$ (i.e., taking values $f(t_i), f'(t_i), \dots, f^{(m_i-1)}(t_i)$) satisfying $m_1 + m_2 + \cdots + m_k = n$?

Exercise 5.18. Use cofactor expansion to explain the determinant of upper and lower triangular matrices.

5.1.5 Cramer's Rule

The cofactor expansion suggests the *adjugate matrix* of a square matrix A

$$\mathrm{adj}(A) = \begin{pmatrix} \det A_{11} & -\det A_{21} & \cdots & (-1)^{n-1}\det A_{n1} \\ -\det A_{12} & \det A_{22} & \cdots & (-1)^{n-2}\det A_{n2} \\ \vdots & \vdots & & \vdots \\ (-1)^{1-n}\det A_{1n} & (-1)^{2-n}\det A_{2n} & \cdots & (-1)^{n-n}\det A_{nn} \end{pmatrix}.$$

The ij-entry of the matrix multiplication $A\,\mathrm{adj}(A)$ is

$$c_{ij} = (-1)^{j-1}a_{i1}\det A_{j1} + (-1)^{j-2}a_{i2}\det A_{j2} + \cdots + (-1)^{j-n}a_{in}\det A_{jn}.$$

The cofactor expansion of $\det A$ along the j-th row shows that the diagonal entries $c_{jj} = \det A$. In case $i \neq j$, we compare with the cofactor expansion of $\det A$ along the

j-th row, and find that c_{ij} is the determinant of the matrix B obtained by replacing the j-th row $(a_{j1}, a_{j2}, \dots, a_{jn})$ of A by the i-th row $(a_{i1}, a_{i2}, \dots, a_{in})$. Since both the i-th and the j-th rows of B are the same as the i-th row of A, by the alternating property of the determinant in the row vectors, we conclude that the off-diagonal entry $c_{ij} = 0$. For the 3×3 case, the argument is ($\hat{a}_{1i} = a_{1i}$, the hat $\hat{}$ is added to indicate along which row the cofactor expansion is made)

$$\begin{aligned} c_{12} &= -\hat{a}_{11} \det A_{21} + \hat{a}_{12} \det A_{22} - \hat{a}_{13} \det A_{23} \\ &= -\hat{a}_{11} \det \begin{pmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{pmatrix} + \hat{a}_{12} \det \begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix} - \hat{a}_{13} \det \begin{pmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{pmatrix} \\ &= \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \hat{a}_{11} & \hat{a}_{12} & \hat{a}_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = 0. \end{aligned}$$

The calculation of $A\,\text{adj}(A)$ gives

$$A\,\text{adj}(A) = (\det A)I.$$

In case A is invertible, this gives an explicit formula for the inverse

$$A^{-1} = \frac{1}{\det A}\text{adj}(A).$$

A consequence of the formula is the following explicit formula for the solution of $A\vec{x} = \vec{b}$.

Proposition 5.1.6 (Cramer's Rule)**.** *If $A = (\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n)$ is an invertible matrix, then the solution of $A\vec{x} = \vec{b}$ is given by*

$$x_i = \frac{\det(\vec{v}_1\ \cdots\ \overset{(i)}{\vec{b}}\ \cdots\ \vec{v}_n)}{\det A}.$$

Proof. In case A is invertible, the solution of $A\vec{x} = \vec{b}$ is

$$\vec{x} = A^{-1}\vec{b} = \frac{1}{\det A}\text{adj}(A)\vec{b}.$$

Then x_i is the i-th coordinate

$$x_i = \frac{1}{\det A}[(-1)^{1-i}(\det A_{1i})b_1 + (-1)^{2-1}(\det A_{2i})b_2 + \cdots + (-1)^{n-i}(\det A_{ni})b_n],$$

and $(-1)^{1-i}(\det A_{1i})b_1 + (-1)^{2-1}(\det A_{2i})b_2 + \cdots + (-1)^{n-i}(\det A_{ni})b_n$ is the cofactor expansion of the determinant of the matrix $(\vec{v}_1\ \cdots\ \overset{(i)}{\vec{b}}\ \cdots\ \vec{v}_n)$ obtained by replacing the i-th column of A by $\vec{b}$. $\square$

For the system of three equations in three variables, the Cramer's rule is

$$x_1 = \frac{\det\begin{pmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{pmatrix}}{\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}}, \quad x_2 = \frac{\det\begin{pmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{pmatrix}}{\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}}, \quad x_3 = \frac{\det\begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{pmatrix}}{\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}}.$$

Cramer's rule is not a practical way of calculating the solution for two reasons. The first is that it only applies to the case A is invertible (and in particular the number of equations is the same as the number of variables). The second is that the row operation is much more efficient method for finding solutions.

5.2 Geometry

The determinant of a real square matrix is a real number. The number is determined by its absolute value (magnitude) and its sign. The absolute value is the volume of the parallelotope spanned by the column vectors of the matrix. The sign is determined by the orientation of the Euclidean space.

5.2.1 Volume

The parallelogram spanned by (a, b) and (c, d) may be divided into five pieces A, B, B', C, C'. The center piece A is a rectangle. The triangle B has the same area (due to same base and height) as the dotted triangle below A. The triangle B' is identical to B. Therefore the areas of B and B' together is the area of the dotted rectangle below A. By the same reason, the areas of C and C' together is the area of the dotted rectangle on the left of A. The area of the parallelogram is then the sum of the areas of the rectangle A, the dotted rectangle below A, and the dotted rectangle on the left of A. This sum and is clearly

$$ad - bc = \det\begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Strictly speaking, the picture used for the argument assumes that (a, b) and (c, d) are in the first quadrant, and the first vector is "below" the second vector. One may try other possible positions of the two vectors and find that the area (which is always ≥ 0) is always the determinant up to a sign. Alternatively, we may also use the

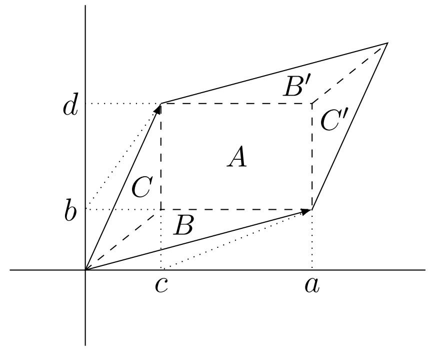

Figure 5.2.1: Area of parallelogram.

formula for the area in Section 4.1.1

$$\begin{aligned}\text{Area}((a,b),(c,d)) &= \sqrt{\|(a,b)\|^2\|(c,d)\|^2 - ((a,b)\cdot(c,d))^2}\\ &= \sqrt{(a^2+b^2)(c^2+d^2)-(ac+bd)^2}\\ &= \sqrt{(a^2c^2+a^2d^2+b^2c^2+b^2d^2)-(a^2c^2+2abcd+b^2d^2)}\\ &= \sqrt{a^2d^2+b^2c^2-2abcd} = |ad-bc|.\end{aligned}$$

Proposition 5.2.1. *The absolute value of the determinant of $A = (\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n)$ is the volume of the parallelotope spanned by column vectors*

$$P(A) = \{x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n : 0 \le x_i \le 1\}.$$

Proof. We show that a column operation has the same effect on the volume $\text{vol}(A)$ of $P(A)$ and on $|\det A|$. We illustrate the idea by only looking at the operations on the first two columns.

The operation $C_1 \leftrightarrow C_2$ is

$$A = (\vec{v}_1\ \vec{v}_2\ \cdots) \mapsto \tilde{A} = (\vec{v}_2\ \vec{v}_1\ \cdots).$$

The operation does not change the parallelotope. Therefore $P(\tilde{A}) = P(A)$, and we get $\text{vol}(\tilde{A}) = \text{vol}(A)$. This is the same as $|\det\tilde{A}| = |-\det A| = |\det A|$.

The operation $C_1 \to aC_2$ is

$$A = (\vec{v}_1\ \vec{v}_2\ \cdots) \mapsto \tilde{A} = (a\vec{v}_1\ \vec{v}_2\ \cdots).$$

The operation stretches the parallelotope by a factor of $|a|$ in the $\vec{v}_1$ direction, and keeps all the other directions fixed. Therefore the volume of $P(\tilde{A})$ is $|a|$ times the volume of $P(A)$, and we get $\text{vol}(\tilde{A}) = |a|\text{vol}(A)$. This is the same as $|\det\tilde{A}| = |a\det A| = |a||\det A|$.

The operation $C_1 \to C_1 + aC_2$ is

$$A = (\vec{v}_1\ \vec{v}_2\ \cdots) \mapsto \tilde{A} = (\vec{v}_1 + a\vec{v}_2\ \vec{v}_2\ \cdots).$$

The operation moves $\vec{v}_1$ vertex of the parallelotope along the direction of $\vec{v}_2$, and keeps all the other directions fixed. Therefore $P(\tilde{A})$ and $P(A)$ have the same base along the subspace $\text{Span}\{\vec{v}_2, \dots, \vec{v}_n\}$ (the $(n-1)$-dimensional parallelotope spanned by $\vec{v}_2, \dots, \vec{v}_n$) and the same height (from the tips of $\vec{v}_1 + a\vec{v}_2$ or $\vec{v}_1$ to the base $\text{Span}\{\vec{v}_2, \dots, \vec{v}_n\}$). This implies that the operation preserves the volume, and we get $\text{vol}(\tilde{A}) = \text{vol}(A)$. This is the same as $|\det \tilde{A}| = |\det A|$.

If A is invertible, then the column operation reduces A to the identity matrix I. The volume of $P(I)$ is 1 and we also have $\det I = 1$. If A is not invertible, then $P(A)$ is degenerate and has volume 0. We also have $\det A = 0$ by Proposition 5.1.4. This completes the proof. □

Example 5.2.1. The volume of the tetrahedron is $\frac{1}{6}$ of the corresponding parallelotope. For example, the volume of the tetrahedron with vertices $\vec{v}_0 = (1, 1, 1), \vec{v}_1 = (1, 2, 3), \vec{v}_2 = (2, 3, 1), \vec{v}_3 = (3, 1, 2)$ is

$$\begin{aligned}\frac{1}{6}|\det(\vec{v}_1 - \vec{v}_0, \vec{v}_2 - \vec{v}_0, \vec{v}_3 - \vec{v}_0)| &= \frac{1}{6}\left|\det\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix}\right| = \frac{1}{6}\left|\det\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 0 & -4 & 1 \end{pmatrix}\right| \\ &= \frac{1}{6}\left|-\det\begin{pmatrix} 1 & 2 \\ -4 & 1 \end{pmatrix}\right| = \frac{1}{6}|-9| = \frac{3}{2}.\end{aligned}$$

In general, the volume of the simplex with vertices $\vec{v}_0, \vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ is

$$\frac{1}{n!}|\det(\vec{v}_1 - \vec{v}_0 \quad \vec{v}_2 - \vec{v}_0 \quad \cdots \quad \vec{v}_n - \vec{v}_0)|.$$

5.2.2 Orientation

The line $\mathbb{R}$ has two orientations: positive and negative. The positive orientation is represented by a positive number (such as 1), and the negative orientation is represented by a negative number (such as -1).

The plane $\mathbb{R}^2$ has two orientations: counterclockwise, and clockwise. The counterclockwise orientation is represented by an ordered pair of directions, such that going from the first to the second direction is counterclockwise. The clockwise orientation is also represented by an ordered pair of directions, such that going from the first to the second direction is clockwise. For example, $\{\vec{e}_1, \vec{e}_2\}$ is counterclockwise, and $\{\vec{e}_2, \vec{e}_1\}$ is clockwise.

The space $\mathbb{R}^3$ has two orientations: right hand, and left hand. The left hand orientation is represented by an ordered triple of directions, such that going from the first to the second and then the third follows the right hand rule. The left hand orientation is also represented by an ordered triple of directions, such that going from the first to the second and then the third follows the left hand rule. For example, $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is right hand, and $\{\vec{e}_2, \vec{e}_1, \vec{e}_3\}$ is left hand.

How can we introduce orientation in a general vector space? For example, the line $H = \mathbb{R}(1,-2,3)$ is a 1-dimensional subspace of $\mathbb{R}^3$. The line has two directions represented by $(1,-2,3)$ and $-(1,-2,3) = (-1,2,-3)$. However, there is no preference as to which direction is positive and which is negative. Moreover, both $(1,-2,3)$ and $2(1,-2,3) = (2,-4,6)$ represent the same directions. In fact, all vectors representing the same direction as $(1,-2,3)$ form the set

$$o_{(1,-2,3)} = \{c(1,-2,3) \colon c > 0\}.$$

We note that any vector $c(1,-2,3) \in o_{(1,-2,3)}$ can be continuously deformed to $(1,-2,3)$ in H without passing through the zero vector

$$\vec{v}(t) = ((1-t)c + t)(1,-2,3) \in H, \quad \vec{v}(t) \neq \vec{0} \text{ for all } 0 \le t \le 1,$$

$$\vec{v}(0) = c(1,-2,3), \quad \vec{v}(1) = (1,-2,3).$$

Similarly, the set

$$o_{(-1,2,-3)} = \{c(-1,2,-3) \colon c > 0\} = \{c(1,-2,3) \colon c < 0\}$$

consists of all the vectors in H that can be continuously deformed to $(-1,2,-3)$ in H without passing through the zero vector.

Suppose $\dim V = 1$. If we fix a nonzero vector $\vec{v} \in V - \vec{0}$, then the two directions of V are represented by the disjoint sets

$$o_{\vec{v}} = \{c\vec{v} \colon c > 0\}, \quad o_{-\vec{v}} = \{-c\vec{v} \colon c > 0\} = \{c\vec{v} \colon c < 0\}.$$

Any two vectors in $o_{\vec{v}}$ can be continuously deformed to each other without passing through the zero vector, and the same can be said about $o_{-\vec{v}}$. Moreover, we have $o_{\vec{v}} \sqcup o_{-\vec{v}} = V - \vec{0}$. An orientation of V is a choice of one of the two sets.

Suppose $\dim V = 2$. An orientation of V is represented by a choice of ordered basis $\alpha = \{\vec{v}_1, \vec{v}_2\}$. Another choice $\beta = \{\vec{w}_1, \vec{w}_2\}$ represents the same orientation if α can be continuously deformed to β through bases (i.e., without passing through linearly dependent pair of vectors). For the special case of $V = \mathbb{R}^2$, it is intuitively clear that, if going from $\vec{v}_1$ to $\vec{v}_2$ is counterclockwise, then going from $\vec{w}_1$ to $\vec{w}_2$ must also be counterclockwise. The same intuition applies to the clockwise direction.

Let $\alpha = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ and $\beta = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$ be ordered bases of a vector space V. We say α and β are *compatibly oriented* if there is

$$\alpha(t) = \{\vec{v}_1(t), \vec{v}_2(t), \ldots, \vec{v}_n(t)\}, \quad t \in [0,1],$$

such that

1. $\alpha(t)$ is continuous, i.e., each $\vec{v}_i(t)$ is a continuous function of t.

2. $\alpha(t)$ is a basis for each t.

3. $\alpha(0)=\alpha$ and $\alpha(1)=\beta$.

In other words, α and β are connected by a continuous family of ordered bases.

We can imagine $\alpha(t)$ to be a "movie" that starts from α and ends at β. If $\alpha(t)$ connects α to β, then the reverse movie $\alpha(1-t)$ connects β to α. If $\alpha(t)$ connects α to β, and $\beta(t)$ connects β to γ, then we may stitch the two movies together and get a movie

$$\begin{cases} \alpha(2t), & \text{for } t\in[0,\frac{1}{2}], \\ \beta(2t-1), & \text{for } t\in[\frac{1}{2},1], \end{cases}$$

that connects α to γ. This shows that the compatible orientation is an equivalence relation.

In $\mathbb{R}^1$, a basis is a nonzero number. Any positive number can be deformed (through nonzero numbers) to 1, and any negative number can be deformed to -1.

In $\mathbb{R}^2$, a basis is a pair of non-parallel vectors. Any such pair can be deformed (through non-parallel pairs) to $\{\vec{e}_1,\vec{e}_2\}$ or $\{\vec{e}_1,-\vec{e}_2\}$, and $\{\vec{e}_1,\vec{e}_2\}$ cannot be deformed to $\{\vec{e}_1,-\vec{e}_2\}$ without passing through parallel pair.

Proposition 5.2.2. *Two ordered bases α,β are compatibly oriented if and only if* $\det[I]_{\beta\alpha}>0$.

Proof. Suppose α,β are compatibly oriented. Then there is a continuous family $\alpha(t)$ of ordered bases connecting the two. The function $f(t)=\det[I]_{\alpha(t)\alpha}$ is a continuous function satisfying $f(0)=\det[I]_{\alpha\alpha}=1$ and $f(t)\neq 0$ for all $t\in[0,1]$. By the intermediate value theorem, f never changes sign. Therefore $\det[I]_{\beta\alpha}=\det[I]_{\alpha(1)\alpha}=f(1)>0$. This proves the necessity.

For the sufficiency, suppose $\det[I]_{\beta\alpha}>0$. We study the effect of the operations on sets of vectors in Proposition 3.1.3 on the orientation. We describe the operations on $\vec{u}$ and $\vec{v}$, which are regarded as $\vec{v}_i$ and $\vec{v}_j$. The other vectors in the set are fixed and are omitted in the formula.

1. The sets $\{\vec{u},\vec{v}\}$ and $\{\vec{v},-\vec{u}\}$ are connected by the "90° rotation" (the quotation mark indicates that we only pretend $\{\vec{u},\vec{v}\}$ to be orthonormal)

   $$\alpha(t)=\{\cos\tfrac{t\pi}{2}\vec{u}+\sin\tfrac{t\pi}{2}\vec{v},-\sin\tfrac{t\pi}{2}\vec{u}+\cos\tfrac{t\pi}{2}\vec{v}\}.$$

   Therefore the first operation plus a sign change preserves the orientation. Combining two such rotations also shows that $\{\vec{u},\vec{v}\}$ and $\{-\vec{u},-\vec{v}\}$ are connected by "180° rotation", and are therefore compatibly oriented.

2. For any $c>0$, $\vec{v}$ and $c\vec{v}$ are connected by $\alpha(t)=((1-t)c+t)\vec{v}$. Therefore the second operation with positive scalar preserves the orientation.

3. The sets $\{\vec{u},\vec{v}\}$ and $\{\vec{u}+c\vec{v},\vec{v}\}$ are connected by the sliding $\alpha(t)=\{\vec{u}+tc\vec{v},\vec{v}\}$. Therefore the third operation preserves the orientation.

By these operations, any ordered basis can be modified to become certain "reduced column echelon set". If $V = \mathbb{R}^n$, then the "reduced column echelon set" is either $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_{n-1}, \vec{e}_n\}$ or $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_{n-1}, -\vec{e}_n\}$. In general, we use the β-coordinate isomorphism $V \cong \mathbb{R}^n$ to translate into a Euclidean space. Then we can use a sequence of operations above to modify α to either $\beta = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_{n-1}, \vec{w}_n\}$ or $\beta' = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_{n-1}, -\vec{w}_n\}$. Correspondingly, we have a continuous family $\alpha(t)$ of ordered bases connecting α to β or β'.

By the just proved necessity, we have $\det[I]_{\alpha(1)\alpha} > 0$. On the other hand, the assumption $\det[I]_{\beta\alpha} > 0$ implies $\det[I]_{\beta'\alpha} = \det[I]_{\beta'\beta}\det[I]_{\beta\alpha} = -\det[I]_{\beta\alpha} < 0$. Therefore $\alpha(1) \neq \beta'$, and we must have $\alpha(1) = \beta$. This proves that α and β are compatibly oriented. □

Proposition 5.2.2 shows that the orientation compatibility gives exactly two equivalence classes. We denote the equivalence class represented by α by

$$o_\alpha = \{\beta\colon \alpha, \beta \text{ compatibly oriented}\} = \{\beta\colon \det[I]_{\beta\alpha} > 0\}.$$

We also denote the other equivalence class by

$$-o_\alpha = \{\beta\colon \alpha, \beta \text{ incompatibly oriented}\} = \{\beta\colon \det[I]_{\beta\alpha} < 0\}.$$

In general, the set of all ordered bases is the disjoint union $o \cup o'$ of two equivalence classes. The choice of o or o' specifies an *orientation* of the vector space. In other words, an *oriented* vector space is a vector space equipped with a preferred choice of the equivalence class. An ordered basis α of an oriented vector space is *positively oriented* if α belongs to the preferred equivalence class, and is otherwise *negatively oriented*.

The *standard (positive) orientation* of $\mathbb{R}^n$ is the equivalence class represented by the standard basis $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_{n-1}, \vec{e}_n\}$. The standard negative orientation is then represented by $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_{n-1}, -\vec{e}_n\}$.

Proposition 5.2.3. *Let A be an $n \times n$ invertible matrix. Then* $\det A > 0$ *if and only if the columns of A form a positively oriented basis of* $\mathbb{R}^n$*, and* $\det A < 0$ *if and only if the columns of A form a negatively oriented basis of* $\mathbb{R}^n$.

5.2.3 Determinant of Linear Operator

A linear transformation $L\colon V \to W$ has the matrix $[L]_{\beta\alpha}$ with respect to bases α, β of V, W. For the matrix to be square, we need $\dim V = \dim W$. Then we can certainly define $\det[L]_{\beta\alpha}$ to be the "determinant with respect to the bases". The problem is that $\det[L]_{\beta\alpha}$ depends on the choice of α, β.

If $L\colon V \to V$ is a linear operator, then we usually choose $\alpha = \beta$ and consider $\det[L]_{\alpha\alpha}$. If we choose another basis α' of V, then $[L]_{\alpha'\alpha'} = P^{-1}[L]_{\alpha\alpha}P$, where

$P = [I]_{\alpha\alpha'}$ is the matrix between the two bases. Further by Proposition 5.1.3, we have

$$\det(P^{-1}[L]_{\alpha\alpha}P) = (\det P)^{-1}(\det[L]_{\alpha\alpha})(\det P) = \det[L]_{\alpha\alpha}.$$

Therefore the determinant of a linear operator is well defined.

Exercise 5.19. Prove that $\det(L \circ K) = \det L \det K$, $\det L^{-1} = \frac{1}{\det L}$, $\det L^* = \det L$.

Exercise 5.20. Prove that $\det(aL) = a^{\dim V} \det L$.

In another case, suppose $L\colon V \to W$ is a linear transformation between oriented inner product spaces. Then we may consider $[L]_{\beta\alpha}$ for positively oriented orthonormal bases α and β. If α' and β' are other orthonormal bases, then $[I]_{\alpha\alpha'}$ and $[I]_{\beta'\beta}$ are orthogonal matrices, and we have $\det[I]_{\alpha\alpha'} = \pm 1$ and $\det[I]_{\beta'\beta} = \pm 1$. If we further know that α' and β' are positively oriented, then these determinants are positive, and we get

$$\det[L]_{\beta'\alpha'} = \det[I]_{\beta'\beta}\ \det[L]_{\beta\alpha}\ \det[I]_{\alpha\alpha'} = 1\ \det[L]_{\beta\alpha}\ 1 = \det[L]_{\beta\alpha}.$$

This shows that the determinant of a linear transformation between oriented inner product spaces is well defined.

The two cases of well defined determinant of linear transformations suggest a common and deeper concept of determinant for linear transformations between vector spaces of the same dimension. The concept will be clarified in the theory of exterior algebra.

5.2.4 Geometric Axiom for Determinant

An $n \times n$ invertible matrix A gives an $(n+k) \times (n+k)$ invertible matrix $\begin{pmatrix} A & O \\ O & I \end{pmatrix}$, called the *stabilisation* of A. The parallelotope $P\begin{pmatrix} A & O \\ O & I \end{pmatrix}$ is the "orthogonal product" of $P(A)$ in $\mathbb{R}^n$ and $P(I)$ in $\mathbb{R}^k$. Moreover, $P(I)$ is the unit cube and should have volume 1. Therefore we expect the *stabilisation property* for the determinant

$$\det \begin{pmatrix} A & O \\ O & I \end{pmatrix} = \det A.$$

The other property we expect from the geometry is the *multiplication property* for invertible $n \times n$ matrices A and B

$$\det AB = \det A \det B.$$

Theorem 5.2.4. *The determinant of invertible matrices is the unique function satisfying the stabilisation property, the multiplication property, and $D(a) = a$.*

Proof. Let D be a function with the properties in the theorem. By taking $A = B = I$ in the multiplication property, we get $D(I) = 1$. By taking $B = A^{-1}$ in the multiplication property, we get $D(A^{-1}) = D(A)^{-1}$.

A row operation of third type is obtained by multiplying an elementary matrix $E_{ij}(c)$ to the left. The following shows that $E_{12}(c) = E_{13}(c)E_{32}(1)E_{13}(c)^{-1}E_{32}(1)^{-1}$.

$$\begin{pmatrix}1 & c & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & c\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & c\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}^{-1}.$$

By thinking of the matrix multiplication as happing only to the i-th, j-th and k-th columns and rows, the equality shows that $E_{ij}(c) = E_{ik}(c)E_{kj}(1)E_{ik}(c)^{-1}E_{kj}(1)^{-1}$. Then by the multiplication property, we have (at least for $n \times n$ matrices, $n \geq 3$)

$$D(E_{ij}(c)) = D(E_{ik}(c))\ D(E_{kj}(1))\ D(E_{ik}(c))^{-1}\ D(E_{kj}(1))^{-1} = 1.$$

By the multiplication property again, we have $D(E_{ij}(c)A) = D(A)$. In other words, row operations of third type do not change D.

Next we argue that third type row operations can change any invertible matrix to a diagonal one. The key is the following "skew row exchange" operation similar to the first type row operation

$$\begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix} \xrightarrow{R_1 - R_2} \begin{pmatrix}\vec{a} - \vec{b}\\ \vec{b}\end{pmatrix} \xrightarrow{R_2 + R_1} \begin{pmatrix}\vec{a} - \vec{b}\\ \vec{a}\end{pmatrix} \xrightarrow{R_1 - R_2} \begin{pmatrix}-\vec{b}\\ \vec{a}\end{pmatrix}.$$

Since the operation is a combination of three third type row operations, it does not change D.

An invertible $n \times n$ matrix A can be reduced to I by row operations. In more detail, we may first use first type row operation to move a nonzero entry in the first column of A into the 11-entry position, so that the 11-entry is nonzero. Of course this can also be achieved by "skew row exchange", with the cost of adding "$-$" sign. In other words, we may use third type row operations to make the 11-entry nonzero. Then we may use more third type row operations to make the other entries in the first column into 0. Therefore we may use only third type row operations to get

$$A \to \begin{pmatrix}a_1 & *\\ O & A_1\end{pmatrix},$$

where A_1 is an invertible $(n-1) \times (n-1)$ matrix. Then we repeat the process for A_1. This means using third type row operations to make the 22-entry nonzero, and all the other entries in the second column into 0. Inductively, we use third type row operations to get a diagonal matrix

$$A \to \hat{A} = \begin{pmatrix}a_1 & 0 & \cdots & 0\\ 0 & a_2 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & a_n\end{pmatrix}, \quad a_i \neq 0.$$

Since only third type row operations are used, we have $D(A) = D(\hat{A})$ and $\det A = \det \hat{A}$.

For a diagonal matrix, we have a sequence of third type row operations (the last $\to$ is a combination of three third type row operations)

$$\begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix} \to \begin{pmatrix} b & 1 \\ 0 & a \end{pmatrix} \to \begin{pmatrix} b & 1 \\ -ab & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ -ab & 0 \end{pmatrix} \to \begin{pmatrix} ab & 0 \\ 0 & 1 \end{pmatrix}.$$

By repreatedly using this, we may use third type row operations to get

$$\hat{A} \to \begin{pmatrix} a & O \\ O & I \end{pmatrix}, \quad a = a_1 a_2 \dots a_n.$$

Then we get $D(A) = D(\hat{A}) = D\begin{pmatrix} a & O \\ O & I \end{pmatrix}$ and $\det A = \det \hat{A} = \det \begin{pmatrix} a & O \\ O & I \end{pmatrix}$. By the stabilisation property, we get $D(A) = D(a) = a$. We also know $\det A = \det(a) = a$. This completes the proof that $D = \det$. $\square$

Behind the theorem is the algebraic K-theory of real numbers. The theory deals with the problem of invertible matrices that are equivalent up to stabilisation and third type operations. The proof of the theorem shows that, for real number matrices, every invertible matrix is equivalent to the diagonal matrix with diagonal entries $a, 1, 1, \dots, 1$.

Chapter 6

General Linear Algebra

A vector space is characterised by addition $V \times V \to V$ and scalar multiplication $\mathbb{R} \times V \to V$. The addition is an *internal* operation, and the scalar multiplication uses $\mathbb{R}$ which is *external* to V. Therefore the concept of vector space can be understood in two steps. First we single out the internal structure of addition.

Definition 6.0.1. An *abelian group* is a set V, together with the operation of addition

$$u + v \colon V \times V \to V,$$

such that the following are satisfied.

1. Commutativity: $u + v = v + u$.

2. Associativity: $(u + v) + w = u + (v + w)$.

3. Zero: There is an element $0 \in V$ satisfying $u + 0 = u = 0 + u$.

4. Negative: For any u, there is v, such that $u + v = 0 = v + u$.

For the external structure, we may use scalars other than the real numbers $\mathbb{R}$, and most of the linear algebra theory remain true. For example, if we use rational numbers $\mathbb{Q}$ in place of $\mathbb{R}$ in Definition 1.1.1 of vector spaces, then all the chapters so far remain valid, with taking square roots (in inner product space) as the only exception. A much more useful scalar is the complex numbers $\mathbb{C}$, for which all chapters remain valid, except a more suitable version of the complex inner product needs to be developed.

Further extension of the scalar could abandon the requirement that nonzero scalars can be divided. This leads to the useful concept of modules over rings.

6.1 Complex Linear Algebra

6.1.1 Complex Number

A complex number is of the form $a+ib$, with $a,b\in\mathbb{R}$ and $i=\sqrt{-1}$ satisfying $i^2=-1$. A complex number has the *real* and *imaginary* parts

$$\mathrm{Re}(a+ib)=a,\quad \mathrm{Im}(a+ib)=b.$$

The addition and multiplication of complex numbers are

$$\begin{aligned}(a+ib)+(c+id)&=(a+c)+i(b+d),\\ (a+ib)(c+id)&=(ac-bd)+i(ad+bc).\end{aligned}$$

It can be easily verified that the operations satisfy the usual properties (such as commutativity, associativity, distributivity) of arithmetic operations. In particular, the subtraction is

$$(a+ib)-(c+id)=(a-c)+i(b-d),$$

and the division is

$$\frac{a+ib}{c+id}=\frac{(a+ib)(c-id)}{(c+id)(c-id)}=\frac{(ac+bd)+i(-ad+bc)}{c^2+d^2}.$$

All complex numbers $\mathbb{C}$ can be identified with the Euclidean space $\mathbb{R}^2$, with the real part as the first coordinate and the imaginary part as the second coordinate. The corresponding real vector has length r and angle θ (i.e., polar coordinate), and we have

$$a+ib=r\cos\theta+ir\sin\theta=re^{i\theta}.$$

The first equality is trigonometry, and the second equality uses the expansion (the theoretical explanation is the complex analytic continuation of the exponential function of real numbers)

$$\begin{aligned}e^{i\theta}&=1+\frac{1}{1!}i\theta+\frac{1}{2!}(i\theta)^2+\cdots+\frac{1}{n!}(i\theta)^n+\cdots\\ &=\left(1-\frac{1}{2!}\theta^2+\frac{1}{4!}\theta^4+\cdots\right)+i\left(\theta-\frac{1}{3!}\theta^3+\frac{1}{5!}\theta^5+\cdots\right)\\ &=\cos\theta+i\sin\theta.\end{aligned}$$

The complex exponential has the usual properties of the real exponential (because of complex analytic continuation), and we can easily get the multiplication and division of complex numbers

$$(re^{i\theta})(r'e^{i\theta'})=rr'e^{i(\theta+\theta')},\quad \frac{re^{i\theta}}{r'e^{i\theta'}}=\frac{r}{r'}e^{i(\theta-\theta')}.$$

The polar viewpoint easily shows that multiplying $re^{i\theta}$ means scaling by r and rotation by θ.

The *complex conjugation* $\overline{a+bi} = a - bi$ preserves the four arithmetic operations

$$\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2, \quad \overline{z_1-z_2} = \bar{z}_1 - \bar{z}_2, \quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2, \quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}.$$

Therefore the conjugation an *automorphism* (self-isomorphism) of $\mathbb{C}$. Geometrically, the conjugation means reflection with respect to the x-axis. This gives the conjugation in polar coordinates

$$\overline{re^{i\theta}} = re^{-i\theta}.$$

The length can also be expressed in terms of the conjugation

$$z\bar{z} = a^2 + b^2 = r^2, \quad |z| = r = \sqrt{z\bar{z}}.$$

This suggests that the positivity of the inner product can be extended to complex vector spaces, as long as we modify the inner product by using the complex conjugation.

A major difference between $\mathbb{R}$ and $\mathbb{C}$ is that the polynomial $t^2 + 1$ has no root in $\mathbb{R}$ but has a pair of roots $\pm i$ in $\mathbb{C}$. In fact, complex numbers has the following so called *algebraically closed* property.

Theorem 6.1.1 (Fundamental Theorem of Algebra)**.** *Any non-constant complex polynomial has roots.*

The real number $\mathbb{R}$ is not algebraically closed.

6.1.2 Complex Vector Space

By replacing $\mathbb{R}$ with $\mathbb{C}$ in the definition of vector spaces, we get the defintion of complex vector spaces.

Definition 6.1.2. A *(complex) vector space* is a set V, together with the operations of addition and scalar multiplication

$$\vec{u} + \vec{v} \colon V \times V \to V, \quad a\vec{u} \colon \mathbb{C} \times V \to V,$$

such that the following are satisfied.

1. Commutativity: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.

2. Associativity for addition: $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$.

3. Zero: There is an element $\vec{0} \in V$ satisfying $\vec{u} + \vec{0} = \vec{u} = \vec{0} + \vec{u}$.

4. Negative: For any $\vec{u}$, there is $\vec{v}$ (to be denoted $-\vec{u}$), such that $\vec{u}+\vec{v} = \vec{0} = \vec{v}+\vec{u}$.

5. One: $1\vec{u} = \vec{u}$.

6. Associativity for scalar multiplication: $(ab)\vec{u} = a(b\vec{u})$.

7. Distributivity in the scalar: $(a+b)\vec{u} = a\vec{u} + b\vec{u}$.

8. Distributivity in the vector: $a(\vec{u}+\vec{v}) = a\vec{u} + a\vec{v}$.

The complex Euclidean space $\mathbb{C}^n$ has the usual addition and scalar multiplication

$$(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n),$$
$$a(x_1, x_2, \dots, x_n) = (ax_1, ax_2, \dots, ax_n).$$

All the material in the chapters on vector space, linear transformation, subspace remain valid for complex vector spaces. The key is that complex numbers $\mathbb{C}$ has four arithmetic operations like real numbers $\mathbb{R}$, and all the properties of the arithmetic operations remain valid. The most important is the division that is used in the cancelation property and the proof of results such as Proposition 1.3.8.

The *conjugate vector space* $\bar{V}$ of a complex vector space V is the same set V with the same addition, but with the scalar multiplication modified by the conjugation

$$a\bar{\vec{v}} = \overline{\bar{a}\vec{v}}.$$

Here $\bar{\vec{v}}$ is the vector $\vec{v} \in V$ regarded as a vector in $\bar{V}$. Then $a\bar{\vec{v}}$ is the scalar multiplication in $\bar{V}$. On the right side is the vector $\bar{a}\vec{v} \in V$ regarded as a vector in $\bar{V}$. The definition means that multiplying a in $\bar{V}$ is the same as multiplying $\bar{a}$ in V. For example, the scalar multiplication in the conjugate Euclidean space $\bar{\mathbb{C}}^n$ is

$$a(x_1, x_2, \dots, x_n) = (\bar{a}x_1, \bar{a}x_2, \dots, \bar{a}x_n).$$

Let $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be a basis of V. Let $\bar{\alpha} = \{\bar{\vec{v}}_1, \bar{\vec{v}}_2, \dots, \bar{\vec{v}}_n\}$ be the same set considered as being inside $\bar{V}$. Then $\bar{\alpha}$ is a basis of $\bar{V}$ (see Exercise 6.2). The "identity map" $V \to \bar{V}$ is

$$\vec{v} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n \in V \mapsto \bar{x}_1\bar{\vec{v}}_1 + \bar{x}_2\bar{\vec{v}}_2 + \cdots + \bar{x}_n\bar{\vec{v}}_n = \bar{\vec{v}} \in \bar{V}.$$

Note that the scalar multiplication on the right is in $\bar{V}$. This means

$$[\bar{\vec{v}}]_{\bar{\alpha}} = \overline{[\vec{v}]_{\alpha}}.$$

Exercise 6.1. Prove that a complex subspace of V is also a complex subspace of $\bar{V}$. Moreover, sum and direct sum of subspaces in V and $\bar{V}$ are the same.

Exercise 6.2. Suppose α is a set of vectors in V, and $\bar{\alpha}$ is the corresponding set in $\bar{V}$.

1. Prove that α and $\bar{\alpha}$ span the same subspace.

2. Prove that α is linearly independent in V if and only if $\bar{\alpha}$ is linearly independent in $\bar{V}$.

If we restrict the scalars to real numbers, then a complex vector space becomes a real vector space. For example, the complex vector space $\mathbb{C}$ becomes the real vector space $\mathbb{R}^2$, with $z = x + iy \in \mathbb{C}$ identified with $(x, y) \in \mathbb{R}^2$. In general, $\mathbb{C}^n$ becomes $\mathbb{R}^{2n}$, and $\dim_{\mathbb{R}} V = 2 \dim_{\mathbb{C}} V$ for any complex vector space V.

Conversely, how can a real vector space V be obtained by restricting the scalars of a complex vector space to real numbers? Clearly, we need to add the scalar multiplication by i. This is a special real linear operator $J \colon V \to V$ satisfying $J^2 = -I$. Given such an operator, we define the complex multiplication by

$$(a + ib)\vec{v} = a\vec{v} + bJ(\vec{v}).$$

Then we can verify all the axioms of the complex vector space. For example,

$$\begin{aligned}
(a + ib)((c + id)\vec{v}) &= (a + ib)(c\vec{v} + dJ(\vec{v})) \\
&= a(c\vec{v} + dJ(\vec{v})) + bJ(c\vec{v} + dJ(\vec{v})) \\
&= ac\vec{v} + adJ(\vec{v}) + bcJ(\vec{v}) + bdJ^2(\vec{v}) \\
&= (ac - bd)\vec{v} + (ad + bc)J(\vec{v}), \\
((a + ib)(c + id))\vec{v} &= ((ac - bd) + i(ad + bc))\vec{v} \\
&= (ac - bd)\vec{v} + (ad + bc)J(\vec{v}).
\end{aligned}$$

Therefore the operator J is a *complex structure* on the real vector space.

Proposition 6.1.3. *A complex vector space is a real vector space equipped with a linear operator J satisfying $J^2 = -I$.*

Exercise 6.3. Prove that $\mathbb{R}$ has no complex structure. In other words, there is no real linear operator $J \colon \mathbb{R} \to \mathbb{R}$ satisfying $J^2 = -I$.

Exercise 6.4. Suppose J is a complex structure on a real vector space V, and $\vec{v} \in V$ is nonzero. Prove that $\vec{v}, J(\vec{v})$ are linearly independent.

Exercise 6.5. Suppose J is a complex structure on a real vector space V. Suppose $\vec{v}_1$, $J(\vec{v}_1)$, $\vec{v}_2$, $J(\vec{v}_2)$, $\dots$, $\vec{v}_k$, $J(\vec{v}_k)$ are linearly independent. Prove that if $\vec{v}$ is not in the span of these vectors, then $\vec{v}_1, J(\vec{v}_1), \vec{v}_2, J(\vec{v}_2), \dots, \vec{v}_k, J(\vec{v}_k), \vec{v}, J(\vec{v})$ is still linearly independent.

Exercise 6.6. Suppose J is a complex structure on a real vector space V. Prove that there is a set $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$, and we get $J(\alpha) = \{J(\vec{v}_1), J(\vec{v}_2), \dots, J(\vec{v}_n)\}$, such that $\alpha \cup J(\alpha)$ is a real basis of V. Moreover, prove that α is a complex basis of V (with complex structure given by J) if and only if $\alpha \cup J(\alpha)$ is a real basis of V.

Exercise 6.7. If a real vector space has an operator J satisfying $J^2 = -I$, prove that the real dimension of the space is even.

6.1.3 Complex Linear Transformation

A map $L\colon V \to W$ of complex vector spaces is *(complex) linear* if

$$L(\vec{u}+\vec{v}) = L(\vec{u}) + L(\vec{v}), \quad L(a\vec{v}) = aL(\vec{v}).$$

Here a can be any complex number. It is *conjugate linear* if

$$L(\vec{u}+\vec{v}) = L(\vec{u}) + L(\vec{v}), \quad L(a\vec{v}) = \bar{a}L(\vec{v}).$$

Using the conjugate vector space, the following are equivalent

1. $L\colon V \to W$ is conjugate linear.
2. $L\colon \bar{V} \to W$ is linear.
3. $L\colon V \to \bar{W}$ is linear.

We have the vector space $\mathrm{Hom}(V,W)$ of linear transformations, and also the vector space $\overline{\mathrm{Hom}}(V,W)$ of conjugate linear transformations. They are related by

$$\overline{\mathrm{Hom}}(V,W) = \mathrm{Hom}(\bar{V},W) = \mathrm{Hom}(V,\bar{W}).$$

A conjugate linear transformation $L\colon \mathbb{C}^n \to \mathbb{C}^m$ is given by a matrix

$$\begin{aligned} L(\vec{x}) &= L(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n) \\ &= \bar{x}_1 L(\vec{e}_1) + \bar{x}_2 L(\vec{e}_2) + \cdots + \bar{x}_n L(\vec{e}_n) = A\bar{\vec{x}}. \end{aligned}$$

The matrix of L is

$$A = (L(\vec{e}_1)\ L(\vec{e}_2)\ \ldots\ L(\vec{e}_n)) = [L]_{\epsilon\epsilon}.$$

If we regard L as a linear transformation $L\colon \mathbb{C}^n \to \bar{\mathbb{C}}^m$, then the formula becomes $L(\vec{x}) = \overline{A\bar{\vec{x}}} = \bar{A}\vec{x}$, or $\bar{A} = [L]_{\bar{\epsilon}\epsilon}$. If we regard L as $L\colon \bar{\mathbb{C}}^n \to \mathbb{C}^m$, then the formula becomes $L(\bar{\vec{x}}) = A\bar{\vec{x}}$, or $A = [L]_{\epsilon\bar{\epsilon}}$.

In general, the matrix of a conjugate linear transformation $L\colon V \to W$ is

$$[L]_{\beta\alpha} = [L(\alpha)]_{\beta}, \quad [L(\vec{v})]_{\beta} = [L]_{\beta\alpha}\overline{[\vec{v}]_{\alpha}}.$$

Then the matrices of the two associated linear transformations are

$$[L\colon V \to \bar{W}]_{\bar{\beta}\alpha} = \overline{[L]_{\beta\alpha}}, \quad [L\colon \bar{V} \to W]_{\beta\bar{\alpha}} = [L]_{\beta\alpha}.$$

We see that conjugation on the target adds conjugation to the matrix, and the conjugation on the source preserves the matrix.

Due to two types of linear transformations, there are two dual spaces

$$V^* = \mathrm{Hom}(V, \mathbb{C}), \quad \bar{V}^* = \overline{\mathrm{Hom}}(V, \mathbb{C}) = \mathrm{Hom}(V, \bar{\mathbb{C}}) = \mathrm{Hom}(\bar{V}, \mathbb{C}).$$

We note that $\mathrm{Hom}(\bar{V}, \mathbb{C})$ means the dual space $(\bar{V})^*$ of the conjugate space $\bar{V}$. Moreover, we have a conjugate linear isomorphism (i.e., invertible conjugate linear transformation)

$$l(\vec{x}) \in V^* \mapsto \bar{l}(\vec{x}) = \overline{l(\vec{x})} \in \bar{V}^*,$$

which can also be regarded as a usual linear isomorphism between the conjugate $\overline{V^*}$ of the dual space V^* and the dual $\bar{V}^*$ of the conjugate space $\bar{V}$. In this sense, there is no ambiguity about the notation $\bar{V}^*$.

A basis $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ of V has the corresponding conjugate basis $\bar{\alpha}$ of $\bar{V}$ and the dual basis α^* of V^*. Both further have the same corresponding basis $\bar{\alpha}^* = \{\bar{\vec{v}}_1^*, \bar{\vec{v}}_2^*, \dots, \bar{\vec{v}}_n^*\}$ of $\bar{V}^*$, given by

$$\bar{\vec{v}}_i^*(x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n) = \bar{x}_i = \overline{\vec{v}_i^*(\vec{x})}.$$

We can use this to justify the matrices of linear transformations

$$\begin{aligned}
&[L]_{\bar{\beta}\alpha} = [L(\alpha)]_{\bar{\beta}} = \bar{\beta}^*(L(\alpha)) = \overline{\beta^*(L(\alpha))} = \overline{[L(\alpha)]}_{\beta} = \overline{[L]}_{\beta\alpha},\\
&[L]_{\beta\bar{\alpha}} = [L(\bar{\alpha})]_{\beta} = [L(\alpha)]_{\beta} = [L]_{\beta\alpha}.
\end{aligned}$$

In the second line, L in $L(\bar{\alpha})$ means $L\colon \bar{V} \to W$, and L in $L(\alpha)$ means $L\colon V \to W$. Therefore they are the same in W.

A conjugate linear transformation $L\colon V \to W$ induces a conjugate linear transformation $L^*\colon V^* \to W^*$. We have $(aL)^* = aL^*$, which means that the map

$$\overline{\mathrm{Hom}}(V, W) \to \overline{\mathrm{Hom}}(W^*, V^*)$$

is a linear transformation. Then the equivalent viewpoints

$$\overline{\mathrm{Hom}}(V, W) \to \mathrm{Hom}(\bar{W}^*, V^*) = \mathrm{Hom}(W^*, \bar{V}^*)$$

are conjugate linear, which means $(aL)^* = \bar{a}L^*$.

Exercise 6.8. What is $\bar{\bar{V}}$? What is $\overline{\bar{V}^*}$? What is $\mathrm{Hom}(\bar{V}, \bar{W})$? What is $\overline{\mathrm{Hom}}(\bar{V}, W)$?

Exercise 6.9. We have $\mathrm{Hom}(V, W) = \mathrm{Hom}(\bar{V}, \bar{W})$? What is the relation between $[L]_{\beta\alpha}$ and $[L]_{\bar{\beta}\bar{\alpha}}$?

Exercise 6.10. What is the composition of a (conjugate) linear transformation with another (conjugate) linear transformation? Interpret this as induced (conjugate) linear transformations L_*, L^* and repeat Exercises 2.12, 2.13.

6.1.4 Complexification and Conjugation

For any real vector space W, we may construct its *complexification* $V = W \oplus iW$. Here iW is a copy of W in which a vector $\vec{w} \in W$ is denoted $i\vec{w}$. Then V becomes a complex vector space by

$$\begin{aligned}(\vec{w}_1 + i\vec{w}_2) + (\vec{w}_1' + i\vec{w}_2') &= (\vec{w}_1 + \vec{w}_1') + i(\vec{w}_2 + \vec{w}_2'),\\ (a + ib)(\vec{w}_1 + i\vec{w}_2) &= (a\vec{w}_1 - b\vec{w}_2) + i(b\vec{w}_1 + a\vec{w}_2).\end{aligned}$$

A typical example is $\mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n$.

Conversely, when is a complex vector space V the complexification of a real vector space W? This is the same as finding a real subspace $W \subset V$ (i.e., $\vec{u}, \vec{v} \in W$ and $a, b \in \mathbb{R}$ implying $a\vec{u} + b\vec{v} \in W$), such that $V = W \oplus iW$. By Exercise 6.6, for any (finite dimensional) complex vector space V, such W is exactly the real span

$$\begin{aligned}\mathrm{Span}^{\mathbb{R}}\alpha &= \{x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n \colon x_i \in \mathbb{R}\}\\ &= \mathbb{R}\vec{v}_1 + \mathbb{R}\vec{v}_2 + \cdots + \mathbb{R}\vec{v}_n\end{aligned}$$

of a complex basis $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ of V. For example, if we take the standard basis ϵ of $V = \mathbb{C}^n$, then we get $W = \mathrm{Span}^{\mathbb{R}}\epsilon = \mathbb{R}^n$. This makes $\mathbb{C}^n$ into the complexification of $\mathbb{R}^n$.

Due to the many possible choices of complex basis α, the subspace W is not unique. For example, $V = \mathbb{C}^n$ is also the complexification of $W = i\mathbb{R}^n$ (real span of $i\epsilon = \{i\vec{e}_1, i\vec{e}_2, \dots, i\vec{e}_n\}$). Next we show that such W are in one-to-one correspondence with conjugation operators on V.

A *conjugation* of a complex vector space is a conjugate linear operator $C \colon V \to V$ satisfying $C^2 = I$. The basic example of conjugation is

$$\overline{(z_1, z_2, \dots, z_n)} = (\bar{z}_1, \bar{z}_2, \dots, \bar{z}_n) \colon \mathbb{C}^n \to \mathbb{C}^n.$$

If $V = W \oplus iW$, then

$$C(\vec{w}_1 + i\vec{w}_2) = \vec{w}_1 - i\vec{w}_2$$

is a conjugation on V. Conversely, if C is a conjugation on W, then we introduce

$$W = \mathrm{Re}V = \{\vec{w} \colon C(\vec{w}) = \vec{w}\}.$$

For $\vec{u} = i\vec{w} \in iW$, we have $C(\vec{u}) = C(i\vec{w}) = \bar{i}C(\vec{w}) = -i\vec{w} = -\vec{u}$, and we get

$$iW = \mathrm{Im}V = \{\vec{u} \colon C(\vec{u}) = -\vec{u}\}.$$

Moreover, any $\vec{v} \in V$ has the unique decomposition

$$\vec{v} = \vec{w}_1 + i\vec{w}_2, \quad \vec{w}_1 = \tfrac{1}{2}(\vec{v} + C(\vec{v})) \in W, \quad \vec{w}_2 = \tfrac{1}{2}(\vec{v} - C(\vec{v})) \in iW.$$

This gives the direct sum $V = W \oplus iW = \mathrm{Re}V \oplus \mathrm{Im}V$.

For a given complexification $V = W \oplus iW$, or equivalently a conjugation C, we will not use C and instead denote the conjugation by

$$\overline{\vec{w}_1 + i\vec{w}_2} = \vec{w}_1 - i\vec{w}_2, \quad \vec{w}_1, \vec{w}_2 \in W.$$

A complex subspace $H \subset V$ has the corresponding *conjugate subspace*

$$\bar{H} = \{\bar{\vec{v}} \colon \vec{v} \in H\} = \{\vec{w}_1 - i\vec{w}_2 \colon \vec{w}_1, \vec{w}_2 \in W,\ \vec{w}_1 + i\vec{w}_2 \in H\}.$$

Note that we also use $\bar{H}$ to denote the conjugate space of H. For the given conjugation on V, the two notations are naturally isomorphic.

Example 6.1.1. Let $C \colon \mathbb{C} \to \mathbb{C}$ be a conjugation. Then we have a fixed complex number $c = C(1)$, and $C(z) = C(z1) = \bar{z}C(1) = c\bar{z}$. The formula satisfies the first two properties of the conjugation. Moreover, we have $C^2(z) = C(c\bar{z}) = c\overline{c\bar{z}} = |c|^2 z$. Therefore the third condition means $|c| = 1$. We find that conjugations on $\mathbb{C}$ are in one-to-one correspondence with points $c = e^{i\theta}$ on the unit circle.

For $C(z) = e^{i\theta}\bar{z}$, we have (the real part with respect to c)

$$\begin{aligned} \mathrm{Re}_\theta \mathbb{C} &= \{z \colon e^{i\theta}\bar{z} = z\} = \{re^{\rho} \colon e^{i\theta}e^{-\rho} = e^{\rho},\ r \geq 0\} \\ &= \{re^{\rho} \colon 2\rho = \theta \bmod 2\pi,\ r \geq 0\} = \mathbb{R}e^{i\frac{\theta}{2}}. \end{aligned}$$

This is the real line of angle $\frac{\theta}{2}$. The imaginary part

$$\mathrm{Im}_\theta \mathbb{C} = i\mathbb{R}e^{i\frac{\theta}{2}} = e^{\frac{\pi}{2}}\mathbb{R}e^{i\frac{\theta}{2}} = \mathbb{R}e^{i\frac{\theta+\pi}{2}}$$

is the real line of angle $\frac{\theta+\pi}{2}$, and is orthogonal to $\mathrm{Re}_\theta \mathbb{C}$.

Exercise 6.11. Suppose $V = W \oplus iW$ is a complexification. What is the conjugation with respect to the complexification $V = iW \oplus W$?

Exercise 6.12. Suppose $V = W \oplus iW$ is a complexification, and $\alpha \subset W$ is a set of real vectors.

1. Prove that $\mathrm{Span}^{\mathbb{C}}\alpha$ is the complexification of $\mathrm{Span}^{\mathbb{R}}\alpha$.

2. Prove that α is $\mathbb{R}$-linearly independent if and only if it is $\mathbb{C}$-linearly independent.

3. Prove that α is an $\mathbb{R}$-basis of W if and only if it is a $\mathbb{C}$-basis of V.

Exercise 6.13. For a complex subspace H of a complexification $V = W \oplus iW$, prove that $\bar{H} = H$ if and only if $H = U \oplus iU$ for a real subspace U of W.

Exercise 6.14. Suppose V is a complex vector space with conjugation. Suppose α is a set of vectors, and $\bar{\alpha}$ is the set of conjugations of vectors in α.

1. Prove that $\text{Span}\bar{\alpha} = \overline{\text{Span}\alpha}$.
2. Prove that α is linearly independent if and only if $\bar{\alpha}$ is linearly independent.
3. Prove that α is a basis of V if and only if $\bar{\alpha}$ is a basis of V.

Exercise 6.15. Prove that

$$\text{Ran}\bar{A} = \overline{\text{Ran}A}, \quad \text{Nul}\bar{A} = \overline{\text{Nul}A}, \quad \text{rank}\bar{A} = \text{rank}A.$$

Consider a complex linear transformation between complexifications of real vector spaces

$$L\colon V \oplus iV \to W \oplus iW.$$

We have $\mathbb{R}$-linear transformations $L_1 = \text{Re}L, L_2 = \text{Im}L\colon V \to W$ given by

$$L(\vec{v}) = L_1(\vec{v}) + iL_2(\vec{v}), \quad \vec{v} \in V.$$

Then the whole L is determined by L_1, L_2

$$\begin{aligned} L(\vec{v}_1 + i\vec{v}_2) &= L(\vec{v}_1) + iL(\vec{v}_2) \\ &= L_1(\vec{v}_1) + iL_2(\vec{v}_1) + i(L_1(\vec{v}_2) + iL_2(\vec{v}_2)) \\ &= L_1(\vec{v}_1) - L_2(\vec{v}_2) + i(L_2(\vec{v}_1) + L_1(\vec{v}_2)). \end{aligned}$$

This means that, in the block form, we have (using $iV \cong V$ and $iW \cong W$)

$$L = \begin{pmatrix} L_1 & -L_2 \\ L_2 & L_1 \end{pmatrix} : V \oplus (i)V \to W \oplus (i)W.$$

Let α and β be $\mathbb{R}$-bases of V and W. Then we have real matrices $[L_1]_{\beta\alpha}$ and $[L_2]_{\beta\alpha}$. Moreover, we know α and β are also $\mathbb{C}$-bases of $V \oplus iV$ and $W \oplus W$, and

$$[L]_{\beta\alpha} = [L_1]_{\beta\alpha} + i[L_2]_{\beta\alpha}.$$

The conjugation on the target space also gives the conjugation of L

$$\bar{L}(\vec{v}) = \overline{L(\bar{\vec{v}})}\colon V \oplus iV \to W \oplus iW.$$

We have

$$\bar{L}(\vec{v}) = L_1(\vec{v}) - iL_2(\vec{v}), \quad \vec{v} \in V,$$

and

$$[\bar{L}]_{\beta\alpha} = [L_1]_{\beta\alpha} - i[L_2]_{\beta\alpha} = \overline{[L]}_{\beta\alpha}.$$

Exercise 6.16. Prove that a complex linear transformation $L\colon V \oplus iV \to W \oplus iW$ satisfies $L(W) \subset W$ if and only if it preserves the conjugation: $L(\bar{\vec{v}}) = \overline{L(\vec{v})}$ (this is the same as $L = \bar{L}$).

Exercise 6.17. Suppose $V = W \oplus iW = W' \oplus iW'$ are two complexifications. What can you say about the real and imaginary parts of the identity $I\colon W \oplus iW \to W' \oplus iW'$?

6.1.5 Conjugate Pair of Subspaces

We study a complex subspace H of a complixification $V = W \oplus iW$ satisfying $V = H \oplus \bar{H}$, where $\bar{H}$ is the conjugation subspace.

The complex subspace H has a complex basis

$$\alpha = \{\vec{u}_1 - i\vec{w}_1, \vec{u}_2 - i\vec{w}_2, \dots, \vec{u}_m - i\vec{w}_m\}, \quad \vec{u}_j, \vec{w}_j \in W.$$

Then $\bar{\alpha} = \{\vec{u}_1 + i\vec{w}_1, \vec{u}_2 + i\vec{w}_2, \dots, \vec{u}_m + i\vec{w}_m\}$ is a complex basis of $\bar{H}$, and $\alpha \cup \bar{\alpha}$ is a (complex) basis of $V = H \oplus \bar{H}$. We introduce the real and imaginary parts of $\bar{\alpha}$

$$\beta = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}, \quad \beta^\dagger = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m\}.$$

Since vectors in $\alpha \cup \bar{\alpha}$ and vectors in $\beta \cup \beta^\dagger$ are (complex) linear combinations of each other, and the two sets have the same number of vectors, we know $\beta \cup \beta^\dagger$ is also a complex basis of V.

Since the vectors in $\beta \cup \beta^\dagger$ are in W, the set is actually a (real) bases of the real subspace W. We introduce real subspaces

$$E = \text{Span}\beta, \quad E^\dagger = \text{Span}\beta^\dagger, \quad W = E \oplus E^\dagger.$$

We also introduce a real isomorphism by taking $\vec{u}_j$ to $\vec{w}_j$

$$\dagger\colon E \cong E^\dagger, \quad \vec{u}_j^\dagger = \vec{w}_j.$$

Because $\vec{u}_j - i\vec{u}_j^\dagger = \vec{u}_j - i\vec{w}_j \in \alpha \subset H$, the isomorphism has the property that $\vec{u} - i\vec{u}^\dagger \in H$ for any $\vec{u} \in E$.

Proposition 6.1.4. *Suppose $V = W \oplus iW = H \oplus \bar{H}$ for a real vector space W and a complex subspace H. Then there are real subspaces $E, E^\dagger$ and an isomorphism $\vec{u} \leftrightarrow \vec{u}^\dagger$ between E and $E^\dagger$, such that*

$$W = E \oplus E^\dagger, \quad H = \{(\vec{u}_1 + \vec{u}_2^\dagger) + i(\vec{u}_2 - \vec{u}_1^\dagger)\colon \vec{u}_1, \vec{u}_2 \in E\}.$$

Conversely, any complex subspace H constructed in this way satisfies $V = H \oplus \bar{H}$.

We note that E and $E^\dagger$ are not unique. For example, given a basis α of H, the following is also a basis of H

$$\begin{aligned}\alpha' &= \{i(\vec{u}_1 - i\vec{w}_1), \vec{u}_2 - i\vec{w}_2, \dots, \vec{u}_m - i\vec{w}_m\} \\ &= \{\vec{w}_1 + i\vec{u}_1, \vec{u}_2 - i\vec{w}_2, \dots, \vec{u}_m - i\vec{w}_m\}.\end{aligned}$$

Therefore we may also choose

$$E = \mathbb{R}\vec{w}_1 + \mathbb{R}\vec{u}_2 + \cdots + \mathbb{R}\vec{u}_m, \quad E^\dagger = \mathbb{R}(-\vec{u}_1) + \mathbb{R}\vec{w}_2 + \cdots + \mathbb{R}\vec{w}_m,$$

in our construction and take $\vec{w}_1^\dagger = -\vec{u}_1$, $\vec{u}_j^\dagger = \vec{w}_j$ for $j \geq 2$.

Proof. First we need to show that our construction gives the formula of H in the proposition. For any $\vec{u}_1, \vec{u}_2 \in E$, we have

$$\vec{h}(\vec{u}_1, \vec{u}_2) = \vec{u}_1 + \vec{u}_2^\dagger + i(\vec{u}_2 - \vec{u}_1^\dagger) = \vec{u}_1 - i\vec{u}_1^\dagger + i(\vec{u}_2 - i\vec{u}_2^\dagger) \in H.$$

Conversely, we want to show that any $\vec{h} \in H$ is of the form $\vec{h}(\vec{u}_1, \vec{u}_2)$. We have the decomposition

$$\vec{h} = \vec{u} + i\vec{w}, \quad \vec{u} = \vec{u}_1 + \vec{u}_2^\dagger \in W, \ \vec{w} \in W, \ \vec{u}_1, \vec{u}_2 \in E.$$

Then

$$\vec{w} - \vec{u}_2 + \vec{u}_1^\dagger = -i(\vec{h} - \vec{h}(\vec{u}_1, \vec{u}_2)) \in H.$$

However, we also know $\vec{w} - \vec{u}_2 + \vec{u}_1^\dagger \in W$. By

$$\vec{v} \in H \cap W \implies \vec{v} = \bar{\vec{v}} \in \bar{H} \implies \vec{v} \in H \cap \bar{H} = \{\vec{0}\},$$

we conclude $\vec{w} - \vec{u}_2 + \vec{u}_1^\dagger = \vec{0}$, and

$$\vec{h} = \vec{u} + i\vec{w} = \vec{u}_1 + \vec{u}_2^\dagger + i(\vec{u}_2 - \vec{u}_1^\dagger) = \vec{h}(\vec{u}_1, \vec{u}_2).$$

Now we show the converse, that H as constructed in the proposition is a complex subspace satisfying $V = H \oplus \bar{H}$. First, since E and $E^\dagger$ are real subspaces, H is closed under addition and scalar multiplication by real numbers. To show H is a complex subspace, therefore, it is sufficient to prove $\vec{h} \in H$ implies $i\vec{h} \in H$. This follows from

$$i(\vec{u}_1 + \vec{u}_2^\dagger) + i(\vec{u}_2 - \vec{u}_1^\dagger)) = -\vec{u}_2 + \vec{u}_1^\dagger + i(\vec{u}_1 + \vec{u}_2^\dagger) = \vec{h}(-\vec{u}_2, \vec{u}_1).$$

Next we prove $W \subset H + \bar{H}$, which implies $iW \subset H + \bar{H}$ and $V = W + iW \subset H + \bar{H}$. For $\vec{u} \in E$, we have

$$\begin{aligned}
\vec{u} &= \tfrac{1}{2}(\vec{u} - i\vec{u}^\dagger) + \tfrac{1}{2}(\overline{\vec{u} - i\vec{u}^\dagger}) = \vec{h}(\tfrac{1}{2}\vec{u}, 0) + \overline{\vec{h}(\tfrac{1}{2}\vec{u}, 0)} \in H,\\
\vec{u}^\dagger &= \tfrac{1}{2}(\vec{u}^\dagger + i\vec{u}) + \tfrac{1}{2}(\overline{\vec{u}^\dagger + i\vec{u}}) = \vec{h}(0, \tfrac{1}{2}\vec{u}) + \overline{\vec{h}(0, \tfrac{1}{2}\vec{u})} \in H.
\end{aligned}$$

This shows that $E \subset H$ and $E^\dagger \subset H$, and therefore $W = E + E^\dagger \subset H$.

Finally, we prove that $H + \bar{H}$ is a direct sum by showing that $H \cap \bar{H} = \{\vec{0}\}$. A vector in the intersection is of the form $\vec{h}(\vec{u}_1, \vec{u}_2) = \overline{\vec{h}(\vec{w}_1, \vec{w}_2)}$. By $V = E \oplus E^\dagger \oplus iE \oplus iE^\dagger$, the equality gives

$$\vec{u}_1 = \vec{w}_1, \quad \vec{u}_2^\dagger = \vec{w}_2^\dagger, \quad \vec{u}_2 = -\vec{w}_2, \quad -\vec{u}_1^\dagger = \vec{w}_1^\dagger.$$

Since $\dagger$ is an isomorphism, $\vec{u}_2^\dagger = \vec{w}_2^\dagger$ and $-\vec{u}_1^\dagger = \vec{w}_1^\dagger$ imply $\vec{u}_2 = \vec{w}_2$ and $-\vec{u}_1 = \vec{w}_1$. Then it is easy to show that all component vectors vanish. $\square$

Exercise 6.18. Prove that E in Proposition 6.1.4 uniquely determines $E^\dagger$ by

$$E^\dagger = \{\vec{w} \in W : \vec{u} - i\vec{w} \in H \text{ for some } \vec{u} \in E\}.$$

In the setup of Proposition 6.1.4, we consider a linear operator $L\colon V \to V$ satisfying $L(W) \subset W$ and $L(H) \subset H$. The condition $L(W) \subset W$ means L is a *real operator* with respect to the complex conjugation structure on V. The property is equivalent to L commuting with the conjugation operation (see Exercise 6.16). The condition $L(H) \subset H$ means that H is an *invariant subspace* of L.

By $L(H) \subset H$ and the description of H in Proposition 6.1.4, there are linear transformations $L_1, L_2\colon E \to E$ defined by

$$L(\vec{u}) - iL(\vec{u}^\dagger) = L(\vec{u} - i\vec{u}^\dagger) = (L_1(\vec{u}) + L_2(\vec{u})^\dagger) + i(L_2(\vec{u}) - L_1(\vec{u})^\dagger), \quad \vec{u} \in E.$$

By $V = W \oplus iW$ and $L(\vec{u}), L(\vec{u}^\dagger) \in L(W) \subset W$, we get

$$L(\vec{u}) = L_1(\vec{u}) + L_2(\vec{u})^\dagger, \quad L(\vec{u}^\dagger) = -L_2(\vec{u}) + L_1(\vec{u})^\dagger,$$

or

$$L\begin{pmatrix} \vec{u}_1 \\ \vec{u}_2^\dagger \end{pmatrix} = \begin{pmatrix} L_1(\vec{u}_1) \\ L_2(\vec{u}_1)^\dagger \end{pmatrix} + \begin{pmatrix} -L_2(\vec{u}_2) \\ L_1(\vec{u}_2)^\dagger \end{pmatrix} = \begin{pmatrix} L_1 & -L_2 \\ L_2 & L_1 \end{pmatrix} \begin{pmatrix} \vec{u}_1 \\ \vec{u}_2^\dagger \end{pmatrix}.$$

In other words, with respect to the direct sum $W = E \oplus E^\dagger$ and using $E \cong E^\dagger$, the restriction $L|_W\colon W \to W$ has the block matrix form

$$L|_W = \begin{pmatrix} L_1 & -L_2 \\ L_2 & L_1 \end{pmatrix}.$$

6.1.6 Complex Inner Product

The complex dot product between $\vec{x}, \vec{y} \in \mathbb{C}^n$ cannot be $x_1y_1 + x_2y_2 + \cdots + x_ny_n$ because complex numbers do not satisfy $x_1^2 + x_2^2 + \cdots + x_n^2 \geq 0$. The correct dot product that has the positivity property is

$$(x_1, x_2, \dots, x_n) \cdot (y_1, y_2 \dots, y_n) = x_1\bar{y}_1 + \cdots + x_n\bar{y}_n = \vec{x}^T\bar{\vec{y}}.$$

More generally, a bilinear function b on $V \times V$ satisfies $b(i\vec{v}, i\vec{v}) = i^2 b(\vec{v}, \vec{v}) = -b(\vec{v}, \vec{v})$. Therefore the bilinearity is contradictory to the positivity.

Definition 6.1.5. A *(Hermitian) inner product* on a complex vector space V is a function

$$\langle \vec{u}, \vec{v} \rangle\colon V \times V \to \mathbb{C},$$

such that the following are satisfied.

1. Sesquilinearity: $\langle a\vec{u} + b\vec{u}', \vec{v} \rangle = a\langle \vec{u}, \vec{v} \rangle + b\langle \vec{u}', \vec{v} \rangle$, $\langle \vec{u}, a\vec{v} + b\vec{v}' \rangle = \bar{a}\langle \vec{u}, \vec{v} \rangle + \bar{b}\langle \vec{u}, \vec{v}' \rangle$.

2. Conjugate symmetry: $\langle \vec{v}, \vec{u} \rangle = \overline{\langle \vec{u}, \vec{v} \rangle}$.

3. Positivity: $\langle \vec{u}, \vec{u} \rangle \geq 0$ and $\langle \vec{u}, \vec{u} \rangle = 0$ if and only if $\vec{u} = \vec{0}$.

The sesquilinear (sesqui is Latin for "one and half") property is the linearity in the first vector and the conjugate linearity in the second vector. Using the conjugate vector space $\bar{v}$, this means that the function is bilinear on $V \times \bar{V}$.

The length of a vector is still $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$. Due to the complex value of the inner product, the angle between nonzero vectors is not defined, and the area is not defined. The Cauchy-Schwarz inequality (Proposition 4.1.2) still holds, so that the length still has the three properties in Proposition 4.1.3.

Exercise 6.19. Suppose a function $b(\vec{u}, \vec{v})$ is linear in $\vec{u}$ and is conjugate symmetric. Prove that $b(\vec{u}, \vec{v})$ is conjugate linear in $\vec{v}$.

Exercise 6.20. Prove the complex version of the Cauchy-Schwarz inequality.

Exercise 6.21. Prove the polarisation identity in the complex inner product space (compare Exercise 4.12)

$$\langle \vec{u}, \vec{v} \rangle = \frac{1}{4}(\|\vec{u}+\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2 + i\|\vec{u}+i\vec{v}\|^2 - i\|\vec{u}-i\vec{v}\|^2).$$

Exercise 6.22. Prove the parallelogram identity in the complex inner product space (compare Exercise 4.14)

$$\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = 2(\|\vec{u}\|^2 + \|\vec{v}\|^2).$$

Exercise 6.23. Prove that $\langle \vec{u}, \vec{v} \rangle_{\bar{V}} = \langle \vec{v}, \vec{u} \rangle_V$ is a complex inner product on the conjugate space $\bar{V}$. The "identity map" $V \to \bar{V}$ is a conjugate linear isomorphism that preserves the length, but changes the inner product by conjugation.

The orthogonality $\vec{u} \perp \vec{v}$ is still defined by $\langle \vec{u}, \vec{v} \rangle = 0$, and hence the concepts of orthogonal set, orthonormal set, orthonormal basis, orthogonal complement, orthogonal projection, etc. Due to conjugate linearity in the second variable, one needs to be careful with the order of vectors in formulae. For example, the formula for the coefficient in Proposition 4.2.8 must have $\vec{x}$ as the first vector in inner product.

The Gram-Schmidt process still works, with the formula for the real Gram-Schmidt process still valid. Consequently, any finite dimensional complex inner product space is isometrically isomorphic to the complex Euclidean space with the dot product.

The inner product induces a linear isomorphism

$$\vec{v} \in V \mapsto \langle \vec{v}, \cdot \rangle \in \bar{V}^*$$

and a conjugate linear isomorphism

$$\vec{v} \in V \mapsto \langle \cdot, \vec{v} \rangle \in V^*.$$

A basis is *self-dual* if it is mapped to its dual basis. Under either isomorphism, a basis is self-dual if and only if it is orthonormal.

A linear transformation $L\colon V \to W$ has the dual linear transformation $L^*\colon W^* \to V^*$. Using the conjugate linear isomorphisms induced by the inner product, the dual L^* is translated into the *adjoint* $L^*\colon W \to V$

$$\begin{array}{ccc} W^* & \xrightarrow{L^*(\text{dual})} & V^* \\ {\scriptstyle\langle\cdot,\vec{w}\rangle}\uparrow\cong & & \cong\uparrow{\scriptstyle\langle\cdot,\vec{v}\rangle} \\ W & \xrightarrow{L^*(\text{adjoint})} & V \end{array} \qquad \langle L(\vec{v}), \vec{w}\rangle = \langle \vec{v}, L^*(\vec{w})\rangle.$$

Due to the combination of two conjugate linear isomorphisms, the adjoint L^* is a (non-conjugate, i.e., usual) complex linear transformation.

The adjoint satisfies

$$I^* = I,\ (L+K)^* = L^* + K^*,\ (aL)^* = \bar{a}L^*,\ (L\circ K)^* = K^*\circ L^*,\ (L^*)^* = L.$$

In particular, the adjoint is a conjugate linear isomorphism

$$L \mapsto L^*\colon \operatorname{Hom}(V,W) \cong \operatorname{Hom}(W,V).$$

The conjugate linearity can be verified directly

$$\langle \vec{v}, (aL)^*(\vec{w})\rangle = \langle aL(\vec{v}), \vec{w}\rangle = a\langle L(\vec{v}), \vec{w}\rangle = a\langle \vec{v}, L^*(\vec{w})\rangle = \langle \vec{v}, \bar{a}L^*(\vec{w})\rangle.$$

Let $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ and $\beta = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m\}$ be bases of V and W. By Proposition 2.3.1, we have

$$[L^*\colon W^* \to V^*]_{\alpha^*\beta^*} = [L]_{\beta\alpha}^T.$$

If we further know that α and β are orthonormal, then $\alpha \subset V$ is taken to $\alpha^* \subset V^*$ and $\alpha \subset W$ is taken to $\beta^* \subset W$. The calculation in Section 6.1.3 shows that the composition with the conjugate linear isomorphism $W \cong W^*$ does not change the matrix, but the composition with the conjugate linear isomorphism $V \cong V^*$ adds complex conjugation to the matrix. Therefore we get

$$[L^*\colon W \to V]_{\alpha\beta} = \overline{[L^*\colon W^* \to V^*]}_{\alpha^*\beta^*} = \overline{[L]}_{\beta\alpha}^T = [L]_{\beta\alpha}^*.$$

Here we denote the conjugate transpose of a matrix by $A^* = \bar{A}^T$.

For the special case of complex Euclidean spaces with dot products and the standard bases, the matrix for the adjoint can be verified directly

$$A\vec{x}\cdot\vec{y} = (A\vec{x})^T\overline{\vec{y}} = \vec{x}^T A^T\overline{\vec{y}} = \vec{x}^T\overline{A^*\vec{y}} = \vec{x}\cdot A^*\vec{y}.$$

A linear transformation $L\colon V \to W$ is an isometry (i.e., preserves the inner product) if and only if $L^*L = I$. In case $\dim V = \dim W$, this implies L is invertible, $L^{-1} = L^*$, and $LL^* = I$.

In terms of matrix, a square complex matrix U is called a *unitary matrix* if it satisfies $U^*U = I$. A real unitary matrix is an orthogonal matrix. A unitary matrix is always invertible with $U^{-1} = U^*$. Unitary matrices are precisely the matrices of isometric isomorphisms with respect to orthonormal bases.

Exercise 6.24. Prove

$$(\mathrm{Ran}L)^{\perp} = \mathrm{Ker}L^*, \quad (\mathrm{Ran}L^*)^{\perp} = \mathrm{Ker}L, \quad (\mathrm{Ker}L)^{\perp} = \mathrm{Ran}L^*, \quad (\mathrm{Ker}L^*)^{\perp} = \mathrm{Ran}L.$$

Exercise 6.25. Prove that the formula in Exercise 4.58 extends to linear operator L on complex inner product space

$$\langle L(\vec{u}), \vec{v}\rangle + \langle L(\vec{v}), \vec{u}\rangle = \langle L(\vec{u}+\vec{v}), \vec{u}+\vec{v}\rangle - \langle L(\vec{u}), \vec{u}\rangle - \langle L(\vec{v}), \vec{v}\rangle.$$

Then prove that the following are equivalent

1. $\langle L(\vec{v}), \vec{v}\rangle$ is real for all $\vec{v}$.
2. $\langle L(\vec{u}), \vec{v}\rangle + \langle L(\vec{v}), \vec{u}\rangle$ is real for all $\vec{u}, \vec{v}$.
3. $L = L^*$.

Exercise 6.26. Prove that $\langle L(\vec{v}), \vec{v}\rangle$ is imaginery for all $\vec{v}$ if an donly if $L^* = -L$.

Exercise 6.27. Define the adjoint $L^*\colon W \to V$ of a conjugate linear transformation $L\colon V \to W$ by $\langle L(\vec{v}), \vec{w}\rangle = \langle \vec{v}, L^*(\vec{w})\rangle$. Prove that L^* is conjugate linear, and find the matrix of L^* with respect to orthonormal bases.

Exercise 6.28. Prove that the following are equivalent for a linear transformation $L\colon V \to W$.

1. L is an isometry.
2. L preserves length.
3. L takes an orthonormal basis to an orthonormal set.
4. $L^*L = I$.
5. The matrix A of L with respect to orthonormal bases of V and W satisfies $A^*A = I$.

Exercise 6.29. Prove that the columns of a complex matrix A form an orthogonal set with respect to the dot product if and only if A^*A is diagonal. Moreover, the columns form an orthonormal set if and only if $A^*A = I$.

Exercise 6.30. Prove that P is an orthogonal projection if and only if $P^2 = P = P^*$.

Suppose W is a real vector space with real inner product. Then we may extend the inner product to the complexification $V = W \oplus iW$ in the unique way

$$\langle \vec{u}_1 + i\vec{u}_2, \vec{w}_1 + i\vec{w}_2\rangle = \langle \vec{u}_1, \vec{w}_1\rangle + \langle \vec{u}_2, \vec{w}_2\rangle + i\langle \vec{u}_2, \vec{w}_1\rangle - i\langle \vec{u}_1, \vec{w}_2\rangle.$$

In particular, the length satisfies the Pythagorean theorem

$$\|\vec{w}_1 + i\vec{w}_2\|^2 = \|\vec{w}_1\|^2 + \|\vec{w}_2\|^2.$$

The other property is that $\vec{v} = \vec{u} + i\vec{w}$ and $\bar{\vec{v}} = \vec{u} - i\vec{w}$ are orthogonal if and only if

$$0 = \langle \vec{u} + i\vec{w}, \vec{u} - i\vec{w} \rangle = \|\vec{u}\|^2 - \|\vec{w}\|^2 + 2i\langle \vec{u}, \vec{w} \rangle.$$

This means that

$$\|\vec{u}\| = \|\vec{w}\|, \quad \langle \vec{u}, \vec{w} \rangle = 0.$$

In other words, $\vec{u}$, $\vec{w}$ is the scalar multiple of an orthonormal pair.

Conversely, we may restrict the inner product of a complexification $V = W \oplus iW$ to the real subspace W. If the restriction always takes the real value, then it is an inner product on W, and the inner product on V is the complexification of the inner product on W.

Problem: What is the condition for the restriction of inner product to be real?

Exercise 6.31. Suppose H_1, H_2 are subspaces of real inner product space V. Prove that $H_1 \oplus iH_1, H_2 \oplus iH_2$ are orthogonal subspaces in $V \oplus iV$ if and only if H_1, H_2 are orthogonal subspaces of V.

6.2 Field and Polynomial

6.2.1 Field

The scalars is external to vector spaces. In the theory of vector spaces we developed so far, only the four arithmetic operations of the scalars are used until the introduction of the inner product. If we regard any system with four arithmetic operations as "generalised numbers", then the similar linear algebra theory without inner product can be developed over such generalised numbers.

Definition 6.2.1. A *field* is a set with arithmetic operations $+, -, \times, \div$ satisfying the usual properties.

In fact, only two operations $+, \times$ are required in a field, and $-, \div$ are regarded as the "opposite" or "inverse" of the two operations. Here are the axioms for $+$ and $\times$.

1. Commutativity: $a + b = b + a$, $ab = ba$.
2. Associativity: $(a + b) + c = a + (b + c)$, $(ab)c = a(bc)$.
3. Distributivity: $a(b + c) = ab + ac$.
4. Unit: There are $0, 1$, such that $a + 0 = a = 0 + a$, $a1 = a = 1a$.
5. Inverse: For any a, there is b, such that $a + b = 0 = b + a$. For any $a \neq 0$, there is c, such that $ac = 1 = ca$.

The real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$ and the rational numbers $\mathbb{Q}$ are examples of fields.

Example 6.2.1. The field of $\sqrt{2}$-rational numbers is $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} \colon a, b \in \mathbb{Q}\}$. The four arithmetic operations are

$$\begin{aligned}
(a + b\sqrt{2}) + (c + d\sqrt{2}) &= (a + c) + (b + d)\sqrt{2}, \\
(a + b\sqrt{2}) - (c + d\sqrt{2}) &= (a - c) + (b - d)\sqrt{2}, \\
(a + b\sqrt{2})(c + d\sqrt{2}) &= (ac + 2bd) + (ad + bc)\sqrt{2}, \\
\frac{a + b\sqrt{2}}{c + d\sqrt{2}} = \frac{(a + b\sqrt{2})(c - d\sqrt{2})}{(c + d\sqrt{2})(c - d\sqrt{2})} &= \frac{ac - 2bd}{c^2 - 2d^2} + \frac{-ad + bc}{c^2 - 2d^2}\sqrt{2}.
\end{aligned}$$

The field is a *subfield* of $\mathbb{R}$, just like a subspace.

In general, for any integer p with no square factors, we have the field of $\sqrt{p}$-rational numbers $\mathbb{Q}[\sqrt{p}] = \{a + b\sqrt{p} \colon a, b \in \mathbb{Q}\}$, with

$$\begin{aligned}
(a + b\sqrt{p}) + (c + d\sqrt{p}) &= (a + c) + (b + d)\sqrt{p}, \\
(a + b\sqrt{p})(c + d\sqrt{p}) &= (ac + pbd) + (ad + bc)\sqrt{p}.
\end{aligned}$$

Example 6.2.2. We have the field of $(\sqrt{2}, \sqrt{3})$-rational numbers

$$\mathbb{Q}[\sqrt{2}, \sqrt{3}] = \{a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} \colon a, b, c, d \in \mathbb{Q}\}.$$

The field can be understood as similar to $(\mathbb{Q}[\sqrt{2}])[\sqrt{3}]$, where $\mathbb{Q}[\sqrt{2}]$ plays the role of $\mathbb{Q}$ in $\mathbb{Q}[\sqrt{3}]$

$$\begin{aligned}
(\mathbb{Q}[\sqrt{2}])[\sqrt{3}] &= \{a + b\sqrt{3} \colon a, b \in \mathbb{Q}[\sqrt{2}]\} \\
&= \{(a_1 + a_2\sqrt{2}) + (b_1 + b_2\sqrt{2})\sqrt{3} \colon a_1, a_2, b_1, b_2 \in \mathbb{Q}\} \\
&= \{a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} \colon a, b, c, d \in \mathbb{Q}\}.
\end{aligned}$$

In particular, we know how to divide numbers in this field. They key is the reciprocal

$$\begin{aligned}
\frac{1}{a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}} &= \frac{(a + b\sqrt{2}) - (c + d\sqrt{2})\sqrt{3}}{[(a + b\sqrt{2}) + (c + d\sqrt{2})\sqrt{3}][(a + b\sqrt{2}) - (c + d\sqrt{2})\sqrt{3}]} \\
&= \frac{(a + b\sqrt{2}) - (c + d\sqrt{2})\sqrt{3}}{(a + b\sqrt{2})^2 - 3(c + d\sqrt{2})^2}.
\end{aligned}$$

The rest of the division calculation is similar to $\mathbb{Q}[\sqrt{2}]$.

Example 6.2.3. For any integer p with no cubic factors, we have the field of $\sqrt[3]{p}$-rational numbers

$$\mathbb{Q}[\sqrt[3]{p}] = \{a + b\sqrt[3]{p} + c\sqrt[3]{p^2} \colon a, b, c \in \mathbb{Q}\}.$$

The $+, -, \times$ operations in $\mathbb{Q}[\sqrt[3]{p}]$ are obvious. We only need to explain $\div$. Specifically, we need to explain the existence of the reciprocal of $x = a + b\sqrt[3]{p} + c\sqrt[3]{p^2} \neq 0$. The idea is that $\mathbb{Q}[\sqrt[3]{p}]$ is a $\mathbb{Q}$-vector space spanned by 1, $\sqrt[3]{p}$, $\sqrt[3]{p^2}$. Therefore the four vectors $1, x, x^2, x^3$ are $\mathbb{Q}$-linearly dependent (this is the linear algebra theory prior to inner product)

$$a_0 + a_1x + a_2x^2 + a_3x^3 = 0, \quad a_i \in \mathbb{Q}.$$

If $a_0 \neq 0$, then the equality implies

$$\begin{aligned}\frac{1}{x} &= -\frac{a_1}{a_0} - \frac{a_2}{a_0}x - \frac{a_3}{a_0}x^2 \\ &= -\frac{a_1}{a_0} - \frac{a_2}{a_0}(a + b\sqrt[3]{p} + c\sqrt[3]{p^2}) - \frac{a_3}{a_0}(a + b\sqrt[3]{p} + c\sqrt[3]{p^2})^2 \\ &= c_0 + c_1\sqrt[3]{p} + c_2\sqrt[3]{p^2}, \quad c_0, c_1, c_2 \in \mathbb{Q}.\end{aligned}$$

If $x_0 = 0$, then we get $a_1 + a_2x + a_3x^2 = 0$ and ask whether $a_1 \neq 0$. The precess goes on and eventually gives the formula for $\frac{1}{x}$ in all cases.

Example 6.2.4. The field of integers modulo a prime number 5 is the set of mod 5 congruence classes

$$\mathbb{Z}_5 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}.$$

For example,

$$\bar{2} = \{2 + 5k \colon k \in \mathbb{Z}\} = \{\dots, -8, -3, 2, 7, 12, \dots\}$$

is the set of all integers n such that $n - 2$ is divisible by 5. In particular, $\bar{n} = \bar{0}$ means n is divisible by 5.

The addition and multiplication are the obvious operations, such as

$$\bar{3} + \bar{4} = \overline{3+4} = \bar{7} = \bar{2}, \quad \bar{3} \cdot \bar{4} = \overline{3 \cdot 4} = \overline{12} = \bar{2}.$$

The two operations satisfy the usual properties. The addition has the obvious opposite operation of subtraction, such as $\bar{3} - \bar{4} = \overline{3-4} = \overline{-1} = \bar{4}$.

The division is a bit more complicated. This means that, for any $\bar{x} \in \mathbb{Z}_5^* = \mathbb{Z}_5 - \bar{0}$, we need to find $\bar{y} \in \mathbb{Z}_5$ satisfying $\bar{x} \cdot \bar{y} = \bar{1}$. Then we have $\bar{y} = \bar{x}^{-1}$. To find $\bar{y}$, we consider the following map

$$\mathbb{Z}_5 \to \mathbb{Z}_5, \quad \bar{y} \mapsto \bar{x}\bar{y} = \overline{xy}.$$

Since $\bar{x} \neq \bar{0}$ means that x is not divisible by 5, the following shows the map is one-to-one

$$\begin{aligned}\overline{xy} = \overline{xz} &\implies \overline{x(y-z)} = \overline{xy - xz} = \overline{xy} - \overline{xz} = \bar{0} \\ &\implies x(y-z) \text{ is divisible by 5} \\ &\implies y - z \text{ is divisible by 5} \\ &\implies \bar{y} = \bar{z}.\end{aligned}$$

Here the reason for the third $\implies$ is that x is not divisible by 5, and 5 is a prime number. Since both sides of the map $\bar{y} \mapsto \overline{xy}$ are finite sets of the same size, the one-to-one property implies the onto property. In particular, we have $\overline{xy} = \bar{1}$ for some $\bar{y}$.

In general, for any prime number p, we have the field

$$\mathbb{F}_p = \{\bar{0}, \bar{1}, \bar{2}, \dots, \overline{p-1}\}.$$

We use $\mathbb{F}$ instead of $\mathbb{Z}$ to emphasise field.

Exercise 6.32. A homomorphism of fields is a nonzero map between two fields preserving the arithmetic operations. Prove that a homomorphism of fields is always one-to-one.

Exercise 6.33. Prove that the conjugation $a+b\sqrt{2} \mapsto a-b\sqrt{2}$ is a homomorphism of $\mathbb{Q}[\sqrt{2}]$ to itself. Moreover, this is the only non-trivial self-homomorphism of $\mathbb{Q}[\sqrt{2}]$.

Exercise 6.34. Find all the self-homomorphism of $\mathbb{Q}[\sqrt{2}, \sqrt{3}]$.

Exercise 6.35. Show that it makes sense to introduce the field $\mathbb{F}_5[\sqrt{2}]$. When would you have difficulty introducing $\mathbb{F}_p[\sqrt{n}]$? Here p is a prime number and $1 \le n < p$.

Exercise 6.36. Prove that $(x+y)^p = x^p + y^p$ in $\mathbb{F}_p$. In particular, $x \mapsto x^p$ is a self-homomorphism of $\mathbb{F}_p$. In fact, this is a self-homomorphism of any field of characteristic p.

Given a field $\mathbb{F}$, we consider the natural map $\mathbb{Z} \to \mathbb{F}$ that takes n to $n = 1 + 1 + \dots + 1$, where 1 is the unit element of $\mathbb{F}$. The map preserves addition and multiplication. If the map is injective, then we have a copy of $\mathbb{Z}$ inside $\mathbb{F}$. By using the division in $\mathbb{F}$, we get $\mathbb{Q} \subset \mathbb{F}$. In this case, we say $\mathbb{F}$ has *characteristic* 0.

If the map is not injective, we let p is the smallest natural number that is mapped to 0 (if $-p$ is mapped to 0, then p is mapped to 0). If $p = p_1p_2$ has two integral factors, then $p_1p_2 = 0$ in a field $\mathbb{F}$ implies $p_1 = 0$ or $p_2 = 0$. Therefore p is a prime number, and $\mathbb{F}_p \subset \mathbb{F}$. We say $\mathbb{F}$ has *characteristic* p.

6.2.2 Vector Space over Field

By changing $\mathbb{R}$ in Definition 1.1.1 to a field $\mathbb{F}$, we get the definition of a vector space over a field. Chapters 1, 2, 3 and Section 5.1 remain valid over any field. The part of linear algebra related to geometric measurement may not extend to any field. This includes inner product space, and the geometric aspect of the determinant. Another minor exception is expressing any matrix as the sum of a symmetric and a skew-symmetric matrix. Since dividing 2 is used, the fact is no longer true over a field of characteristic 2.

If one field is contained in another field $\mathbb{F} \subset \mathbb{E}$, then we say $\mathbb{F}$ is a *subfield* of $\mathbb{E}$, and $\mathbb{E}$ is a *field extension* of $\mathbb{F}$. For example, $\mathbb{Q}$ is a subfield of $\mathbb{Q}[\sqrt{2}]$, and $\mathbb{Q}[\sqrt{2}]$ is a subfield of $\mathbb{Q}[\sqrt{2}, \sqrt{3}]$. Moreover, both $\mathbb{Q}[\sqrt{2}]$ and $\mathbb{Q}[\sqrt{2}, \sqrt{3}]$ are field extensions of $\mathbb{Q}$.

Suppose $\mathbb{F} \subset \mathbb{E}$ is a field extension. Then for $a, b \in \mathbb{F}$ and $u, v \in \mathbb{E}$, we have $au + bv \in \mathbb{E}$. This makes $\mathbb{E}$ into a vector space over $\mathbb{F}$. For example, $\mathbb{Q}[\sqrt{2}]$ has basis $\{1, \sqrt{2}\}$ over $\mathbb{Q}$, and $\mathbb{Q}[\sqrt{2}, \sqrt{3}]$ has basis $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$ over $\mathbb{Q}$. Therefore we have the $\mathbb{Q}$-dimensions

$$\dim_{\mathbb{Q}} \mathbb{Q}[\sqrt{2}] = 2, \quad \dim_{\mathbb{Q}} \mathbb{Q}[\sqrt{2}, \sqrt{3}] = 4.$$

Since any element in $\mathbb{Q}[\sqrt{2}, \sqrt{3}]$ has the expression

$$a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} = (a + b\sqrt{2}) + (c + d\sqrt{2})\sqrt{3}$$

for unique "coefficients" $a + b\sqrt{2}, c + d\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$, we find that $\{1, \sqrt{3}\}$ is a basis of the vector space $\mathbb{Q}[\sqrt{2}, \sqrt{3}]$ over the field $\mathbb{Q}[\sqrt{2}]$

$$\dim_{\mathbb{Q}[\sqrt{2}]} \mathbb{Q}[\sqrt{2}, \sqrt{3}] = 2.$$

Usually we use more convenient notation $[\mathbb{E} \colon \mathbb{F}]$ for $\dim_{\mathbb{F}} \mathbb{E}$. Then

$$[\mathbb{Q}[\sqrt{2}] \colon \mathbb{Q}] = 2, \quad [\mathbb{Q}[\sqrt{2}, \sqrt{3}] \colon \mathbb{Q}] = 4, \quad [\mathbb{Q}[\sqrt{2}, \sqrt{3}] \colon \mathbb{Q}[\sqrt{2}]] = 2.$$

Proposition 6.2.2. *Suppose three fields satisfy $\mathbb{F}_1 \subset \mathbb{F}_2 \subset \mathbb{F}_3$. Then*

$$[\mathbb{F}_3 \colon \mathbb{F}_1] = [\mathbb{F}_3 \colon \mathbb{F}_2][\mathbb{F}_2 \colon \mathbb{F}_1].$$

Proof. Suppose $v_1, v_2, \dots, v_m \in \mathbb{F}_3$ is a basis of $\mathbb{F}_3$ as $\mathbb{F}_2$-vector space. Suppose $w_1, w_2, \dots, w_n \in \mathbb{F}_2$ is a basis of $\mathbb{F}_2$ as $\mathbb{F}_1$-vector space.

Any element $x \in \mathbb{F}_3$ is a linear combination

$$x = y_1 v_1 + y_2 v_2 + \cdots + y_m v_m, \quad y_i \in \mathbb{F}_2.$$

The elements y_i of $\mathbb{F}_2$ are also linear combinations

$$y_i = z_{i1} w_1 + z_{i2} w_2 + \cdots + z_{in} w_n, \quad z_{ij} \in \mathbb{F}_1.$$

Then we get

$$x = \sum_i y_i v_i = \sum_i \left(\sum_j z_{ij} w_j \right) v_i = \sum_{ij} z_{ij} w_j v_i.$$

This is a linear combination of $w_j v_i \in \mathbb{F}_3$ with $z_{ij} \in \mathbb{F}_1$ as coefficients. Therefore all $w_j v_i$ span $\mathbb{F}_3$ as an $\mathbb{F}_1$-vector space.

Next we show that w_jv_i are $\mathbb{F}_1$-linearly independent. Consider the equality of linear combinations in $\mathbb{F}_3$ with $\mathbb{F}_1$-coefficients

$$\sum_{ij} z_{ij}w_jv_i = \sum_{ij} z'_{ij}w_jv_i, \quad z_{ij}, z'_{ij} \in \mathbb{F}_1.$$

Let $y_i = \sum_j z_{ij}w_j$ and $y'_i = \sum_j z'_{ij}w_j$. Then $y_i, y'_i \in \mathbb{F}_2$, and the above becomes an equality of linear combinations in $\mathbb{F}_3$ with $\mathbb{F}_2$-coefficients

$$\sum_i y_iv_i = \sum_i y'_iv_i, \quad y_i, y'_i \in \mathbb{F}_2.$$

Since $v_1, v_2, \dots, v_m$ is a basis of $\mathbb{F}_3$ as $\mathbb{F}_2$-vector space, the $\mathbb{F}_2$-coefficients are equal. The equality $y_i = y'_i$ is then an equality of linear combinations in $\mathbb{F}_2$ with $\mathbb{F}_1$-coefficients

$$\sum_j z_{ij}w_j = \sum_j z'_{ij}w_j, \quad z_{ij}, z'_{ij} \in \mathbb{F}_1.$$

Since $w_1, w_2, \dots, w_n$ is a basis of $\mathbb{F}_2$ as $\mathbb{F}_1$-vector space, we get $z_{ij} = z'_{ij}$. This proves the linear independence of all w_jv_i.

Therefore the mn vectors $w_1v_1, w_1v_2, \dots, w_nv_m$ form a basis of $\mathbb{F}_3$ as $\mathbb{F}_1$-vector space, and we get $[\mathbb{F}_3 \colon \mathbb{F}_1] = mn = [\mathbb{F}_3 \colon \mathbb{F}_2][\mathbb{F}_2 \colon \mathbb{F}_1]$. □

6.2.3 Polynomial over Field

We denote all the polynomials over a field $\mathbb{F}$ by

$$\mathbb{F}[t] = \{f(t) = a_0 + a_1t + a_2t^2 + \cdots + a_nt^n \colon a_i \in \mathbb{F}\}.$$

The polynomial has degree n if $a_n \neq 0$, and is *monic* if $a_n = 1$.

We have $+, -, \times$ operations in $\mathbb{F}[t]$ but not $\div$. In fact, $\mathbb{F}[t]$ is very similar to the integers $\mathbb{Z}$. Both belong to the concept of *integral domain*, which is a set R with two operations $+, \times$ satisfying the following.

1. Commutativity: $a + b = b + a$, $ab = ba$.

2. Associativity: $(a + b) + c = a + (b + c)$, $(ab)c = a(bc)$.

3. Distributivity: $a(b + c) = ab + ac$.

4. Unit: There are $0, 1$, such that $a + 0 = a = 0 + a$, $a1 = a = 1a$.

5. Negative: For any a, there is b, such that $a + b = 0 = b + a$.

6. No zero divisor: $ab = 0$ implies $a = 0$ or $b = 0$.

The first four axioms are the same as the field. The only modification is that the existence of the multiplicative inverse (allowing $\div$ operation) is replaced by the no zero divisor condition. The condition is equivalent to the cancelation property: $ab = ac$ and $a \neq 0 \implies b = c$.

Due to the cancelation property, we may construct the field of rational numbers as the quotients of integers

$$\mathbb{Q} = \{\tfrac{a}{b} \colon a, b \in \mathbb{Z},\ b \neq 0\}.$$

Similarly, we have the field of *rational functions*

$$\mathbb{F}(t) = \{\tfrac{f(t)}{g(t)} \colon f(t), g(t) \in \mathbb{F}[t],\ g(t) \neq 0\}.$$

In general, an integral domain can be regarded as a system with $+, -, \times$ that can be embedded into a field.

Integers and polynomials are special kinds of integral domain, in that they have certain division process. For an integer a and a nonzero integer b, there are unique integers q and r, such that

$$a = qb + r, \quad 0 \leq r < |b|.$$

We say the division of a by the *divisor* b has the *quotient* q and the *remainder* r. When the remainder $r = 0$, we say b *divides* a or is a *factor* of a, and we denote $b|a$.

Polynomials over a field have the similar division process. For example, let

$$\begin{aligned} a(t) &= (t+1)(t-1)^2(t^2 - t + 1) = t^5 - 2t^4 + t^3 + t^2 - 2t + 1, \\ b(t) &= (t+1)^3(t-1) = t^4 + 2t^3 - 2t - 1. \end{aligned}$$

The following calculation

$$\begin{array}{r|l} & \hphantom{-t^5 - 2t^4 + t^3 + t^2 - 2}t - 4 \\ \cline{2-2} t^4 + 2t^3 - 2t - 1 & \hphantom{-}t^5 - 2t^4 + t^3 + t^2 - 2t + 1 \\ & -t^5 - 2t^4 \hphantom{+t^3} + 2t^2 + t \\ \cline{2-2} & \hphantom{-t^5} - 4t^4 + t^3 + 3t^2 - t + 1 \\ & \hphantom{-t^5 -} 4t^4 + 8t^3 \hphantom{+3t^2} - 8t - 4 \\ \cline{2-2} & \hphantom{-t^5 - 4t^4 +} 9t^3 + 3t^2 - 9t - 3 \end{array}$$

shows that

$$\begin{aligned} a(t) &= t^5 - 2t^4 + t^3 + t^2 - 2t + 1 \\ &= (t-4)(t^4 + 2t^3 - 2t - 1) + 9t^3 + 3t^2 - 9t - 3 \\ &= q(t)b(t) + r(t), \end{aligned}$$

where

$$q(t) = t - 4,\ r(t) = 9t^3 + 3t^2 - 9t - 3.$$

In general, we have the following.

Proposition 6.2.3. *Suppose $f(t), g(t)$ are polynomials over a field $\mathbb{F}$. If $g(t) \neq 0$, then there are unique polynomials $q(t)$ and $r(t)$, such that*

$$f(t) = q(t)g(t) + r(t), \quad \deg r(t) < \deg q(t).$$

Again, the division of $f(t)$ by the divisor $g(t)$ has the quotient $q(t)$ and the remainder $r(t)$. If $r(t) = 0$, the $g(t)$ *divides* (is a *factor* of) $f(t)$, and we denote $g(t)|f(t)$.

Definition 6.2.4. A *Euclidean domain* is an integral domain R, with a function $d\colon R - 0 \to$ non-negative integers, such that the following division process happens: For any $a, b \in R$, with $b \neq 0$, there are unique $q, r \in R$, such that

$$a = qb + r, \quad r = 0 \text{ or } d(r) < d(b).$$

We may take $d(a) = |a|$ for $R = \mathbb{Z}$ and $d(f(t)) = \deg f(t)$ for $R = \mathbb{F}[t]$. All discussion of the rest of this section applies to the Euclidean domain.

An integer $d \in \mathbb{Z}$ is a *common divisor* of a and b if $d|a$ and $d|b$. We say d is a *greatest common divisor*, and denote $d = \gcd(a, b)$, if

$$c|a \text{ and } c|b \iff c|d.$$

As defined, the greatest common divisor is unique up to $\pm$ sign. We usually prefer choosing the positive one. Moreover, it is easy to extend the concept of greatest common divisor to more than two integers.

The greatest common divisor can be calculated by the *Euclidean algorithm*. The algorithm is based on the fact that, if $a = qb + r$, then $c|a$ and $c|b$ is equivalent to $c|b$ and $c|r$. For example, by $96 = 2 \times 42 + 12$, we have $c|96$ and $c|42 \iff c|42$ and $c|12$. The process repeats until we reach complete division (i.e., zero remainder).

$$\begin{aligned} c|96 \text{ and } c|42 &\iff c|42 \text{ and } c|12 && \text{by } 96 = 2 \times 42 + 12 \\ &\iff c|12 \text{ and } c|6 && \text{by } 42 = 3 \times 12 + 6 \\ &\iff c|6 \text{ and } c|0 && \text{by } 12 = 2 \times 6 + 0 \\ &\iff c|6. \end{aligned}$$

We conclude $6 = \gcd(96, 42)$.

The Euclidean algorithm is based on the division process, and is therefore valid in any Euclidean domain. In particular, this proves the existence of greatest common divisor in any Euclidean domain. For polynomials over a field, the following divisions

$$\begin{aligned} t^5 - 2t^4 + t^3 + t^2 - 2t + 1 &= (t-4)(t^4 + 2t^3 - 2t - 1) + 3(3t^3 + t^2 - 3t - 1), \\ t^4 + 2t^3 - 2t - 1 &= \tfrac{1}{9}(3t+5)(3t^3 + t^2 - 3t - 1) + \tfrac{4}{9}(t^2 - 1), \\ 3t^3 + t^2 - 3t - 1 &= (3t+1)(t^2 - 1) \end{aligned}$$

imply

$$\begin{aligned}
&\gcd(t^5 - 2t^4 + t^3 + t^2 - 2t + 1, t^4 + 2t^3 - 2t - 1)\\
=&\gcd(t^4 + 2t^3 - 2t - 1, 3t^3 + t^2 - 3t - 1)\\
=&\gcd(3t^3 + t^2 - 3t - 1, t^2 - 1)\\
=&\gcd(t^2 - 1, 0)\\
=&t^2 - 1.
\end{aligned}$$

We note that, for polynomials, the greatest common divisor is unique up to multiplying a nonzero element in the field. Therefore we may ignore the field coefficients such as 3 and $\frac{4}{9}$ in the calculation above. In particular, we usually choose a monic polynomial as the preferred greatest common divisor.

The algorithm can be applied to more than two polynomials. Among several polynomials, we may always choose the polynomial of smallest degree to divide the other polynomials. We gather this divisor polynomial and replace all the other polynomials by the remainders. Then we repeat the process. This proves the existence of the greatest common divisor among several polynomials.

Proposition 6.2.5. *Suppose $f_1(t), f_2(t), \dots, f_k(t) \in \mathbb{F}[t]$ are nonzero polynomials. Then there is a unique monic polynomial $d(t)$, such that $g(t)$ divides every one of $f_1(t), f_2(t), \dots, f_k(t)$ if and only if $g(t)$ divides $d(t)$.*

The Euclidean algorithm can also be used to express the greatest common divisor as a combination of the original numbers or polynomials. For example, we have

$$\begin{aligned}
6 &= 42 - 3\times 12 = 42 - 3 \times (96 - 2\times 42)\\
&= -3\times 96 + (1 + 3\times 2)\times 42 = -3\times 96 + 7\times 42.
\end{aligned}$$

Similarly,

$$\begin{aligned}
t^2 - 1 &= \tfrac{9}{4}(t^4 + 2t^3 - 2t - 1) - \tfrac{1}{4}(3t+5)(3t^3 + t^2 - 3t - 1)\\
&= \tfrac{9}{4}(t^4 + 2t^3 - 2t - 1)\\
&\quad - \tfrac{1}{12}(3t+5)[(t^5 - 2t^4 + t^3 + t^2 - 2t + 1) - (t-4)(t^4 + 2t^3 - 2t - 1)]\\
&= (-\tfrac{1}{4}t - \tfrac{5}{12})(t^5 - 2t^4 + t^3 + t^2 - 2t + 1)\\
&\quad + (\tfrac{1}{4}t^2 - \tfrac{17}{20}t + \tfrac{33}{20})(t^4 + 2t^3 - 2t - 1).
\end{aligned}$$

This also extends to more than two polynomials.

Proposition 6.2.6. *Suppose $f_1(t), f_2(t), \dots, f_k(t) \in \mathbb{F}[t]$ are nonzero polynomials. Then there are polynomials $u_1(t), u_2(t), \dots, u_k(t)$, such that*

$$\gcd(f_1(t), f_2(t), \dots, f_k(t)) = f_1(t)u_1(t) + f_2(t)u_2(t) + \cdots + f_k(t)u_k(t).$$

Several polynomials are *coprime* if their greatest common divisor is 1. In other words, the only polynomials dividing every one of $f_1(t), f_2(t), \dots, f_k(t)$ are nonzero elements of the field. In this case, we can find polynomials $u_1(t), u_2(t), \dots, u_k(t)$ satisfying

$$f_1(t)u_1(t) + f_2(t)u_2(t) + \cdots + f_k(t)u_k(t) = 1.$$

6.2.4 Unique Factorisation

We know any nonzero integer is a unique product of prime numbers. We analyse the detailed reason behind this fact, and justify the similar property for polynomials over a field.

An element a of an integral domain R is called *invertible* if there is b, such that $ab = 1 = ba$. It is easy to show that b is unique. Then we denote $b = a^{-1}$, and we denote the set of all invertibles by R^*. We have $\mathbb{Z}^* = \{1, -1\}$. For any field $\mathbb{F}$, we have $\mathbb{F}^* = \mathbb{F} - 0$, and $\mathbb{F}[t]^* = \mathbb{F}^*$.

Definition 6.2.7. Let p be a non-invertible element in an integral domain R.

1. p is *irreducible* if $p = ab$ implies either a or b is invertible.

2. p is *prime* if $p|ab$ implies $p|a$ or $p|b$.

The concept of irreducible is used to construct factorisation. The concept of prime is used to justify the uniqueness of factorisation. It happens that the two concepts are equivalent in a Euclidean domain, such as $\mathbb{Z}$ and $\mathbb{F}[t]$.

First, we construct factorisation in $\mathbb{Z}$. We start with a nonzero integer a and ask whether $a = bc$ for some $b, c \neq \pm 1$. If not, then a is irreducible. If yes, then $|b| < |a|$ and we further ask the same question for b in place of a. Since $|b| < |a|$ in each step, the precess eventually stops, and we get an irreducible factor p of a. Now we repeat the precess with $\frac{a}{p} \in \mathbb{Z}$ in place of a, and find another irreducible factor q of $\frac{a}{p}$, and so on. Since $|\frac{a}{p}| < |a|$, the precess eventually stops, and we express a as a product of irreducibles. The expression is $a = \pm p_1^{m_1} p_2^{m_2} \dots p_k^{m_k}$, where p_i are distinct positive irreducible numbers, and m_i are natural numbers.

Second, we argue about the uniqueness of factorisation in $\mathbb{Z}$. Therefore we assume $p_1^{m_1} p_2^{m_2} \dots p_k^{m_k} = q_1^{n_1} q_2^{n_2} \dots q_l^{n_l}$. Suppose we also know that the irreducibles p_i, q_j are also primes. Then $p_1 | q_1^{n_1} q_2^{n_2} \dots q_l^{n_l}$ implies p_1 divides some q_j. In other words, we have $q_i = cp_1$. Since q_i is irreducible, and p_1 is not invertible, we find $c = \pm 1$ to be invertible. Since we choose $p_1, q_j > 0$, we get $c = 1$. Without loss of generality, therefore, we may assume $p_1 = q_1$. Then we get $p_1^{m_1-1} p_2^{m_2} \dots p_k^{m_k} = q_1^{n_1-1} q_2^{n_2} \dots q_l^{n_l}$, and we may use induction to prove the uniqueness of factorisation.

Finally, we compare irreducible integers and prime integers. In fact, the second step above assumed that any irreducible p is also a prime. To see this, we assume $p|ab$ and $p \nmid a$, and wish to prove $p|b$. If b divides p and a, then by p irreducible, we have $b = \pm 1$ or $\pm p$. Since $b = \pm p \nmid a$, we have $b = \pm 1$. Therefore the greatest common divisor of p and a is 1. By the Euclidean algorithm, we have $pu + av = 1$ for some integers u, v. Then p divides $b = pbu + abv$.

We note that the proof of irreducible implying prime uses the Euclidean algorithm, and is therefore valid in any Euclidean domain. Conversely, assume a prime $p = ab$. Then $p|a$ or $p|b$. If $p|a$, then $a = pc$ for some integer c. Then $p = ab = pcb$, and we get $cb = 1$. This implies that b is invertible, and proves that any prime is irreducible. We note that the key to the proof is the cancelation of p from the equality $p = pcb$. The cancelation holds in any integral domain.

With the degree of polynomial in place of the absolute value, the argument above for the integers $\mathbb{Z}$ is also valid for the polynomials $\mathbb{F}[t]$ over a field.

Proposition 6.2.8. *For any nonzero polynomial $f(t) \in \mathbb{F}[t]$, there are $c \in \mathbb{F}^*$, unique monic irreducible polynomials $p_1(t), p_2(t), \dots, p_k(t)$, and natural numbers $m_1, m_2, \dots, m_k$, such that*

$$f(t) = cp_1(t)^{m_1}p_2(t)^{m_2}\dots p_k(t)^{m_k}.$$

The unique factorisation can be used to obtain the greatest common divisor. For example, we have

$$\gcd(96, 42) = \gcd(2^5 \cdot 3, 2 \cdot 3 \cdot 7) = 2^{\min\{5,1\}} \cdot 3^{\min\{1,1\}} \cdot 7^{\min\{0,1\}} = 2 \cdot 3 = 6,$$

and

$$\begin{aligned}
&\gcd(x^5 - 2x^4 + x^3 + x^2 - 2x + 1, x^4 + 2x^3 - 2x - 1) \\
=&\gcd((x+1)(x-1)^2(x^2 - x + 1), (x+1)^3(x-1)) \\
=&(x+1)(x-1) = x^2 - 1.
\end{aligned}$$

Exercise 6.37. The Fundamental Theorem of Algebra (Theorem 6.1.1) says that any nonconstant complex polynomial has root. Use this to show that complex irreducible polynomials are linear functions.

Exercise 6.38. Show that if a complex number r is a root of a real polynomial, then the complex conjugate $\bar{r}$ is also a root of the polynomial. Then use this to explain real irreducible polynomials are the following

- Linear: $a + bt$, with $a, b \in \mathbb{R}$ and $b \neq 0$.
- Quadratic: $a + bt + ct^2$, with $a, b, c \in \mathbb{R}$ and the discriminant $b^2 - 4ac < 0$.

6.2.5 Field Extension

The complex field is obtained from the attempt of solving the equations $at^2+bt+c = 0$ for $a, b, c \in \mathbb{R}$. This can be achieved by adding the solution of $t^2 + 1 = 0$ to $\mathbb{R}$.

The field $\mathbb{Q}[\sqrt{2}]$ is obtained from the attempt of solving the polynomial equation $t^2-2=0$, when the only known numbers are the rational numbers $\mathbb{Q}$.

In general, let $\mathbb{F}\subset\mathbb{C}$ be a number field (a subfield of complex numbers). Then $\mathbb{F}$ has characteristic 0 and contains at least all the rationals $\mathbb{Q}$. For a complex number $r\in\mathbb{C}$, we wish to extend $\mathbb{F}$ to a bigger field containing r. The bigger field should at least contain all the polynomials of r with coefficient in $\mathbb{F}$

$$\mathbb{F}[r]=\{f(r)\colon f(t)\in\mathbb{F}[t]\}=\{a_0+a_1r+a_2r^2+\cdots+a_nr^n\colon r_i\in\mathbb{F}\}.$$

For any polynomials $f(t),g(t)\in\mathbb{F}[t]$, we have $f(r)+g(r),f(r)g(r)\in\mathbb{F}[r]$. Therefore $\mathbb{F}[r]$ is an integral domain. It remains to introduce division.

We need to consider two possibilities:

1. r is *algebraic* over $\mathbb{F}$: $f(r)=0$ for some $f(t)\in\mathbb{F}[t]$.
2. r is *transcendental* over $\mathbb{F}$: If $f(t)\in\mathbb{F}[t]$ is nonzero, then $f(r)\neq 0$.

A number is transcendental if it is not the root of a nonzero polynomial. This means that $1,r,r^2,\dots,r^n$ are linearly independent over $\mathbb{F}$ for any n, so that $F[r]$ is an infinite dimensional vector space over $\mathbb{F}$. An example is $e=2.71828\cdots$ over $\mathbb{F}=\mathbb{Q}$. In this case, the map

$$f(t)\mapsto f(r)\colon F[t]\to F[r]\subset\mathbb{C}$$

is injective. As a consequence, we conclude that the map on rational functions

$$\frac{f(t)}{g(t)}\mapsto\frac{f(r)}{g(r)}\colon F(t)\to F(r)\subset\mathbb{C}$$

is also injective. In particular, the smallest field extension of $\mathbb{F}$ containing r is $F(r)$, which is strictly bigger than $F[r]$.

A number is algebraic if $1,r,r^2,\dots,r^n$ are linearly dependent for sufficiently large n. Suppose n is the smallest number, such that $1,r,r^2,\dots,r^n$ are linearly dependent over $\mathbb{F}$. Then we have a monic polynomial $m(t)\in\mathbb{F}[t]$ of degree n, such that $m(r)=0$. Moreover, since n is the smallest number, we know $1,r,r^2,\dots,r^{n-1}$ are linearly independent over $\mathbb{F}$. This means that

$$s(t)\in\mathbb{F}[t],\ s(r)=0,\ \deg r<n\implies s(t)=0.$$

Now suppose $f(t)\in\mathbb{F}[t]$ satisfies $f(r)=0$. We have $f(t)=q(t)m(t)+s(t)$ with $\deg s<\deg f=n$. Then $0=f(r)=q(r)m(r)+s(r)=s(r)$. By the property about $s(t)$ above, we have $s(t)=0$. Therefore we get

$$f(r)=0\implies m(t)|f(t).$$

Of course, we also have

$$m(t)|f(t)\implies f(t)=g(t)m(t)\implies f(r)=g(r)m(r)=g(r)0=0.$$

In other words, a polynomial vanishes at r if and only if it is divisible by $m(t)$. Therefore we call $m(t)$ the *minimal polynomial* of the algebraic number r.

The number $\sqrt{-1}$ is algebraic over $\mathbb{R}$, with the minimal polynomial t^2+1. If an integer $a \neq 0, 1, -1$ has no square factor, then $\sqrt{a}$ is algebraic over $\mathbb{Q}$, with the minimal polynomial $t^2 - a$. Moreover, $\sqrt[3]{2}$ is algebraic over $\mathbb{Q}$, with the minimal polynomial $t^3 - 2$.

Examples 6.2.1, 6.2.2, 6.2.3 suggest that for algebraic r, $\mathbb{F}[r]$ admits division, and is therefore the smallest field extension containing r.

Theorem 6.2.9. *Suppose r is algebraic over a field $\mathbb{F}$. Suppose $m(t) \in \mathbb{F}[t]$ is the minimal polynomial of r, and $\deg m(t) = n$. Then $\mathbb{F}[r]$ is a field, and $[\mathbb{F}[r] : \mathbb{F}] = n$.*

Proof. Let

$$m(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1} t^{n-1} + t^n, \quad a_i \in \mathbb{F}.$$

Then $m(r) = 0$ implies that r^n is a linear combination of $1, r, r^2, \ldots, r^{n-1}$

$$r^n = -a_0 - a_1 r - a_2 r^2 - \cdots - a_{n-1} r^{n-1}.$$

This implies that r^{n+1} is also a linear combination of $1, r, r^2, \ldots, r^{n-1}$

$$\begin{aligned} r^{n+1} &= -a_0 r - a_1 r^2 - a_2 r^3 - \cdots - a_{n-2} r^{n-1} - a_{n-1} r^n \\ &= -a_0 r - a_1 r^2 - a_2 r^3 - \cdots - a_{n-2} r^{n-1} \\ &\quad + a_{n-1}(a_0 + a_1 r + a_2 r^2 + \cdots + a_{n-1} r^{n-1}) \\ &= b_0 + b_1 r + b_2 r^2 + \cdots + b_{n-1} r^{n-1}, \quad b_i \in \mathbb{F}. \end{aligned}$$

Inductively, we know r^k is a linear combination of $1, r, r^2, \ldots, r^{n-1}$. Therefore

$$\mathbb{F}[r] = \{c_0 + c_1 r + c_2 r^2 + \cdots + c_{n-1} r^{n-1} \colon c_i \in \mathbb{F}\}.$$

Moreover, since $1, r, r^2, \ldots, r^{n-1}$ are linearly independent, we get $\dim_{\mathbb{F}} \mathbb{F}[r] = n$.

It remains to show $\mathbb{F}[r]$ is a field. It is sufficient to argue that any

$$0 \neq x = c_0 + c_1 r + c_2 r^2 + \cdots + c_{n-1} r^{n-1} \in \mathbb{F}[r]$$

has inverse in $\mathbb{F}[r]$. We may use the argument in Example 6.2.3. Specifically, since $\dim_{\mathbb{F}} \mathbb{F}[r] = n$, the $n+1$ vectors $1, x, x^2, \ldots, x^n$ in $\mathbb{F}[r]$ must be linearly dependent. Therefore there is a polynomial $f(t) \in \mathbb{F}[t]$ of degree $\leq n$, such that $f(x) = 0$. Without loss of generality, we may further assume that $f(x)$ is a monic polynomial, and has the smallest degree among all polynomials satisfying $f(x) = 0$ (i.e., $f(t)$ is the minimal polynomial of x). Let

$$f(t) = b_0 + b_1 t + b_2 t^2 + \cdots + b_{k-1} t^{k-1} + t^k, \quad k \leq n.$$

If $b_0=0$, then $f(x)=0$ implies $b_1+b_2x+\cdots+b_{k-1}x^{k-2}+x^{k-1}=0$, contradicting the smallest degree assumption. Then $f(x)=0$ implies that

$$\frac{1}{x}=-\frac{b_1}{b_0}-\frac{b_2}{b_0}r-\cdots-\frac{b_{k-1}}{b_0}r^{k-2}-\frac{1}{b_0}r^{k-1}.$$

Substituting $x=c_0+c_1r+c_2r^2+\cdots+c_{n-1}r^{n-1}$ into above and expand, we find the above is in $\mathbb{F}[r]$. $\square$

Exercise 6.39. Suppose $\mathbb{E}$ is a finite dimensional field extension of $\mathbb{F}$. Prove that a number is algebraic over $\mathbb{E}$ if and only if it is algebraic over $\mathbb{F}$. How are the degrees of minimal polynomials over respective fields related?

Exercise 6.40. Suppose a and b are algebraic over $\mathbb{F}$. Prove that $a+b, a-b, a\times b, a\div b$ are also algebraic over $\mathbb{F}$.

Exercise 6.41. Explain $[\mathbb{Q}[\sqrt[3]{2}]:\mathbb{Q}]=3$ and $[\mathbb{Q}[\sqrt[4]{5}]:\mathbb{Q}]=4$. Then explain $[\mathbb{Q}[\sqrt[3]{2},\sqrt[4]{5}]:\mathbb{Q}]=12$.

Exercise 6.42. Suppose $\mathbb{E}$ is a field extension of $\mathbb{F}$, such that $[\mathbb{E}:\mathbb{F}]$ is a prime number. If $r\in\mathbb{E}-\mathbb{F}$, prove that $\mathbb{E}=\mathbb{F}[r]$.

6.2.6 Trisection of Angle

A classical Greek problem is to use ruler and compass to divide any angle into three equal parts. For example, when the angle is $\frac{1}{3}\pi$ (60 degrees), this means drawing an angle of $\frac{1}{9}\pi$ (20 degrees).

Here is the precise meaning of construction by ruler and compass:

1. We start with two points on the plane. The two points are considered as constructed.

2. If two points are constructed, then the straight line passing through the two points is constructed.

3. If two points are constructed, then the circle centered at one point and passing through the other point is constructed.

4. The intersection of two constructed lines or circles is constructed.

Denote all the constructed points, lines and circles by $\mathcal{C}$. We present some basic constructions.

- Given a line l and a point p in $\mathcal{C}$, the line passing through p and perpendicular to l is in $\mathcal{C}$.

- Given a line l and a point p in $\mathcal{C}$, the line passing through p and parallel to l is in $\mathcal{C}$.
- Given a line l and three points p, x, y in $\mathcal{C}$, such that $p \in l$. Then there are two points on l, such that the distance to p is the same as the distance between x and y. The two points are constructed.

Figure 6.2.6 gives the constructions. The first construction depends on whether p is on l or not. The numbers indicate the order of constructions.

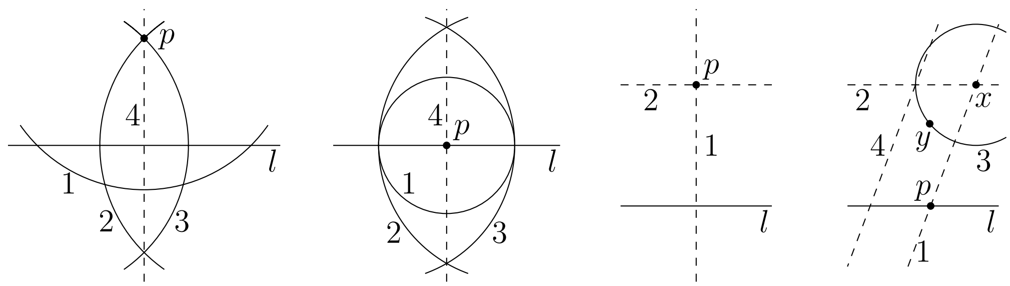

Figure 6.2.1: Basic constructions.

Now we parameterise the construction by complex numbers. We set the two initial points to be 0 and 1. The construction of perpendicular line shows that a complex number can be constructed if and only if the real and imaginary parts can be constructed. Therefore we only need to consider the set $\mathcal{C}_{\mathbb{R}}$ of all the points on the real line that can be constructed. The numbers in $\mathcal{C}_{\mathbb{R}}$ are constructible numbers. The trisection problem of the angle $\frac{1}{3}\pi$ is the same as whether $\cos\frac{1}{9}\pi$ is in $\mathcal{C}_{\mathbb{R}}$.

Proposition 6.2.10. *The constructible numbers $\mathcal{C}_{\mathbb{R}}$ is a field. Moreover, if $a \in \mathcal{C}_{\mathbb{R}}$ is positive, then $\sqrt{a} \in \mathcal{C}_{\mathbb{R}}$.*

The proposition is proved by Figure 6.2.6.

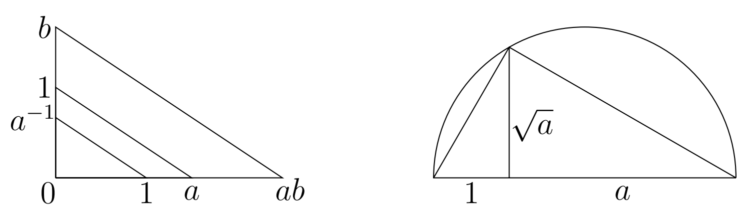

Figure 6.2.2: Constructible numbers is a field, and is closed under square root.

The constructible numbers can be considered as a field extension of $\mathbb{Q}$. From this viewpoint, we start with $\mathbb{Q}$. Then each ruler and compass construction adds new constructible numbers one by one. Each new constructible number gives a field extension. Then finitely many constructions gives a finite sequence of real number field extensions $\mathbb{Q} \subset \mathbb{F}_1 \subset \mathbb{F}_2 \subset \ldots \subset \mathbb{F}_k$.

Suppose we get a field $\mathbb{F}$ after certain number of steps. The next constructible number is obtained from the intersection of two lines or circles based on the numbers in $\mathbb{F}$. In other words, a line is $ax + by = c$ with $a, b, c \in \mathbb{F}$, and a circle is $(x-a)^2 + (y-b)^2 = c^2$ with $a, b, c \in \mathbb{F}$. The following are the possible intersections.

1. The intersection of two lines $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ is obtained by solving the system of two equations. The solution is obtained by arithmetic operations of the coefficients, and therefore still lies in $\mathbb{F}$.

2. The intersection of $a_1x + b_1y = c_1$ and $(x-a_2)^2 + (y-b_2)^2 = c_2^2$ is reduced to quadratic equations of single variables with coefficients in $\mathbb{F}$. Therefore the solution is in $\mathbb{F}$ or $\mathbb{F}[\sqrt{a}]$ for some positive $a \in \mathbb{F}$.

3. The intersection of $(x-a_1)^2 + (y-b_1)^2 = c_1^2$ and $(x-a_2)^2 + (y-b_2)^2 = c_2^2$ also gives a number in $\mathbb{F}$ or $\mathbb{F}[\sqrt{a}]$.

This proves the following.

Proposition 6.2.11. *If a is a constructible number, then there is a sequence of field extensions $\mathbb{Q} = \mathbb{F}_0 \subset \mathbb{F}_1 \subset \mathbb{F}_2 \subset \ldots \subset \mathbb{F}_k$, such that $\mathbb{F}_{i+1} = \mathbb{F}_i[\sqrt{a_i}]$ for some $a_i \in \mathbb{F}_i$, and $a \in \mathbb{F}_k$.*

Propositions 6.2.10 and 6.2.11 are complementary, and actually gives the necessary and sufficient condition for a real number to be constructible. Now we are ready to tackle the trisection problem.

Theorem 6.2.12. $\cos\frac{1}{9}\pi$ *is not constructible by ruler and compass.*

Proof. By $\cos\frac{1}{3}\pi = \frac{1}{2}$ and $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, we find that $r = \cos\frac{1}{9}\pi$ is a root of $f(t) = 8t^3 - 6t - 1$. If $f(t)$ is not irreducible, then it has a linear factor $at + b$. We may arrange to have a, b to be coprime integers. The linear factor means that $f(-\frac{b}{a}) = 0$, which is the same as $a(6b - a) = 8b^2$. Since a, b are coprime, we have $a|8$. This implies $b|(6b-a)$, or $b|a$. By the coprime assumption again, we have $b = \pm 1$, and $a(\pm 6 - a) = 8$. Then it is easy to see that no integer a satisfies $a(\pm 6 - a) = 8$.

Therefore $f(t)$ is irreducible. By Theorem 6.2.9, we get $[\mathbb{Q}[r] : \mathbb{Q}] = 3$. On the other hand, if r is constructible, then $r \in \mathbb{F}_k$ for a sequence of field extensions in Proposition 6.2.11. This implies $\mathbb{Q}[r] \subset \mathbb{F}_k$.

By applying Proposition 6.2.2 to $\mathbb{Q} \subset \mathbb{Q}[r] \subset \mathbb{F}_k$, we find that

$$[\mathbb{F}_k : \mathbb{Q}] = [\mathbb{F}_k : \mathbb{Q}[r]]\,[\mathbb{Q}[r] : \mathbb{Q}] = 3[\mathbb{F}_k : \mathbb{Q}[r]].$$

Therefore the $\mathbb{Q}$-dimension $[\mathbb{F}_k : \mathbb{Q}]$ of F_k is divisible by 3. On the other hand, applying Theorem 6.2.9 to $\mathbb{F}_{i+1} = \mathbb{F}_i[\sqrt{a_i}]$, we get $[\mathbb{F}_{i+1} : \mathbb{F}_i] = 2$. Then we apply

Proposition 6.2.2 to $\mathbb{Q} = \mathbb{F}_0 \subset \mathbb{F}_1 \subset \mathbb{F}_2 \subset \ldots \subset \mathbb{F}_k$ to get

$$[\mathbb{F}_k : \mathbb{Q}] = [\mathbb{F}_k : \mathbb{F}_{k-1}] \cdots [\mathbb{F}_2 : \mathbb{F}_1] [\mathbb{F}_1 : \mathbb{F}_0] = 2^k.$$

Since 3 does not divide 2^k, we get a contradiction. $\square$

Chapter 7

Spectral Theory

The famous Fibonacci numbers $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots$ is defined through the recursive relation

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2}.$$

Given a specific number, say 100, we can certainly calculate F_{100} by repeatedly applying the recursive relation 100 times. However, it is not obvious what the general formula for F_n should be.

The difficulty for finding the general formula is due to the lack of understanding of the *structure* of the recursion process. The Fibonacci numbers is a linear system because it is governed by a linear equation $F_n = F_{n-1} + F_{n-2}$. Many differential equations such as Newton's second law $F = m\vec{x}''$ are also linear. Understanding the structure of linear operators inherent in linear systems helps us solving problems about the system.

7.1 Eigenspace

We illustrate how understanding the geometric structure of a linear operator helps solving problems.

Example 7.1.1. Suppose a pair of numbers x_n, y_n is defined through the recursive relation

$$x_0 = 1, \quad y_0 = 0, \quad x_n = x_{n-1} - y_{n-1}, \quad y_n = x_{n-1} + y_{n-1}.$$

To find the general formula for x_n and y_n, we rewrite the recursive relation as a linear transformation

$$\vec{x}_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \vec{x}_{n-1} = A\vec{x}_{n-1}, \quad \vec{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \vec{e}_1.$$

By

$$A = \sqrt{2} \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix},$$

the linear operator is the rotation by $\frac{\pi}{4}$ and scalar multiplication by $\sqrt{2}$. Therefore $\vec{x}_n$ is obtained by rotating $\vec{e}_1$ by $n\frac{\pi}{4}$ and multiplication by $(\sqrt{2})^n$. We conclude that

$$x_n = 2^{\frac{n}{2}} \cos \tfrac{n\pi}{4}, \quad y_n = 2^{\frac{n}{2}} \sin \tfrac{n\pi}{4}.$$

For example, we have $(x_{8k}, y_{8k}) = (2^{4k}, 0)$ and $(x_{8k+3}, y_{8k+3}) = (-2^{4k+1}\sqrt{2}, 2^{4k+1}\sqrt{2})$.

Example 7.1.2. Suppose a linear system is obtained by repeatedly applying the matrix

$$A = \begin{pmatrix} 13 & -4 \\ -4 & 7 \end{pmatrix}.$$

To find A^n, we note that

$$A\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad A\begin{pmatrix} -2 \\ 1 \end{pmatrix} = 15\begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

This implies

$$A^n\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 5^n\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad A^n\begin{pmatrix} -2 \\ 1 \end{pmatrix} = 15^n\begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Then we may apply A^n to the other vectors by expressing as linear combinations of $\vec{v}_1 = (1,2), \vec{v}_2 = (-2,1)$. For example, by $(0,1) = \frac{2}{5}\vec{v}_1 + \frac{1}{5}\vec{v}_2$, we get

$$A^n\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{2}{5}A^n\vec{v}_1 + \frac{1}{5}A^n\vec{v}_2 = \frac{2}{5}5^n\vec{v}_1 + \frac{1}{5}15^n\vec{v}_2 = 5^{n-1}\begin{pmatrix} 2 - 2\cdot 3^n \\ 4 + 3^n \end{pmatrix}.$$

Exercise 7.1. In Example 7.1.1, what do you get if you start with $x_0 = 0$ and $y_0 = 1$?

Exercise 7.2. In Example 7.1.2, find the matrix A^n.

7.1.1 Invariant Subspace

We have better understanding of a linear operator L on V if it decomposes with respect to a direct sum $V = H_1 \oplus H_2 \oplus \cdots \oplus H_k$

$$\vec{v} = \vec{h}_1 + \vec{h}_2 + \cdots + \vec{h}_k \implies L(\vec{v}) = L_1(\vec{h}_1) + L_2(\vec{h}_2) + \cdots + L_k(\vec{h}_k).$$

In block notation, this means

$$L = L_1 \oplus L_2 \oplus \cdots \oplus L_n = \begin{pmatrix} L_1 & & & O \\ & L_2 & & \\ & & \ddots & \\ O & & & L_k \end{pmatrix}.$$

The linear operator in Example 7.1.2 is decomposed into multiplying 5 and 15 with respect to the direct sum $\mathbb{R}^2 = \mathbb{R}(1,2) \oplus \mathbb{R}(-2,1)$.

In the decomposition above, we have

$$\vec{h} \in H_i \implies L(\vec{h}) \in H_i.$$

This leads to the following concept.

Definition 7.1.1. For a linear operator $L\colon V \to V$, a subspace $H \subset V$ is *invariant* if $\vec{h} \in H \implies L(\vec{h}) \in H$.

The zero space $\{\vec{0}\}$ and the whole space V are trivial examples of invariant subspaces. We wish to express V as a direct sum of invariant subspaces.

It is easy to see that a 1-dimensional subspace $\mathbb{R}\vec{x}$ is invariant if and only if $L(\vec{x}) = \lambda\vec{x}$ for some scalar λ. In this case, we say λ is an *eigenvalue* and $\vec{x}$ is an *eigenvector*.

Example 7.1.3. Both $\mathbb{R}\vec{v}_1$ and $\mathbb{R}\vec{v}_2$ in Example 7.1.2 are invariant subspaces of the matrix A. In fact, $\vec{v}_1$ is an eigenvector of A of eigenvalue 5, and $\vec{v}_2$ is an eigenvector of A of eigenvalue 15.

Are there any other eigenvectors? If $a\vec{v}_1 + b\vec{v}_2$ is an eigenvector of eigenvalue λ, then

$$A(a\vec{v}_1 + b\vec{v}_2) = 5a\vec{v}_1 + 15b\vec{v}_2 = \lambda(a\vec{v}_1 + b\vec{v}_2) = \lambda a\vec{v}_1 + \lambda b\vec{v}_2.$$

Since $\vec{v}_1, \vec{v}_2$ are linearly independent, we get $\lambda a = 5a$ and $\lambda b = 15b$. This implies either $a = 0$ or $b = 0$. Therefore there are no other eigenvectors except non-zero multiples of $\vec{v}_1, \vec{v}_2$. In other words, $\mathbb{R}\vec{v}_1$ and $\mathbb{R}\vec{v}_2$ are the only 1-dimensional invariant subspaces.

Example 7.1.4. Consider the derivative operator $D\colon C^\infty \to C^\infty$. If $f \in P_n \subset C^\infty$, then $D(f) \in P_{n-1}$. Therefore the polynomial subspaces P_n are D-invariant.

Example 7.1.5. Let L be a linear operator on V. Then for any $\vec{v} \in V$, the subspace

$$H = \mathbb{R}\vec{v} + \mathbb{R}L(\vec{v}) + \mathbb{R}L^2(\vec{v}) + \cdots$$

is L-invariant, called the *L-cyclic subspace* generated by $\vec{v}$. This is clearly the smallest invariant subspace of L containing $\vec{v}$.

If $\vec{v}, L(\vec{v}), L^2(\vec{v}), \dots, L^n(\vec{v})$ are linearly dependent for some n (this happens when V is finite dimensional, as we can take any $n > \dim V$), then we let k be the smallest number, such that $\alpha = \{\vec{v}, L(\vec{v}), L^2(\vec{v}), \dots, L^{k-1}(\vec{v})\}$ is linearly independent. This implies $L^k(\vec{v})$ is a linear combination of α. In other words, we have

$$L^k(\vec{v}) + a_{k-1}L^{k-1}(\vec{v}) + a_{k-2}L^{k-2}(\vec{v}) + \cdots + a_1L(\vec{v}) + a_0\vec{v} = \vec{0}.$$

This further implies that $L^n(\vec{v})$ is a linear combination of α for any $n \geq k$. The minimality of k implies that α is linearly independent. Therefore α is a basis of the cyclic subspace H.

Exercise 7.3. Show that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has only one 1-dimensional invariant subspace.

Exercise 7.4. For the derivative operator $D\colon C^\infty \to C^\infty$, find the smallest invariant subspace containing t^n, and the smallest invariant subspace containing $\sin t$.

Exercise 7.5. Show that if a rotation of $\mathbb{R}^2$ has a 1-dimensional invariant subspace, then the rotation is either I or $-I$.

Exercise 7.6. Prove that $\text{Ran}L$ is L-invariant. What about $\text{Ker}L$?

Exercise 7.7. Suppose $LK = KL$. Prove that $\text{Ran}K$ and $\text{Ker}K$ are L-invariant.

Exercise 7.8. Suppose H is an invariant subspace of L and K. Prove that H is an invariant subspace of $L+K, aL, L\circ K$. In particular, H is an invariant subspace of any polynomial $f(L) = a_nL^n + a_{n-1}L^{n-1} + \cdots + a_1L + a_0I$ of L.

Exercise 7.9. Suppose $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ span H. Prove that H is L-invariant if and only if $L(\vec{v}_i) \in H$.

Exercise 7.10. Prove that sum and intersection of L-invariant subspaces are still L-invariant.

Exercise 7.11. Suppose L is a linear operator on a complex vector space with conjugation. If H is an invariant subspace of L, prove that $\bar{H}$ is an invariant subspace of $\bar{L}$.

Exercise 7.12. What is the relation between the invariant subspaces of $K^{-1}LK$ and L?

Exercise 7.13. Suppose $V = H \oplus H'$. Prove that H is an invariant subspace of a linear operator L on V if and only if the block form of L with respect to the direct sum is

$$L = \begin{pmatrix} L_1 & * \\ O & L_2 \end{pmatrix}.$$

What about H' being an invariant subspace of L?

Exercise 7.14. Prove that the L-cyclic subspace generated by $\vec{v}$ is the smallest invariant subspace containing $\vec{v}$.

Exercise 7.15. For the cyclic subspace H in Example 7.1.5, we have the smallest number k, such that $\alpha = \{\vec{v}, L(\vec{v}), L^2(\vec{v}), \ldots, L^{k-1}(\vec{v})\}$ is linearly independent. Let

$$f(t) = t^k + a_{k-1}t^{k-1} + a_{k-2}t^{k-2} + \cdots + a_1t + a_0, \quad f(L)(\vec{v}) = \vec{0}.$$

We also note that $L|_H$ is a linear operator on the subspace H.

1. Prove that $f(L)(\vec{h}) = \vec{0}$ for all $\vec{h} \in H$. This means $f(L|_H) = O$.

2. If $f(t)=g(t)h(t)$, prove that $\text{Ker}g(L|_H)\supset \text{Ran}h(L|_H)\neq\{\vec{0}\}$.

Exercise 7.16. In Example 7.1.5, show that the matrix of the restriction of L to the cyclic subspace H with respect to the basis α is

$$[L|_H]_{\alpha\alpha}=\begin{pmatrix}0 & 0 & \cdots & 0 & 0 & -a_0\\ 1 & 0 & \cdots & 0 & 0 & -a_1\\ 0 & 1 & \cdots & 0 & 0 & -a_2\\ \vdots & \vdots & & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0 & -a_{k-2}\\ 0 & 0 & \cdots & 0 & 1 & -a_{k-1}\end{pmatrix}.$$

7.1.2 Eigenspace

The simplest direct sum decomposition $L=L_1\oplus L_2\oplus\cdots\oplus L_n$ is that L multiplies a scalar λ_i on each H_i (this may or may not exist)

$$\vec{v}=\vec{h}_1+\vec{h}_2+\cdots+\vec{h}_k,\ \vec{h}_i\in H_i \implies L(\vec{v})=\lambda_1\vec{h}_1+\lambda_2\vec{h}_2+\cdots+\lambda_k\vec{h}_k.$$

In other words, we have

$$L=\lambda_1 I\oplus\lambda_2 I\oplus\cdots\oplus\lambda_k I,$$

where I is the identity operator on subspaces. If λ_i are distinct, then the equality means exactly $H_i=\text{Ker}(L-\lambda_i I)$, so that

$$V=\text{Ker}(L-\lambda_1 I)\oplus\text{Ker}(L-\lambda_2 I)\oplus\cdots\oplus\text{Ker}(L-\lambda_k I).$$

We note that, for finite dimensional V, the subspace $H_i\neq\{\vec{0}\}$ if and only if $L-\lambda_i I$ is not invertible. This means $L(\vec{v})=\lambda_i\vec{v}$ for some nonzero $\vec{v}$.

Definition 7.1.2. A number λ is an *eigenvalue* of a linear operator $L\colon V\to V$ if $L-\lambda I$ is not invertible. The associated kernel subspace is the *eigenspace*

$$\text{Ker}(L-\lambda I)=\{\vec{v}\colon L(\vec{v})=\lambda\vec{v}\}.$$

For finite dimensional V, the non-invertiblity of $L-\lambda I$ is equivalent to that the eigenspace $\text{Ker}(L-\lambda I)\neq\{\vec{0}\}$. Any nonzero vector in the eigenspace is an *eigenvector*.

Example 7.1.6. For the orthogonal projection P of $\mathbb{R}^3$ to the plane $x+y+z=0$ in Example 2.1.13, the plane is the eigenspace of eigenvelue 1, and the line $\mathbb{R}(1,1,1)$ orthogonal to the plane is the eigenspace of eigenvelue 0. The fact is used in the example (and Example 2.3.9) to get the matrix of P.

Example 7.1.7. By Exercise 3.65, a projection P on V induces

$$V = \text{Ran}P \oplus \text{Ran}(I-P) = \text{Ker}(I-P) \oplus \text{Ker}P = \text{Ker}(P - 1\cdot I) \oplus \text{Ker}(P - 0\cdot I).$$

Therefore P has eigenvalues 1 and 0, with respective eigenspaces Ran$P$ and Ran$(I-P)$.

Example 7.1.8. The transpose operation of square matrices has eigevalues 1 and -1, with symmetric matrices and skew-symmetric matrices as respective eigenspaces.

Example 7.1.9. Consider the derivative linear transformation $D(f) = f'$ on the space of all real valued smooth functions $f(t)$ on $\mathbb{R}$. The eigenspace of eigenvalue λ consists of all functions f satisfying $f' = \lambda f$. This means exactly $f(t) = ce^{\lambda t}$ is a constant multiple of the exponential function $e^{\lambda t}$. Therefore any real number λ is an eigenvalue, and the eigenspace $\text{Ker}(D - \lambda I) = \mathbb{R}e^{\lambda t}$.

Example 7.1.10. We may also consider the derivative linear transformation on the space V of all complex valued smooth functions $f(t)$ on $\mathbb{R}$ of period 2π. In this case, the eigenspace $\text{Ker}^{\mathbb{C}}(D - \lambda I)$ still consists of $ce^{\lambda t}$, but c and λ can be complex numbers. For the function to have period 2π, we further need $e^{\lambda 2\pi} = e^{\lambda 0} = 1$. This means $\lambda = in \in i\mathbb{Z}$. Therefore the (relabeled) eigenspaces are $\text{Ker}^{\mathbb{C}}(D - inI) = \mathbb{C}e^{int}$. The "eigenspace decomposition" $V = \oplus_n \mathbb{C}e^{int}$ essentially means the Fourier series. More details will be given in Example 7.2.2.

Exercise 7.17. Prove that 0 is the eigenvalue of a linear operator if and only if the linear operator is not invertible.

Exercise 7.18. Suppose $L = \lambda_1 I \oplus \lambda_2 I \oplus \cdots \oplus \lambda_k I$ on $V = H_1 \oplus H_2 \oplus \cdots \oplus H_k$, and λ_i are distinct. Prove that $\lambda_1, \lambda_2, \dots, \lambda_k$ are all the eigenvalues of L, and any invariant subspace of L is of the form $W_1 \oplus W_2 \oplus \cdots \oplus W_k$, with $W_i \subset H_i$.

Exercise 7.19. Suppose L is a linear operator on a complex vector space with conjugation. Prove that $L(\vec{v}) = \lambda\vec{v}$ implies $\bar{L}(\bar{\vec{v}}) = \bar{\lambda}\bar{\vec{v}}$. In particular, if $H = \text{Ker}(L - \lambda I)$ is an eigenspace of L, then the conjugate subspace $\bar{H} = \text{Ker}(\bar{L} - \bar{\lambda} I)$ is an eigenspace of $\bar{L}$.

Exercise 7.20. What is the relation between the eigenvalues and eigenspaces of $K^{-1}LK$ and L?

For any polynomial $f(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0 \in \mathbb{F}[t]$, we define

$$f(L) = a_n L^n + a_{n-1}L^{n-1} + \cdots + a_1 L + a_0 I.$$

The set of all polynomials of L

$$\mathbb{F}[L] = \{f(L) \colon f \text{ is a polynomial}\}$$

is a commutative algebra in the sense that

$$K_1, K_2 \in \mathbb{F}[L] \implies a_1K_1 + a_2K_2 \in \mathbb{F}[L],\ K_1K_2 = K_2K_1 \in \mathbb{F}[L].$$

If $L(\vec{h}) = \lambda\vec{h}$, then we easily get $L^i(\vec{h}) = \lambda^i\vec{h}$, and

$$\begin{aligned} f(L)(\vec{h}) &= a_nL^n(\vec{h}) + a_{n-1}L^{n-1}(\vec{h}) + \cdots + a_1L(\vec{h}) + a_0I(\vec{h}) \\ &= a_n\lambda^n\vec{h} + a_{n-1}\lambda^{n-1}\vec{h} + \cdots + a_1\lambda\vec{h} + a_0\vec{h} = f(\lambda)(\vec{h}). \end{aligned}$$

Therefore we have the following result, which implies that a simplest direct sum decomposition for L is also simplest for all polynomials of L.

Proposition 7.1.3. *If L is multiplying λ on a subspace H and $f(t)$ is a polynomial, then $f(L)$ is multiplying $f(\lambda)$ on H.*

The maximal subspace on which L is multiplying λ is the eigenspace $\mathrm{Ker}(L-\lambda I)$. The following shows that, in order to get the simplest direct sum decomposition, it is sufficient to show that V is a sum of eigenspaces.

Proposition 7.1.4. *The sum of eigenspaces with distinct eigenvalues is a direct sum.*

Proof. Suppose L is multiplying λ_i on H_i, and λ_i are distinct. Suppose

$$\vec{h}_1 + \vec{h}_2 + \cdots + \vec{h}_k = \vec{0}, \quad \vec{h}_i \in H_i.$$

Inspired by Example 2.2.13, we take

$$f(t) = (t - \lambda_2)(t - \lambda_3)\cdots(t - \lambda_k).$$

By Proposition 7.1.3, we have $f(L)(\vec{h}_i) = f(\lambda_i)\vec{h}_i$ for $\vec{h}_i \in H_i$. Applying $f(L)$ to the equality $\vec{0} = \vec{h}_1 + \vec{h}_2 + \cdots + \vec{h}_k$ and using $f(\lambda_2) = \cdots = f(\lambda_k) = 0$, we get

$$\vec{0} = f(L)(\vec{h}_1 + \vec{h}_2 + \cdots + \vec{h}_k) = f(\lambda_1)\vec{h}_1 + f(\lambda_2)\vec{h}_2 + \cdots + f(\lambda_k)\vec{h}_k = f(\lambda_1)\vec{h}_1.$$

By $f(\lambda_1) \neq 0$, we get $\vec{h}_1 = \vec{0}$. By taking f to be the other similar polynomials, we get all $\vec{h}_i = \vec{0}$. □

The following is a very useful property about eigenspaces when there are two commuting operators.

Proposition 7.1.5. *Suppose L and K are linear operators on V. If $LK = KL$, then any eigenspace of L is an invariant subspace of K.*

Proof. If $\vec{v} \in \mathrm{Ker}(L - \lambda I)$, then $L(\vec{v}) = \lambda\vec{v}$. This implies $L(K(\vec{v})) = K(L(\vec{v})) = K(\lambda\vec{v}) = \lambda K(\vec{v})$. Therefore $K(\vec{v}) \in \mathrm{Ker}(L - \lambda I)$. □

Exercise 7.21. Suppose $L = \lambda_1 I \oplus \lambda_2 I \oplus \cdots \oplus \lambda_k I$ on $V = H_1 \oplus H_2 \oplus \cdots \oplus H_k$. Prove that

$$\det L = \lambda_1^{\dim H_1} \lambda_2^{\dim H_2} \cdots \lambda_k^{\dim H_k},$$

and

$$(L - \lambda_1 I)(L - \lambda_2 I) \ldots (L - \lambda_k I) = O.$$

Exercise 7.22. Suppose $L = L_1 \oplus L_2 \oplus \cdots \oplus L_k$ and $K = K_1 \oplus K_2 \oplus \cdots \oplus K_k$ with respect to the same direct sum decomposition $V = H_1 \oplus H_2 \oplus \cdots \oplus H_k$.

1. Prove that $LK = KL$ if and only if $L_i K_i = K_i L_i$ for all i.
2. Prove that if $L_i = \lambda_i I$ for all i, then $LK = KL$.

The second statement is the converse of Proposition 7.1.5.

Exercise 7.23. Suppose a linear operator satisfies $L^2 + 3L + 2 = O$. What can you say about the eigenvalues of L?

Exercise 7.24. A linear operator L is *nilpotent* if it satisfies $L^n = O$ for some n. Show that the derivative operator from P_n to itself is nilpotent. What are the eigenvalues of nilpotent operators?

Exercise 7.25. Prove that $f(K^{-1}LK) = K^{-1} f(L) K$ for any polynomial f.

7.1.3 Characteristic Polynomial

By Proposition 5.1.4, λ is an eigenvalue of a linear operator L on a finite dimensional vector space if and only if $\det(L - \lambda I) = 0$. This is the same as $\det(\lambda I - L) = 0$.

Definition 7.1.6. The *characteristic polynomial* of a linear operator L is $\det(tI - L)$.

Let α be a basis of the vector space, and the vector space has dimension n. Let $A = [L]_{\alpha\alpha}$. Then

$$\det(tI - A) = t^n - \sigma_1 t^{n-1} + \sigma_2 t^{n-2} + \cdots + (-1)^{n-1} \sigma_{n-1} t + (-1)^n \sigma_n.$$

The polynomial is independent of the choice of basis because the characteristic polynomials of $P^{-1}AP$ and A are the same.

Exercise 7.26. Prove that the eigenvalues of an upper or lower triangular matrix are the diagonal entries.

Exercise 7.27. Suppose L_1 and L_2 are linear operators. Prove that the characteristic polynomial of $\begin{pmatrix} L_1 & * \\ O & L_2 \end{pmatrix}$ is $\det(tI - L_1)\det(tI - L_2)$. Prove that the same is true for $\begin{pmatrix} L_1 & O \\ * & L_2 \end{pmatrix}$.

Exercise 7.28. What is the characteristic polynomial of the derivative operator on P_n?

Exercise 7.29. What is the characteristic polynomial of the transpose operation on $n \times n$ matrices?

Exercise 7.30. For 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, show that

$$\det(tI - A) = t^2 - (\mathrm{tr}A)t + \det A, \quad \mathrm{tr}A = a + d, \quad \det A = ad - bc.$$

Exercise 7.31. Prove that $\sigma_n = \det A$, and

$$\sigma_1 = \mathrm{tr} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = a_{11} + a_{22} + \cdots + a_{nn}.$$

Exercise 7.32. Let $I = (\vec{e}_1 \ \vec{e}_2 \ \cdots \ \vec{e}_n)$ and $A = (\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n)$. Then the term $(-1)^{n-1}\sigma_{n-1}t$ in $\det(tI - A)$ is

$$\begin{aligned} (-1)^{n-1}\sigma_{n-1}t &= \sum_{1 \le i < j \le n} \det(-\vec{v}_1 \ \cdots \ t\vec{e}_i \ \cdots \ -\vec{v}_n) \\ &= \sum_{1 \le i < j \le n} (-1)^{n-1} t \det(\vec{v}_1 \ \cdots \ \vec{e}_i \ \cdots \ \vec{v}_n), \end{aligned}$$

where the i-th columns of A is replaced by $t\vec{e}_i$. Using the argument similar to the cofactor expansion to show that

$$\sigma_{n-1} = \det A_{11} + \det A_{22} + \cdots + \det A_{nn}.$$

Exercise 7.33. For an $n \times n$ matrix A and $1 \le i_1 < i_2 < \cdots < i_k \le n$, let $A(i_1, i_2, \ldots, i_k)$ be the $k \times k$ submatrix of A of the $i_1, i_2, \ldots, i_k$ rows and $i_1, i_2, \ldots, i_k$ columns. Prove that

$$\sigma_k = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \det A(i_1, i_2, \ldots, i_k).$$

Exercise 7.34. Suppose $L = \lambda_1 I \oplus \lambda_2 I \oplus \cdots \oplus \lambda_k I$ on $V = H_1 \oplus H_2 \oplus \cdots \oplus H_k$. Prove that

$$\det(tI - L) = (t - \lambda_1)^{\dim H_1} (t - \lambda_2)^{\dim H_2} \cdots (t - \lambda_k)^{\dim H_k}.$$

Then verify directly that the polynomial $f(t) = \det(tI - L)$ satisfies $f(L) = O$.

Exercise 7.35. Show that the characteristic polynomial of the matrix in Example 7.16 (for the restriction of a linear operator to a cyclic subspace) is $t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$.

Proposition 7.1.7. *If H is an invariant subspace of a linear operator L on V, then the characteristic polynomial $\det(tI - L|_H)$ of the restriction $L|_H$ divides the characteristic polynomial $\det(tI - L)$ of L.*

Proof. Let H' be a direct summand of H: $V = H \oplus H'$. Since H is L-invariant, we have (see Exercise 7.13)

$$L = \begin{pmatrix} L|_H & * \\ O & L' \end{pmatrix}.$$

Here the linear operator L' is the H'-component of the restriction of L on H'. Then by Exercise 7.27, we have

$$\det(tI - L) = \det \begin{pmatrix} tI - L|_H & * \\ O & tI - L' \end{pmatrix} = \det(tI - L|_H)\det(tI - L'). \qquad \square$$

Theorem 7.1.8 (Cayley-Hamilton Theorem)**.** *Let $f(t) = \det(tI - L)$ be the characteristic polynomial of a linear operator L. Then $f(L) = O$.*

By Exercise 7.30, for a 2×2 matrix, the theorem says

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 - (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} + (ad - bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

One could imagine that a direct computational proof would be very complicated.

Proof. For any vector $\vec{v}$, construct the cyclic subspace in Example 7.1.5

$$H = \mathbb{R}\vec{v} + \mathbb{R}L(\vec{v}) + \mathbb{R}L^2(\vec{v}) + \cdots.$$

The subspace is L-invariant and has a basis $\alpha = \{\vec{v}, L(\vec{v}), L^2(\vec{v}), \dots, L^{k-1}(\vec{v})\}$. Moreover, the fact that $L^k(\vec{v})$ is a linear combination of α

$$L^k(\vec{v}) = -a_{k-1}L^{k-1}(\vec{v}) - a_{k-2}L^{k-2}(\vec{v}) - \cdots - a_1L(\vec{v}) - a_0\vec{v}$$

means that $g(t) = t^k + a_{k-1}t^{k-1} + a_{k-2}t^{k-2} + \cdots + a_1t + a_0$ satisfies $g(L)(\vec{v}) = \vec{0}$.

By Exercises 7.16 and 7.35, the characteristic polynomial $\det(tI - L|_H)$ is exactly the polynomial $g(t)$ above. By Proposition 7.1.7, we know $f(t) = h(t)g(t)$ for a polynomial $h(t)$. Then

$$f(L)(\vec{v}) = h(L)(g(L)(\vec{v})) = h(L)(\vec{0}) = \vec{0}.$$

Since this is proved for all $\vec{v}$, we get $f(L) = O$. $\square$

Example 7.1.11. The characteristic polynomial of the matrix in Example 2.2.18 is

$$\det \begin{pmatrix} t-1 & -1 & -1 \\ 1 & t & -1 \\ 0 & 1 & t-1 \end{pmatrix} = t(t-1)^2 - 1 + (t-1) + (t-1) = t^3 - 2t^2 + 3t - 3.$$

By Cayley-Hamilton Theorem, this implies

$$A^3 - 2A^2 + 3A - 3I = O.$$

This is the same as $A(A^2-2A+3I)=3I$ and gives

$$\begin{aligned}
A^{-1} &= \frac{1}{3}(A^2-2A+3I) \\
&= \frac{1}{3}\left[\begin{pmatrix} 0 & 0 & 3 \\ -1 & -2 & 0 \\ 1 & -1 & 0 \end{pmatrix} - 2\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \\
&= \frac{1}{3}\begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{pmatrix}.
\end{aligned}$$

The result is the same as Example 2.2.18, but the calculation is more complicated.

7.1.4 Diagonalisation

We saw that the simplest case for a linear operator L on V is that V is the direct sum of eigenspaces of L. If we take a basis of each eigenspace, then the union of such bases is a basis of V consisting of eigenvectors of L. Therefore the simplest case is that L has a basis of eigenvectors.

Let $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be a basis of eigenvectors, with corresponding eigenvalues $d_1, d_2, \dots, d_n$. The numbers d_j are the eigenvalues λ_i repeated $\dim \operatorname{Ker}(L-\lambda_i I)$ times. The equalities $L(\vec{v}_j) = d_j\vec{v}_j$ mean that the matrix of L with respect to the basis α is *diagonal*

$$[L]_{\alpha\alpha} = \begin{pmatrix} d_1 & & & O \\ & d_2 & & \\ & & \ddots & \\ O & & & d_n \end{pmatrix} = D.$$

For this reason, we describe the simplest case for a linear operator as follows.

Definition 7.1.9. A linear operator is *diagonalisable* if it has a basis of eigenvectors.

Suppose L is a linear operator on $\mathbb{F}^n$, with corresponding matrix A. If L has a basis α of eigenvectors, then for the standard basis ϵ, we have

$$A = [L]_{\epsilon\epsilon} = [I]_{\epsilon\alpha}[L]_{\alpha\alpha}[I]_{\alpha\epsilon} = PDP^{-1}, \quad P = [\alpha]_\epsilon = (\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n).$$

We call the formula $A = PDP^{-1}$ a *diagonalisation* of A. A diagonalisation is the same as a basis of eigenvectors (which form the columns of P).

The following is a special diagonalisable case.

Proposition 7.1.10. *If a linear operator on an n-dimensional vector space has n distinct eigenvalues, then the linear operator is diagonalisable.*

Proof. The condition means $\det(tI-L)=(t-\lambda_1)(t-\lambda_2)\cdots(t-\lambda_n)$, with λ_i distinct. In this case, we pick one eigenvector $\vec{v}_i$ for each eigenvalue λ_i. By Proposition 7.1.4, the eigenvectors $\vec{v}_1,\vec{v}_2,\dots,\vec{v}_n$ are linearly independent. Since the space has dimension n, the eigenvectors form a basis. This proves the proposition. □

To find diagonalisation, we may first solve $\det(tI-L)=0$ to find eigenvalues λ. Then we solve $(L-\lambda I)\vec{x}=\vec{0}$ to find (a basis of) the eigenspace $\operatorname{Ker}(L-\lambda I)$. The number $\dim\operatorname{Ker}(L-\lambda I)$ is the *geometric multiplicity* of λ, and the operator is diagonalisable if and only if the sum of geometric multiplicities is the dimension of the whole space.

Example 7.1.12. For the matrix in Example 7.1.2, we have

$$\begin{aligned}\det(tI-A)&=\det\begin{pmatrix}t-13&4\\4&t-7\end{pmatrix}\\&=(t-13)(t-7)-16=t^2-20t+75=(t-5)(t-15).\end{aligned}$$

This gives two possible eigenvalues 5 and 15.

The eigenspace $\operatorname{Ker}(A-5I)$ is the solutions of

$$(A-5I)\vec{x}=\begin{pmatrix}13-5&-4\\-4&7-5\end{pmatrix}\vec{x}=\begin{pmatrix}8&-4\\-4&2\end{pmatrix}\vec{x}=\vec{0}.$$

We get $\operatorname{Ker}(A-5I)=\mathbb{R}(1,2)$, and the geometric multiplicity is 1.

The eigenspace $\operatorname{Ker}(A-15I)$ is the solutions of

$$(A-15I)\vec{x}=\begin{pmatrix}13-15&-4\\-4&7-15\end{pmatrix}\vec{x}=\begin{pmatrix}-2&-4\\-4&-8\end{pmatrix}\vec{x}=\vec{0}.$$

We get $\operatorname{Ker}(A-15I)=\mathbb{R}(2,-1)$, and the geometric multiplicity is 1.

The sum of geometric multiplicities is $2=\dim\mathbb{R}^2$. Therefore the matrix is diagonalisable, with basis of eigenvectors $\{(1,2),(2,-1)\}$. The corresponding diagonalisation is

$$\begin{pmatrix}13&-4\\-4&7\end{pmatrix}=\begin{pmatrix}1&2\\2&-1\end{pmatrix}\begin{pmatrix}5&0\\0&15\end{pmatrix}\begin{pmatrix}1&2\\2&-1\end{pmatrix}^{-1}.$$

If $A=PDP^{-1}$, then $A^n=PD^nP^{-1}$. It is easy to see that D^n simply means taking the n-th power of the diagonal entries. Therefore

$$\begin{aligned}\begin{pmatrix}13&-4\\-4&7\end{pmatrix}^n&=\begin{pmatrix}1&2\\2&-1\end{pmatrix}\begin{pmatrix}5^n&0\\0&15^n\end{pmatrix}\begin{pmatrix}1&2\\2&-1\end{pmatrix}^{-1}\\&=\frac{1}{5}\begin{pmatrix}1&2\\2&-1\end{pmatrix}\begin{pmatrix}5^n&0\\0&15^n\end{pmatrix}\begin{pmatrix}1&2\\2&-1\end{pmatrix}=5^{n-1}\begin{pmatrix}1+4\cdot3^n&2-2\cdot3^n\\2-2\cdot3^n&4+3^n\end{pmatrix}.\end{aligned}$$

By using Taylor expansion, we also have $f(A)=Pf(D)P^{-1}$, where the function f is applied to each diagonal entry of D. For example, we have

$$e^{\begin{pmatrix}13&-4\\-4&7\end{pmatrix}}=\begin{pmatrix}1&2\\2&-1\end{pmatrix}\begin{pmatrix}e^5&0\\0&e^{15}\end{pmatrix}\begin{pmatrix}1&2\\2&-1\end{pmatrix}^{-1}.$$

Example 7.1.13. For the matrix in Example 7.1.1, we have

$$\det(tI - A) = \det\begin{pmatrix} t-1 & 1 \\ -1 & t-1 \end{pmatrix} = (t-1)^2 + 1 = (t-1-i)(t-1+i).$$

This gives two possible eigenvalues $\lambda = 1+i$, $\bar{\lambda} = 1-i$. We need to view the real matrix A as an operator on the complex vector space $\mathbb{C}^2$ because they are complex eigenvalues.

The eigenspace $\text{Ker}(A - \lambda I)$ is the solutions of

$$(A - (1+i)I)\vec{x} = \begin{pmatrix} 1-(1+i) & -1 \\ 1 & 1-(1+i) \end{pmatrix}\vec{x} = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\vec{x} = \vec{0}.$$

We have the complex eigenspace $\text{Ker}^{\mathbb{C}}(A - \lambda I) = \mathbb{C}(1, -i)$, and the geometric multiplicity is 1.

Since A is a real matrix, by Example 7.19, we know $\text{Ker}^{\mathbb{C}}(A - \bar{\lambda} I) = \mathbb{C}\overline{(1, -i)} = \mathbb{C}(1, i)$ is also a complex eigenspace of A, with the same geometric multiplicity 1.

The sum of geometric multiplicities is $2 = \dim_{\mathbb{C}} \mathbb{C}^2$. Therefore the matrix is diagonalisable, with basis of eigenvectors $\{(1, -i), (1, i)\}$. The corresponding diagonalisation is

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}^{-1}.$$

Example 7.1.14. By Example 5.1.4, we have

$$A = \begin{pmatrix} 1 & -2 & -4 \\ -2 & 4 & -2 \\ -4 & -2 & 1 \end{pmatrix}, \quad \det(tI - A) = (t-5)^2(t+4).$$

We get eigenvalues 5 and -4, and

$$A - 5I = \begin{pmatrix} -4 & -2 & -4 \\ -2 & -1 & -2 \\ -4 & -2 & -4 \end{pmatrix}, \quad A + 4I = \begin{pmatrix} 5 & -2 & -4 \\ -2 & 9 & -2 \\ -4 & -2 & 5 \end{pmatrix}.$$

The eigenspaces are $\text{Ker}(A - 5I) = \mathbb{R}(-1, 2, 0) \oplus \mathbb{R}(-1, 0, 1)$ and $\text{Ker}(A + 4I) = \mathbb{R}(2, 1, 2)$. We get a basis of eigenvectors $\{(-1, 2, 0), (-1, 0, 1), (2, 1, 2)\}$, with the corresponding diagonalisation

$$\begin{pmatrix} 1 & -2 & -4 \\ -2 & 4 & -2 \\ -4 & -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} -1 & -1 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}^{-1}.$$

Example 7.1.15. For the matrix

$$A = \begin{pmatrix} 3 & 1 & -3 \\ -1 & 5 & -3 \\ -6 & 6 & -2 \end{pmatrix},$$

we have

$$\det(tI-A)=\det\begin{pmatrix} t-3 & -1 & 3\\ 1 & t-5 & 3\\ 6 & -6 & t+2\end{pmatrix}=\det\begin{pmatrix} t-4 & -1 & 3\\ t-4 & t-5 & 3\\ 0 & -6 & t+2\end{pmatrix}$$
$$=\det\begin{pmatrix} t-4 & -1 & 3\\ 0 & t-4 & 0\\ 0 & -6 & t+2\end{pmatrix}=(t-4)^2(t+2).$$

We get eigenvalues 4 and -2, and

$$A-4I=\begin{pmatrix} -1 & 1 & -3\\ -1 & 1 & -3\\ -6 & 6 & -6\end{pmatrix},\quad A+2I=\begin{pmatrix} 5 & 1 & -3\\ -1 & 7 & -3\\ -6 & 6 & 0\end{pmatrix}.$$

The eigenspaces are $\mathrm{Ker}(A-4I)=\mathbb{R}(1,1,0)$ and $\mathrm{Ker}(A+2I)=\mathbb{R}(1,1,2)$, with geometric multiplicities 1 and 1. Since $1+1<\dim\mathbb{R}^3$, the matrix is not diagonalisable.

Example 7.1.16. By Proposition 7.1.10, the matrix

$$\begin{pmatrix} 1 & 0 & 0\\ \sqrt{2} & 2 & 0\\ \pi & \sin 1 & e\end{pmatrix}$$

has three distinct eigenvalues $1,2,e$, and is therefore diagonalisable. The actual diagonalisation is quite complicated to calculate.

Exercise 7.36. Find eigenspaces and determine whether this is a basis of eigenvectors. Express the diagonalisable matrix as PDP^{-1}.

1. $\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}$.
2. $\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}$.
3. $\begin{pmatrix} 1 & 2 & 3\\ 0 & 4 & 5\\ 0 & 0 & 6\end{pmatrix}$.
4. $\begin{pmatrix} 1 & 0 & 0\\ 2 & 4 & 0\\ 3 & 5 & 6\end{pmatrix}$.
5. $\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}$.
6. $\begin{pmatrix} 3 & 1 & 0\\ -4 & -1 & 0\\ 4 & -8 & -2\end{pmatrix}$.
7. $\begin{pmatrix} 3 & 1 & -3\\ -1 & 5 & -3\\ -6 & 6 & -2\end{pmatrix}$.
8. $\begin{pmatrix} 1 & 1 & 1 & 1\\ 1 & -1 & -1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\end{pmatrix}$.

Exercise 7.37. Suppose a matrix has the following eigenvalues and eigenvectors. Find the matrix.

1. $(1,1,0),\lambda_1=1$; $(1,0,-1),\lambda_2=2$; $(1,1,2),\lambda_3=3$.

2. $(1,1,0), \lambda_1 = 2$; $(1,0,-1), \lambda_2 = 2$; $(1,1,2), \lambda_3 = -1$.

3. $(1,1,0), \lambda_1 = 1$; $(1,0,-1), \lambda_2 = 1$; $(1,1,2), \lambda_3 = 1$.

4. $(1,1,1), \lambda_1 = 1$; $(1,0,1), \lambda_2 = 2$; $(0,-1,2), \lambda_3 = 3$.

Exercise 7.38. Let $A = \vec{v}\vec{v}^T$, where $\vec{v}$ is a nonzero vector regarded as a vertical matrix.

1. Find the eigenspaces of A.

2. Find the eigenspaces of $aI + bA$.

3. Find the eigenspaces of the matrix

$$\begin{pmatrix} b & a & \dots & a \\ a & b & \dots & a \\ \vdots & \vdots & & \vdots \\ a & a & \dots & b \end{pmatrix}.$$

Exercise 7.39. Describe all 2×2 matrices satisfying $A^2 = I$.

Exercise 7.40. What are diagonalisable nilpotent operators? Then use Exercise 7.24 to explain that the derivative operator on P_n is not diagonalisable.

Exercise 7.41. Is the transpose of $n \times n$ matrix diagonalisable?

Exercise 7.42. Suppose H is an invariant subspace of a linear operator L. Prove that if L is diagonalisable, then the restriction operator $L|_H$ is also diagonalisable.

Exercise 7.43. Prove that $L_1 \oplus L_2$ is diagonalisable if and only if L_1 and L_2 are diagonalisable.

Exercise 7.44. Prove that two diagonalisable matrices are similar if and only if they have the same characteristic polynomial.

Exercise 7.45. Suppose two real matrices A, B are complex diagonalisable, and have the same characteristic polynomial. Is it true that $B = PAP^{-1}$ for a real matrix P?

Exercise 7.46. Suppose $\det(tI - L) = (t-\lambda_1)^{n_1}(t-\lambda_2)^{n_2}\cdots(t-\lambda_k)^{n_k}$, with λ_i distinct. The number n_i is the *algebraic multiplicity* of λ_i. Denote the geometric multiplicity by $g_i = \dim \operatorname{Ker}(L - \lambda_i I)$.

1. By taking $H = \operatorname{Ker}(L - \lambda_i I)$ in Proposition 7.1.7, prove that $g_i \le n_i$.

2. Explain that $g_i \ge 1$.

3. Prove that L is diagonalisable if and only if $g_i = n_i$.

Example 7.1.17. To find the general formula for the Fibonacci numbers, we introduce

$$\vec{x}_n = \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix}, \quad \vec{x}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \vec{x}_{n+1} = \begin{pmatrix} F_{n+1} \\ F_{n+1} + F_n \end{pmatrix} = A\vec{x}_n, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then the n-th Fibonacci number F_n is the first coordinate of $\vec{x}_n = A^n\vec{x}_0$.

The characteristic polynomial $\det(tI - A) = t^2 - t - 1$ has two roots

$$\lambda_1 = \frac{1+\sqrt{5}}{2}, \quad \lambda_2 = \frac{1-\sqrt{5}}{2}.$$

By

$$A - \lambda_1 I = \begin{pmatrix} -\lambda_1 & 1 \\ 1 & 1-\lambda_1 \end{pmatrix} = \begin{pmatrix} -\frac{1+\sqrt{5}}{2} & 1 \\ 1 & \frac{1-\sqrt{5}}{2} \end{pmatrix},$$

we get the eigenspace $\mathrm{Ker}(A - \lambda_1 I) = \mathbb{R}(1, \frac{1+\sqrt{5}}{2})$. If we substitute $\sqrt{5}$ by $-\sqrt{5}$, then we get the other eigenspace $\mathrm{Ker}(A - \lambda_2 I) = \mathbb{R}(1, \frac{1-\sqrt{5}}{2})$. To find $\vec{x}_n$, we decompose $\vec{x}_0$ according to the basis of eigenvectors

$$\vec{x}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix} - \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix}.$$

The two coefficients can be obtained by solving a system of linear equations. Then

$$\vec{x}_n = \frac{1}{\sqrt{5}} A^n \begin{pmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix} - \frac{1}{\sqrt{5}} A^n \begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix} = \frac{1}{\sqrt{5}} \lambda_1^n \begin{pmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix} - \frac{1}{\sqrt{5}} \lambda_2^n \begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix}.$$

Picking the first coordinate, we get

$$F_n = \frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n) = \frac{1}{2^n\sqrt{5}} \left[(1+\sqrt{5})^n - (1-\sqrt{5})^n \right].$$

Exercise 7.47. Find the general formula for the Fibonacci numbers that start with $F_0 = 1$, $F_1 = 0$.

Exercise 7.48. Given the recursive relations and initial values. Find the general formula.

1. $x_n = x_{n-1} + 2y_{n-1}$, $y_n = 2x_{n-1} + 3y_{n-1}$, $x_0 = 1$, $y_0 = 0$.
2. $x_n = x_{n-1} + 2y_{n-1}$, $y_n = 2x_{n-1} + 3y_{n-1}$, $x_0 = 0$, $y_0 = 1$.
3. $x_n = x_{n-1} + 2y_{n-1}$, $y_n = 2x_{n-1} + 3y_{n-1}$, $x_0 = a$, $y_0 = b$.
4. $x_n = x_{n-1} + 3y_{n-1} - 3z_{n-1}$, $y_n = -3x_{n-1} + 7y_{n-1} - 3z_{n-1}$, $z_n = -6x_{n-1} + 6y_{n-1} - 2z_{n-1}$, $x_0 = a$, $y_0 = b$, $z_n = c$.

Exercise 7.49. Consider recursive relation $x_n = a_{n-1}x_{n-1} + a_{n-2}x_{n-2} + \cdots + a_1x_1 + a_0x_0$. Prove that if the polynomial $t^n - a_{n-1}t^{n-1} - a_{n-2}t^{n-2} - \cdots - a_1t - a_0$ has distinct roots $\lambda_1, \lambda_2, \dots, \lambda_n$, then

$$x_n = c_1\lambda_1^n + c_2\lambda_2^n + \cdots + c_n\lambda_n^n.$$

Here the coefficients $c_1, c_2, \dots, c_n$ may be calculated from the initial values $x_0, x_1, \dots, x_{n-1}$. Then apply the result to the Fibonacci numbers.

Exercise 7.50. Consider the recursive relation $x_n = a_{n-1}x_{n-1}+a_{n-2}x_{n-2}+\cdots+a_1x_1+a_0x_0$. Prove that if the polynomial $t^n-a_{n-1}t^{n-1}-a_{n-2}t^{n-2}-\cdots-a_1t-a_0 = (t-\lambda_1)(t-\lambda_2)\dots(t-\lambda_{n-1})^2$, and λ_j are distinct (i.e., λ_{n-1} is the only double root), then

$$x_n = c_1\lambda_1^n + c_2\lambda_2^n + \cdots + c_{n-2}\lambda_{n-2}^n + (c_{n-1} + c_n\lambda_n)\lambda_n^n.$$

Can you imagine the general formula in case $t^n - a_{n-1}t^{n-1} - a_{n-2}t^{n-2} - \cdots - a_1t - a_0 = (t-\lambda_1)^{n_1}(t-\lambda_2)^{n_2}\dots(t-\lambda_k)^{n_k}$?

7.1.5 Complex Eigenvalue of Real Operator

In Example 7.1.13, the real matrix is diagonalised by using complex matrices. We wish to know what this means in terms of only real matrices.

An eigenvalue λ of a real matrix A can be real ($\lambda \in \mathbb{R}$) or not real ($\lambda \in \mathbb{C} - \mathbb{R}$). In case λ is not real, the conjugate $\bar{\lambda}$ is another distinct eigenvalue.

Suppose $\lambda \in \mathbb{R}$. Then the we have the real eigenspace $\mathrm{Ker}^{\mathbb{R}}(A - \lambda I)$. It is easy to see that the complex eigenspace

$$\begin{aligned}\mathrm{Ker}^{\mathbb{C}}(A - \lambda I) &= \{\vec{v}_1 + i\vec{v}_2 \colon \vec{v}_1, \vec{v}_2 \in \mathbb{R}^n,\ A(\vec{v}_1 + i\vec{v}_2) = \lambda(\vec{v}_1 + i\vec{v}_2)\} \\ &= \{\vec{v}_1 + i\vec{v}_2 \colon \vec{v}_1, \vec{v}_2 \in \mathbb{R}^n,\ A\vec{v}_1 = \lambda\vec{v}_1,\ A\vec{v}_2 = \lambda\vec{v}_2\}\end{aligned}$$

is the complexification $\mathrm{Ker}^{\mathbb{R}}(A - \lambda I) \oplus i\mathrm{Ker}^{\mathbb{R}}(A - \lambda I)$ of the real eigenspace. If we choose a real basis $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ of $\mathrm{Ker}^{\mathbb{R}}(A - \lambda I)$, then

$$A\vec{v}_j = \lambda\vec{v}_j.$$

Suppose $\lambda \in \mathbb{C} - \mathbb{R}$. By Exercise 7.19, we have a pair of complex eigenspaces $H = \mathrm{Ker}^{\mathbb{C}}(A - \lambda I)$ and $\bar{H} = \mathrm{Ker}^{\mathbb{C}}(A - \bar{\lambda} I)$ in $\mathbb{C}^n$. By Proposition 7.1.4, the sum $H + \bar{H}$ is direct. Let $\alpha = \{\vec{u}_1 - i\vec{w}_1, \vec{u}_2 - i\vec{w}_2, \dots, \vec{u}_m - i\vec{w}_m\}$, $\vec{u}_j, \vec{w}_j \in \mathbb{R}^n$, be a basis of H. Then $\bar{\alpha} = \{\vec{u}_1 + i\vec{w}_1, \vec{u}_2 + i\vec{w}_2, \dots, \vec{u}_m + i\vec{w}_m\}$ is a basis of $\bar{H}$. By the direct sum $H \oplus \bar{H}$, the union $\alpha \cup \bar{\alpha}$ is a basis of $H \oplus \bar{H}$. The set $\beta = \{\vec{u}_1, \vec{w}_1, \vec{u}_2, \vec{w}_2, \dots, \vec{u}_m, \vec{w}_m\}$ of real vectors and the set $\alpha \cup \bar{\alpha}$ can be expressed as each other's complex linear combinations. Since the two sets have the same number of vectors, the real set β is also a complex basis of $H \oplus \bar{H}$. In particular, β is a linearly independent set (over real or over complex, see Exercise 6.12). Therefore β is the basis of a real subspace $\mathrm{Span}^{\mathbb{R}}\beta \subset \mathbb{R}^n$.

The vectors in β appear in pairs $\vec{u}_j, \vec{w}_j$. If $\lambda = \mu + i\nu$, $\mu, \nu \in \mathbb{R}$, then $A(\vec{u}_j - i\vec{w}_j) = \lambda(\vec{u}_j - i\vec{w}_j)$ means

$$\begin{aligned} A\vec{u}_j &= \mu\vec{u}_j + \nu\vec{w}_j, \\ A\vec{w}_j &= -\nu\vec{u}_j + \mu\vec{w}_j. \end{aligned}$$

This means that the restriction of A to the 2-dimensional subspace $\mathbb{R}\vec{u}_j \oplus \mathbb{R}\vec{w}_j$ has the matrix

$$[A|_{\mathbb{R}\vec{u}_j \oplus \mathbb{R}\vec{w}_j}] = \begin{pmatrix} \mu & -\nu \\ \nu & \mu \end{pmatrix} = r\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \lambda = re^{\theta}.$$

In other words, the restriction of the linear transformation is the rotation by θ and scalar multiplication by r. We remark that the rotation does not mean there is an inner product. In fact, the concepts of eigenvalue, eigenspace and eigenvector do not require inner product. The rotation only means that, if we "pretend" $\{\vec{u}_j, \vec{w}_j\}$ to be an orthonormal basis (which may not be the case with respect to the usual dot product, see Example 7.1.18), then the linear transformation is the rotation by θ and scalar multiplication by r.

The pairs in β give a pair of real subspaces that form $\mathrm{Span}^{\mathbb{R}}\beta = E \oplus E^{\dagger}$

$$\begin{aligned} E &= \mathbb{R}\vec{u}_1 + \mathbb{R}\vec{u}_2 + \cdots + \mathbb{R}\vec{u}_m, \\ E^{\dagger} &= \mathbb{R}\vec{w}_1 + \mathbb{R}\vec{w}_2 + \cdots + \vec{w}_m. \end{aligned}$$

Moreover, we have an isomorphism $\dagger\colon E \cong E^{\dagger}$ given by $\vec{u}_j^{\dagger} = \vec{w}_j$. Then we have

$$A|_{E\oplus E^{\dagger}} = \begin{pmatrix} \mu I & -\nu I \\ \nu I & \mu I \end{pmatrix}.$$

The discussion extends to complex eigenvalue of real linear operator on complex vector space with conjugation. See Section 6.1.5.

In Example 7.1.13, we have

$$\lambda = 1 + i = \sqrt{2}e^{i\frac{\pi}{4}}, \quad \vec{u} - i\vec{w} = (1, -i),$$

and

$$E = \mathbb{R}(1,0), \quad E^{\dagger} = \mathbb{R}(0,1), \quad (1,0)^{\dagger} = (0,1).$$

Proposition 7.1.11. *Suppose a real square matrix A is complex diagonalisable, with complex basis of eigenvectors*

$$\dots, \vec{v}_j, \dots, \vec{u}_k - i\vec{u}_k^{\dagger}, \vec{u}_k + i\vec{u}_k^{\dagger}, \dots,$$

and corresponding eigenvalues

$$\dots, d_j, \dots, a_k + ib_k, a_k - ib_k, \dots.$$

Then

$$A = PDP^{-1}, \quad P = (\cdots\ \vec{v}_j\ \cdots\ \vec{u}_k\ \vec{u}_k^\dagger\ \cdots), \quad D = \begin{pmatrix} \ddots & & & & & \\ & d_j & & & O & \\ & & \ddots & & & \\ & & & a_k & -b_k & \\ & O & & b_k & a_k & \\ & & & & & \ddots \end{pmatrix}.$$

In more conceptual language, which is applicable to a complex diagonalisable real linear operator L on a real vector space V, the result is

$$L = \lambda_1 I \oplus \lambda_2 I \oplus \cdots \oplus \lambda_p I \oplus \begin{pmatrix} \mu_1 I & -\nu_1 I \\ \nu_1 I & \mu_1 I \end{pmatrix} \oplus \begin{pmatrix} \mu_2 I & -\nu_2 I \\ \nu_2 I & \mu_2 I \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} \mu_q I & -\nu_q I \\ \nu_q I & \mu_q I \end{pmatrix},$$

with respect to direct sum decomposition

$$V = H_1 \oplus H_2 \oplus \cdots \oplus H_p \oplus (E_1 \oplus E_1) \oplus (E_2 \oplus E_2) \oplus \cdots \oplus (E_q \oplus E_q).$$

Here $E \oplus E$ actually means the direct sum $E \oplus E^\dagger$ of two different subspaces, together with an isomorphism $\dagger$ that identifies the two subspaces.

Example 7.1.18. Consider

$$A = \begin{pmatrix} 1 & -2 & 4 \\ 2 & 2 & -1 \\ 0 & 3 & 0 \end{pmatrix}, \quad \det(tI - A) = (t-3)(t^2+9).$$

We find $\mathrm{Ker}(A - 3I) = \mathbb{R}(1,1,1)$. For the conjugate pair of eigenvalues $3i, -3i$, we have

$$A - 3iI = \begin{pmatrix} 1-3i & -2 & 4 \\ 2 & 2-3i & -1 \\ 0 & 3 & -3i \end{pmatrix}.$$

For complex vector $\vec{x} \in \mathrm{Ker}(A - 3iI)$, the last equation tells us $x_2 = ix_3$. Substituting into the second equation, we get

$$0 = 2x_1 + (2-3i)x_2 - x_3 = 2x_1 + (2-3i)ix_3 - x_3 = 2x_1 + (2+2i)x_3.$$

Therefore $x_1 = -(1+i)x_3$, and we get $\mathrm{Ker}(A - 3iI) = \mathbb{C}(-1-i, i, 1)$. This gives $\vec{u} = (-1, 0, 1)$, $\vec{u}^\dagger = (1, -1, 0)$, and

$$\begin{aligned} A &= \begin{pmatrix} 1 & -1-i & -1+i \\ 1 & i & -i \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3i & 0 \\ 0 & 0 & -3i \end{pmatrix} \begin{pmatrix} 1 & -1-i & -1+i \\ 1 & i & -i \\ 1 & 1 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}^{-1}. \end{aligned}$$

Geometrically, the operator fixes $(1,1,1)$, "rotate" $\{(-1,0,1),(1,-1,0)\}$ by 90°, and then multiply the whole space by 3.

Example 7.1.19. Consider the linear operator L on $\mathbb{R}^3$ that flips $\vec{u}=(1,1,1)$ to its negative and rotate the plane $H=(\mathbb{R}\vec{u})^\perp=\{x+y+z=0\}$ by 90°. Here the rotation of H is compatible with the normal direction $\vec{u}$ of H by the right hand rule.

In Example 4.2.8, we obtained an orthogonal basis $\vec{v}=(1,-1,0), \vec{w}=(1,1,-2)$ of H. By

$$\det(\vec{v}\;\vec{w}\;\vec{u})=\det\begin{pmatrix}1 & 1 & 1\\ -1 & 1 & 1\\ 0 & -2 & 1\end{pmatrix}=6>0,$$

the rotation from $\vec{v}$ to $\vec{w}$ is compatible with $\vec{v}$, and we have

$$L\left(\tfrac{\vec{v}}{\|\vec{v}\|}\right)=\tfrac{\vec{w}}{\|\vec{w}\|},\quad L\left(\tfrac{\vec{w}}{\|\vec{w}\|}\right)=-\tfrac{\vec{v}}{\|\vec{v}\|},\quad L(\vec{u})=-\vec{u}.$$

This means

$$L(\sqrt{3}\vec{v})=\vec{w},\quad L(\vec{w})=-\sqrt{3}\vec{v},\quad L(\vec{u})=-\vec{u}.$$

Therefore the matrix of L is (taking $P=(\sqrt{3}\vec{v}\;\vec{w}\;\vec{u})$)

$$\begin{aligned}&\begin{pmatrix}\sqrt{3} & 1 & 1\\ -\sqrt{3} & 1 & 1\\ 0 & -2 & 1\end{pmatrix}\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}\sqrt{3} & 1 & 1\\ -\sqrt{3} & 1 & 1\\ 0 & -2 & 1\end{pmatrix}^{-1}\\ &=\frac{1}{3}\begin{pmatrix}-1 & \sqrt{3}-1 & \sqrt{3}-1\\ \sqrt{3}-1 & -1 & -\sqrt{3}-1\\ -\sqrt{3}-1 & \sqrt{3}-1 & 1\end{pmatrix}.\end{aligned}$$

From the meaning of the linear operator, we also know that the 4-th power of the matrix is the identity.

Exercise 7.51. Let $\vec{v}=(1,1,1,1)$, $\vec{w}=(1,1,-1,-1)$. Let $H=\mathbb{R}\vec{v}\oplus\mathbb{R}\vec{w}$. Find the matrix.

1. Rotate H by 45° in the direction of $\vec{v}$ to $\vec{w}$. Flip $H^\perp$.

2. Rotate H by 45° in the direction of $\vec{w}$ to $\vec{v}$. Flip $H^\perp$.

3. Rotate H by 45° in the direction of $\vec{v}$ to $\vec{w}$. Identity on $H^\perp$.

4. Rotate H by 45° in the direction of $\vec{v}$ to $\vec{w}$. Rotate $H^\perp$ by 90°. The orientation of $\mathbb{R}^4$ given by the two rotations is the positive orientation.

5. Orthogonal projection to H, then rotate H by 45° in the direction of $\vec{v}$ to $\vec{w}$.

7.2 Orthogonal Diagonalisation

7.2.1 Normal Operator

Let V be a complex vector space with (Hermitian) inner product $\langle\ ,\ \rangle$. The simplest case for a linear operator L on V is an orthogonal sum

$$L = \lambda_1 I \perp \lambda_2 I \perp \cdots \perp \lambda_k I,$$

with respect to

$$V = H_1 \perp H_2 \perp \cdots \perp H_k.$$

This means that L has an orthonormal basis of eigenvectors, or *orthogonally diagonalisable*.

By Exercise 4.60 and $(\lambda I)^* = \bar{\lambda} I$, the simplest case implies

$$L^* = \bar{\lambda}_1 I \perp \bar{\lambda}_2 I \perp \cdots \perp \bar{\lambda}_k I.$$

Then we get $L^*L = LL^*$.

Definition 7.2.1. A linear operator L on an inner product space is *normal* if $L^*L = LL^*$.

The discussion before the definition proves the necessity of the following result. The sufficiency will follow from the much more general Theorem 7.2.4.

Theorem 7.2.2. *A linear operator on a complex inner product space is orthogonally diagonalisable if and only if it is normal.*

The adjoint of a complex matrix A is $A^* = \bar{A}^T$ (with respect to the standard complex dot product). Therefore a normal matrix means $A^*A = AA^*$. The theorem says that A is normal if and only if $A = UDU^{-1} = UDU^*$ for a diagonal matrix D and a unitary matrix U. This is the *unitary diagonalisation* of A.

Exercise 7.52. Prove that $L_1 \perp L_2$ is orthogonally diagonalisable if and only if L_1 and L_2 are orthogonally diagonalisable.

Exercise 7.53. Prove the following are equivalent.

1. L is normal.
2. L^* is normal.
3. $\|L(\vec{v})\| = \|L^*(\vec{v})\|$ for all $\vec{v}$.
4. In the decomposition $L = L_1 + L_2$, $L_1^* = L_1$, $L_2^* = -L_2$, we have $L_1L_2 = L_2L_1$.

Exercise 7.54. Suppose L is a normal operator on V.

1. Use Exercise 4.62 to prove that $\mathrm{Ker}L = \mathrm{Ker}L^*$.
2. Prove that $\mathrm{Ker}(L - \lambda I) = \mathrm{Ker}(L^* - \bar{\lambda} I)$.
3. Prove that eigenspaces of L with distinct eigenvalues are orthogonal.

Exercise 7.55. Suppose L is a normal operator on V.

1. Use Exercises 6.24 and 7.54 to prove that $\mathrm{Ran}L = \mathrm{Ran}L^*$.
2. Prove that $V = \mathrm{Ker}L \perp \mathrm{Ran}L$.
3. Prove that $\mathrm{Ran}L^k = \mathrm{Ran}L$ and $\mathrm{Ker}L^k = \mathrm{Ker}L$.
4. Prove that $\mathrm{Ran}L^j L^{*k} = \mathrm{Ran}L$ and $\mathrm{Ker}L^{*k} L^j = \mathrm{Ker}L$.

Now we apply Theorem 7.2.2 to a real linear operator L of real inner product space V. The normal property means $L^*L = LL^*$, or the corresponding matrix A with respect to an orthonormal basis satisfies $A^TA = AA^T$. By applying the proposition to the natural extension of L to the complexification $V \oplus iV$, we get the diagonalisation as described in Proposition 7.1.11 and the subsequent remark. The new concern here is the orthogonality. The vectors $\vec{v}_j$ are real eigenvectors with real eigenvalues, and can be chosen to be orthonormal. The vector pairs $\vec{u}_k - i\vec{u}_k^\dagger$, $\vec{u}_k + i\vec{u}_k^\dagger$ are eigenvectors corresponding to a conjugate pair of non-real eigenvalues. Since the two conjugate eigenvalues are distinct, by the third part of Exercise 7.54, the two vectors are orthogonal with respect to the complex inner product. As argued earlier in Section 6.1.6, this means that we may also choose the vectors $\dots, \vec{u}_k, \vec{u}_k^\dagger, \dots$ to be orthonormal. Therefore we get

$$L = \lambda_1 I \perp \cdots \perp \lambda_p I \perp \begin{pmatrix} \mu_1 I & -\nu_1 I \\ \nu_1 I & \mu_1 I \end{pmatrix} \perp \cdots \perp \begin{pmatrix} \mu_q I & -\nu_q I \\ \nu_q I & \mu_q I \end{pmatrix},$$

with respect to

$$V = H_1 \perp \cdots \perp H_p \perp (E_1 \perp E_1) \perp \cdots \perp (E_q \perp E_q).$$

Here $E \perp E$ means the direct sum $E \perp E^\dagger$ of two orthogonal subspaces, together with an isometric isomorphism $\dagger$ that identifies the two subspaces.

7.2.2 Commutative $*$-Algebra

In Propositions 7.1.3 and 7.1.5, we already saw the simultaneous diagonalisation of several linear operators. If a linear operator L on an inner product space is normal, then we may consider the collection of polynomials of L and L^*

$$\mathbb{C}[L, L^*] = \{\sum a_{ij} L^i L^{*j} \colon i, j \text{ non-negative integers}\}.$$

By $L^*L = LL^*$, we have $(L^iL^{*j})(L^kL^{*l}) = L^{i+k}L^{*j+l}$. We also have $(L^iL^{*j})^* = L^jL^{*i}$. Therefore $\mathbb{C}[L, L^*]$ is an example of the following concept.

Definition 7.2.3. A collection $\mathcal{A}$ of linear operators on an inner product space is a *commutative $*$-algebra* if the following are satisfied.

1. Algebra: $L, K \in \mathcal{A}$ implies $aL + bK, LK \in \mathcal{A}$.
2. Closed under adjoint: $L \in \mathcal{A}$ implies $L^* \in \mathcal{A}$.
3. Commutative: $L, K \in \mathcal{A}$ implies $LK = KL$.

The definition allows us to consider more general situation. For example, if L, K are operators, such that L, L^*, K, K^* are mutually commutative. Then the collection $\mathbb{C}[L, L^*, K, K^*]$ of polynomials of L, L^*, K, K^* is also a commutative $*$-algebra. We may then get an orthonormal basis of eigenvectors for both L and K.

An eigenvector of $\mathcal{A}$ is the eigenvector of every linear operator in $\mathcal{A}$.

Theorem 7.2.4. *A commutative $*$-algebra of linear operators on a finite dimensional inner product space is orthogonally diagonalisable.*

The proof of the theorem is based on Proposition 7.1.5 and the following result. Both do not require normal operator.

Lemma 7.2.5. *If H is L-invariant, then $H^\perp$ is L^*-invariant.*

Proof. Let $\vec{v} \in H^\perp$. For any $\vec{h} \in H$, we have $L(\vec{h}) \in H$, and therefore

$$\langle \vec{h}, L^*(\vec{v}) \rangle = \langle L(\vec{h}), \vec{v} \rangle = 0.$$

Since this holds for all $\vec{h} \in H$, we get $L^*(\vec{v}) \in H^\perp$. □

Proof of Theorem 7.2.4. If every operator in $\mathcal{A}$ is a constant multiple on the whole V, then any basis is a basis of eigenvectors for $\mathcal{A}$. So we assume an operator $L \in \mathcal{A}$ is not a constant multiple. By the fundamental theorem of algebra (Theorem 6.1.1), the characteristic polynomial of L has a root and gives an eigenvalue λ. Since L is not a constant multiple, the corresponding eigenspace $H = \mathrm{Ker}(L - \lambda I)$ is neither zero nor V.

Since any $K \in \mathcal{A}$ satisfies $KL = LK$, by Proposition 7.1.5, H is K-invariant. Then by Lemma 7.2.5, $H^\perp$ is K^*-invariant. Since we also have $K^* \in \mathcal{A}$, the conclusion remains true if K is replaced by K^*. By $(K^*)^* = K$, we find that $H^\perp$ is also K-invariant.

Since both H and $H^\perp$ are K-invariant for every $K \in \mathcal{A}$, we may restrict $\mathcal{A}$ to the two subspaces and get two sets of operators $\mathcal{A}_H$ and $\mathcal{A}_{H^\perp}$ of the respective inner product spaces H and $H^{\perp}$. Moreover, both $\mathcal{A}_H$ and $\mathcal{A}_{H^{\perp}}$ are still commutative $*$-algebras.

Since H is neither zero nor V, both H and $H^{\perp}$ have strictly smaller dimension than V. By induction on the dimension of underlying space, both $\mathcal{A}_H$ and $\mathcal{A}_{H^{\perp}}$ are orthogonally diagonalisable. This implies that $\mathcal{A}$ is orthogonally diagonalisable (see Exercise 7.52).

It remains to justify the beginning of the induction, which is when $\dim V = 1$. Since every linear operator is a constant multiple in this case, the conclusion follows trivially. □

Theorem 7.2.4 can be greatly extended. The infinite dimensional version of the theorem is the theory of commutative C^*-algebra.

Exercise 7.56. Suppose H is an invariant subspace of a normal operator L. Use Exercise 7.18 and orthogonal diagonalisation to prove that both H and $H^{\perp}$ are invariant subspaces of L and L^*.

Exercise 7.57. Suppose H is an invariant subspace of a normal operator L. Suppose P is the orthogonal projection to H.

1. Prove that H is an invariant subspace of L if and only if $(I-P)LP = O$.
2. For $X = PL(I-P)$, prove that $\mathrm{tr}XX^* = 0$.
3. Prove that $H^{\perp}$ is an invariant subspace of a normal operator L.

This gives an alternative proof of Exercise 7.56 without using the diagonalisation.

7.2.3 Hermitian Operator

An operator L is *Hermitian* if $L = L^*$. This means

$$\langle L(\vec{v}), \vec{w}\rangle = \langle \vec{v}, L(\vec{w})\rangle \text{ for all } \vec{v}, \vec{w}.$$

Therefore we also say that L is *self-adjoint*. Hermitian operators are normal.

Proposition 7.2.6. *A linear operator on a complex inner product space is Hermitian if and only if it is orthogonally diagonalisable and all the eigenvalues are real.*

A matrix A is Hermitian if $A^* = A$, and we get $A = UDU^{-1} = UDU^*$ for a real diagonal matrix D and a unitary matrix U.

Proof. By Theorem 7.2.2, we only need to show that a normal operator is Hermitian if and only if all the eigenvalues are real. Let $L(\vec{v}) = \lambda\vec{v}$, $\vec{v} \neq \vec{0}$. Then

$$\begin{aligned}\langle L(\vec{v}), \vec{v}\rangle &= \langle \lambda\vec{v}, \vec{v}\rangle = \lambda\langle \vec{v}, \vec{v}\rangle,\\ \langle \vec{v}, L(\vec{v})\rangle &= \langle \vec{v}, \lambda\vec{v}\rangle = \bar{\lambda}\langle \vec{v}, \vec{v}\rangle.\end{aligned}$$

By $L = L^*$, the left are equal. Then by $\langle \vec{v}, \vec{v}\rangle \neq 0$, we get $\lambda = \bar{\lambda}$. □

Example 7.2.1. The matrix

$$A = \begin{pmatrix} 2 & 1+i \\ 1-i & 3 \end{pmatrix}$$

is Hermitian, with characteristic polynomial

$$\det \begin{pmatrix} t-2 & -1-i \\ -1+i & t-3 \end{pmatrix} = (t-2)(t-3) - (1^2+1^2) = t^2 - 5t + 4 = (t-1)(t-4).$$

By

$$A - I = \begin{pmatrix} 1 & 1+i \\ 1+i & 2 \end{pmatrix}, \quad A - 4I = \begin{pmatrix} -2 & 1+i \\ 1-i & -1 \end{pmatrix},$$

we get $\mathrm{Ker}(A-I) = \mathbb{C}(1+i,-1)$ and $\mathrm{Ker}(A-4I) = \mathbb{C}(1,1-i)$. By $\|(1+i,-1)\| = \sqrt{3} = \|(1,1-i)\|$, we get

$$\mathbb{C}^2 = \mathbb{C}(1+i,-1) \perp \mathbb{C}(1,1-i) = \mathbb{C}\tfrac{1}{\sqrt{3}}(1+i,-1) \perp \mathbb{C}\tfrac{1}{\sqrt{3}}(1,1-i), \quad A = 1 \perp 4,$$

and the unitary diagonalisation

$$\begin{pmatrix} 2 & 1+i \\ 1-i & 3 \end{pmatrix} = \begin{pmatrix} \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \end{pmatrix}^{-1}.$$

Example 7.2.2 (Fourier Series)**.** We extend Example 4.3.3 to the vector space V of complex valued smooth periodic functions $f(t)$ of period 2π, and we also extend the inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t)\overline{g(t)}dt.$$

By the calculation in Example 4.3.3, the derivative operator $D(f) = f'$ on V satisfies $D^* = -D$. This implies that the operator $L = iD$ is Hermitian.

As pointed out earlier, Theorems 7.2.2, 7.2.4 and Proposition 7.2.6 can be extended to infinite dimensional spaces (*Hilbert spaces*, to be more precise). This suggests that the operator $L = iD$ should have an orthogonal basis of eigenvectors with real eigenvalues. Indeed we have found in Example 7.1.10 that the eigenvalues of L are precisely integers n, and the eigenspace $\mathrm{Ker}(nI - L) = \mathbb{C}e^{int}$. Moreover (by Exercise 7.54, for example), the eigenspaces are always orthogonal. The following is a direct verification of the orthonormal property

$$\langle e^{imt}, e^{int} \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{imt} e^{-int} dt = \begin{cases} \frac{1}{2\pi i(m-n)} e^{i(m-n)t}\big|_0^{2\pi} = 0, & \text{if } m \neq n, \\ \frac{1}{2\pi} 2\pi = 1, & \text{if } m = n. \end{cases}$$

The diagonalisation means that any periodic function of period 2π should be expressed as

$$f(t) = \sum_{n \in \mathbb{Z}} c_n e^{int} = c_0 + \sum_{n=1}^{+\infty} (c_n e^{int} + c_{-n} e^{-int}) = a_0 + \sum_{n=1}^{+\infty} (a_n \cos nt + b_n \sin nt).$$

Here we use

$$e^{int} = \cos nt + i \sin nt, \quad a_n = c_n + c_{-n}, \quad b_n = c_n - c_{-n}.$$

We conclude that the Fourier series grows naturally out of the diagonalisation of the derivative operator. If we apply the same kind of thinking to the derivative operator on the second vector space in Example 4.3.3 and use the eigenvectors in Example 7.1.9, then we get the Fourier transformation.

Exercise 7.58. Prove that a Hermitian operator L satisfies

$$\begin{aligned}\langle L(\vec{u}), \vec{v}\rangle = \tfrac{1}{4}(&\langle L(\vec{u}+\vec{v}), \vec{u}+\vec{v}\rangle - \langle L(\vec{u}-\vec{v}), \vec{u}-\vec{v}\rangle \\ &- i\langle L(\vec{u}+i\vec{v}), \vec{u}+i\vec{v}\rangle - i\langle L(\vec{u}-i\vec{v}), \vec{u}-i\vec{v}\rangle).\end{aligned}$$

Exercise 7.59. Prove that L is Hermitian if and only if $\langle L(\vec{v}), \vec{v}\rangle = \langle \vec{v}, L(\vec{v})\rangle$ for all $\vec{v}$.

Exercise 7.60. Prove that L is Hermitian if and only if $\langle L(\vec{v}), \vec{v}\rangle$ is real for all $\vec{v}$.

Exercise 7.61. Prove that the determinant of a Hermitian operator is real.

Next we apply Proposition 7.2.6 to a real linear operator L of a real inner product space. The self-adjoint property means $L^* = L$, or the corresponding matrix with respect to an orthonormal basis is symmetric. Since all the eigenvalues are real, the complex eigenspaces are complexifications of real eigenspaces. The orthogonality between complex eigenspaces is then the same as the orthogonality between real eigenspaces. Therefore we conclude

$$L = \lambda_1 I \perp \lambda_2 I \perp \cdots \perp \lambda_p I, \quad \lambda_i \in \mathbb{R}.$$

Proposition 7.2.7. *A real matrix is symmetric if and only if it is (real) orthogonally diagonalisable. In other words, we have $A = UDU^{-1} = UDU^T$ for a real diagonal matrix D and an orthogonal matrix U.*

Example 7.2.3. Even without calculation, we know the symmetric matrix in Examples 7.1.2 and 7.1.12 has orthogonal diagonalisation. From the earlier calculation, we have orthogonal decomposition

$$\mathbb{R}^2 = \mathbb{R}\tfrac{1}{\sqrt{5}}(1,2) \perp \mathbb{R}\tfrac{1}{\sqrt{5}}(2,-1), \quad A = 5 \perp 15,$$

and the orthogonal diagonalisation

$$\begin{pmatrix} 13 & -4 \\ -4 & 7 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 15 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}^{-1}.$$

Example 7.2.4. The symmetric matrix in Example 7.1.14 has an orthogonal diagonalisation. The basis of eigenvectors in the earlier example is not orthogonal. We may apply the Gram-Schmidt process to get an orthogonal basis of $\mathrm{Ker}(A-5I)$

$$\begin{aligned}\vec{v}_1 &= (-1,2,0),\\ \vec{v}_2 &= (-1,0,1)-\tfrac{1+0+0}{1+4+0}(-1,2,0)=\tfrac{1}{5}(-4,-2,5).\end{aligned}$$

Together with the basis $\vec{v}_3=(2,1,2)$ of $\mathrm{Ker}(A+4I)$, we get an orthogonal basis of eigenvectors. By further dividing the length, we get the orthogonal diagonalisation

$$A=\begin{pmatrix}-\frac{1}{\sqrt{5}} & -\frac{4}{3\sqrt{5}} & \frac{2}{3}\\ \frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} & \frac{1}{3}\\ 0 & \frac{\sqrt{5}}{3} & \frac{2}{3}\end{pmatrix}\begin{pmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & -4\end{pmatrix}\begin{pmatrix}-\frac{1}{\sqrt{5}} & -\frac{4}{3\sqrt{5}} & \frac{2}{3}\\ \frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} & \frac{1}{3}\\ 0 & \frac{\sqrt{5}}{3} & \frac{2}{3}\end{pmatrix}^{-1}.$$

Example 7.2.5. The matrix

$$\begin{pmatrix}1 & \sqrt{2} & \pi\\ \sqrt{2} & 2 & \sin 1\\ \pi & \sin 1 & e\end{pmatrix}$$

is numerically too complicated to calculate the eigenvalues and eigenspaces. Yet we still know that the matrix is orthogonally diagonalisable.

Exercise 7.62. Find orthogonal diagonalisation.

1. $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$.
2. $\begin{pmatrix}a & b\\ b & a\end{pmatrix}$.
3. $\begin{pmatrix}-1 & 2 & 0\\ 2 & 0 & 2\\ 0 & 2 & 1\end{pmatrix}$.
4. $\begin{pmatrix}0 & 2 & 4\\ 2 & -3 & 2\\ 4 & 2 & 0\end{pmatrix}$.

Exercise 7.63. Suppose a 3×3 real symmetric matrix has eigenvalues 1, 2, 3. Suppose $(1,1,0)$ is an eigenvector with eigenvalue 1. Suppose $(1,-1,0)$ is an eigenvector with eigenvalue 2. Find the matrix.

Exercise 7.64. Suppose A and B are real symmetric matrices satisfying $\det(tI-A)=\det(tI-B)$. Prove that A and B are similar.

Exercise 7.65. What can you say about a real symmetric matrix A satisfying $A^3=O$. What about satisfying $A^2=I$?

Exercise 7.66 (Legendre Polynomial). The *Legendre polynomials* P_n are obtained by applying the Gram-Schmidt process to the polynomials $1,t,t^2,\dots$ with respect to the inner product $\langle f,g\rangle=\int_{-1}^{1} f(t)g(t)dt$. The following steps show that

$$P_n=\frac{1}{2^n n!}\frac{d^n}{dt^n}[(t^2-1)^n].$$

1. Prove $[(1-t^2)P_n']' + n(n+1)P_n = 0$.
2. Prove $\sum_{n=0}^{\infty} P_n x^n = \frac{1}{\sqrt{1-2tx+x^2}}$.
3. Prove Bonnet's recursion formula $(n+1)P_{n+1} = (2n+1)tP_n - nP_{n-1}$.
4. Prove that $\int_{-1}^{1} P_m P_n dt = \frac{2\delta_{m,n}}{2n+1}$. [eigenvectors of hermitian operator used]

....

An operator L is *skew-Hermitian* if $L = -L^*$. This means

$$\langle L(\vec{v}), \vec{w}\rangle = -\langle \vec{v}, L(\vec{w})\rangle \text{ for all } \vec{v}, \vec{w}.$$

Skew-Hermitian operators are normal.

Note that L is skew-Hermitian if and only if iL is Hermitian. By Proposition 7.2.6, therefore, L is skew-Hermitian if and only if it is orthogonally diagonalisable and all the eigenvalues are imaginary.

Now we apply to a real linear operator L of a real inner product space. The skew-self-adjoint property means $L^* = -L$, or the corresponding matrix with respect to an orthonormal basis is skew-symmetric. The eigenvalues of L are either 0 or a pair $\pm i\lambda$ with $\lambda > 0$. Therefore we get

$$L = O \perp \lambda_1 \begin{pmatrix} O & -I \\ I & O \end{pmatrix} \perp \lambda_2 \begin{pmatrix} O & -I \\ I & O \end{pmatrix} \perp \cdots \perp \lambda_q \begin{pmatrix} O & -I \\ I & O \end{pmatrix},$$

with respect to

$$V = H \perp (E_1 \perp E_1) \perp (E_2 \perp E_2) \perp \cdots \perp (E_q \perp E_q).$$

Exercise 7.67. For any linear operator L, prove that $L = L_1 + L_2$ for unique Hermitian L_1 and skew Hermitian L_2. Moreover, L is normal if and only if L_1 and L_2 commute. In fact, the algebras $\mathbb{C}[L, L^*]$ and $\mathbb{C}[L_1, L_2]$ are equal.

Exercise 7.68. What can you say about the determinant of a skew-Hermitian operator? What about a skew-symmetric real operator?

Exercise 7.69. Suppose A and B are real skew-symmetric matrices satisfying $\det(tI - A) = \det(tI - B)$. Prove that A and B are similar.

Exercise 7.70. What can you say about a real skew-symmetric matrix A satisfying $A^3 = O$. What about satisfying $A^2 = -I$?

7.2.4 Unitary Operator

An operator U is *unitary* if it is an isometric isomorphism. The isometry means $U^*U = I$. Then the isomorphism means $U^{-1} = U^*$. Therefore we have $U^*U = I = UU^*$, and unitary operators are normal.

Proposition 7.2.8. *A linear operator on a complex inner product space is unitary if and only if it is orthogonally diagonalisable and all the eigenvalues satisfy $|\lambda| = 1$.*

The equality $|\lambda| = 1$ follows directly from $\|U(\vec{v})\| = \|\vec{v}\|$. Conversely, it is easy to see that, if $|\lambda_i| = 1$, then $U = \lambda_1 I \perp \lambda_2 I \perp \cdots \perp \lambda_p I$ preserves length and is therefore unitary.

Now we may apply Proposition 7.2.8 to an orthogonal operator U of a real inner product space V. The real eigevalues are 1, -1, and complex eigenvalues appear as conjugate pairs $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$. Therefore we get

$$U = I \perp -I \perp R_{\theta_1} I \perp R_{\theta_2} I \perp \cdots \perp R_{\theta_q} I, \quad R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

with respect to

$$V = H_+ \perp H_- \perp (E_1 \perp E_1) \perp (E_2 \perp E_2) \perp \cdots \perp (E_q \perp E_q).$$

The decomposition implies the following understanding of orthogonal operators.

Proposition 7.2.9. *Any orthogonal operator on a real inner product space is the orthogonal sum of identity, reflection, and rotations (on planes).*

A real matrix A is orthogonal if and only if $A^T A = I = AA^T$. The proposition implies that this is equivalent to $A = UDU^{-1} = UDU^T$ for an orthogonal matrix U and

$$D = \begin{pmatrix} \ddots & & & & & & \\ & 1 & & & & O & \\ & & \ddots & & & & \\ & & & -1 & & & \\ & & & & \ddots & & \\ & & & & & \cos\theta & -\sin\theta & \\ & O & & & & \sin\theta & \cos\theta & \\ & & & & & & & \ddots \end{pmatrix}.$$

Note that an orthogonal operator has determinant $\det U = (-1)^{\dim H_-}$. Therefore U preserves orientation (i.e., $\det U = 1$) if and only if $\dim H_-$ is even. This means that the -1 entries in the diagonal can be grouped into pairs and form rotations by 180°

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{pmatrix}.$$

Therefore an orientation preserving orthogonal operator is an orthogonal sum of identities and rotations. For example, an orientation preserving orthogonal operator on $\mathbb{R}^3$ is always a rotation around a fixed axis.

Example 7.2.6. Let $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ be an orthogonal operator. If all eigenvalues of L are real, then L is one of the following.

1. $L = \pm I$, the identity or the antipodal.
2. L fixes a line and is antipodal on the plane orthogonal to the line.
3. L flips a line and is identity on the plane orthogonal to the line.

If L has non-real eigenvalue, then L is one of the following.

1. L fixes a line and is rotation on the plane orthogonal to the line.
2. L flips a line and is rotation on the plane orthogonal to the line.

Exercise 7.71. Describe all the orthogonal operators of $\mathbb{R}^2$.

Exercise 7.72. Describe all the orientation preserving orthogonal operators of $\mathbb{R}^4$.

Exercise 7.73. Suppose an orthogonal operator exchanges $(1,1,1,1)$ and $(1,1,-1,-1)$, and fixes $(1,-1,1,-1)$. What can the orthogonal operator be?

7.3 Canonical Form

Example 7.1.15 shows that there are non-diagonalisable linear operators. We still wish to understand the structure of general linear operators by decomposing into the direct sum of blocks of some standard shape. If the decomposition is unique in some way, then it is *canonical*. The canonical form can be used to answer the question whether two general square matrices are similar.

The most famous canonical form is the Jordan canonical form, for linear operators whose characteristic polynomial completely factorises. This always happens over complex numbers. Over general fields, we always have the rational canonical form.

7.3.1 Generalised Eigenspace

We study the structure of a linear operator L by thinking of the algebra $\mathbb{F}[L]$ of polynomials of L over the base field $\mathbb{F}$. We will take advantage of the division of polynomials and the consequences of division algorithm in Section 6.2.3.

Let $d(t)$ be the greatest common divisor of polynomials $f_1(t), f_2(t), \dots, f_k(t)$. Then we have polynomials $u_1(t), u_2(t), \dots, u_k(t)$, such that

$$d(t) = f_1(t)u_1(t) + f_2(t)u_2(t) + \cdots + f_k(t)u_k(t).$$

Applying the equality to a linear operator L, we get

$$d(L) = f_1(L)u_1(L) + f_2(L)u_2(L) + \cdots + f_k(L)u_k(L),$$

Therefore

$$f_1(L)(\vec{v}) = f_2(L)(\vec{v}) = \cdots = f_k(L)(\vec{v}) = \vec{0} \implies d(L)(\vec{v}) = \vec{0},$$

and

$$\operatorname{Ran} d(L) \subset \operatorname{Ran} f_1(L) + \operatorname{Ran} f_2(L) + \cdots + \operatorname{Ran} f_k(L).$$

In fact, since $f_i(t) = d(t)q_i(t)$ for some polynomial $q_i(t)$, we also have

$$\operatorname{Ran} f_i(L) = \operatorname{Ran} d(L)q_i(L) \subset \operatorname{Ran} d(L).$$

Therefore we conclude

$$\operatorname{Ran} d(L) = \operatorname{Ran} f_1(L) + \operatorname{Ran} f_2(L) + \cdots + \operatorname{Ran} f_k(L).$$

For the special case that the polynomials are coprime, we have $d(t) = 1$, $d(L) = I$, $\operatorname{Ran} d(L) = V$, and therefore the following.

Lemma 7.3.1. *Suppose L is a linear operator on V, and $f_1(t), f_2(t), \ldots, f_k(t)$ are coprime. Then*

$$V = \operatorname{Ran} f_1(L) + \operatorname{Ran} f_2(L) + \cdots + \operatorname{Ran} f_k(L).$$

Moreover, $f_1(L)(\vec{v}) = f_2(L)(\vec{v}) = \cdots = f_k(L)(\vec{v}) = \vec{0}$ implies $\vec{v} = \vec{0}$.

Recall that any monic polynomial is a unique product of monic irreducible polynomials

$$f(t) = p_1(t)^{n_1} p_2(t)^{n_2} \cdots p_k(t)^{n_k}, \quad p_1(t), p_2(t), \ldots, p_k(t) \text{ distinct.}$$

For example, by the Fundamental Theorem of Algebra (Theorem 6.1.1), the irreducible polynomials over $\mathbb{C}$ are $t - \lambda$. Therefore we have

$$f(t) = (t - \lambda_1)^{n_1}(t - \lambda_2)^{n_2} \cdots (t - \lambda_k)^{n_k}, \quad \lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{C} \text{ distinct.}$$

The irreducible polynomials over $\mathbb{R}$ are $t - \lambda$ and $t^2 + at + b$, with $a^2 < 4b$. Therefore we have

$$f(t) = (t-\lambda_1)^{n_1}(t-\lambda_2)^{n_2} \cdots (t-\lambda_k)^{n_k}(t^2+a_1t+b_1)^{m_1}(t^2+a_2t+b_2)^{m_2} \cdots (t^2+a_lt+b_l)^{m_l}.$$

The quadratic factor $t^2 + at + b$ has a conjugate pair of complex roots $\lambda = \mu + i\nu$, $\bar{\lambda} = \mu - i\nu$, $\mu, \nu \in \mathbb{R}$. We will use specialised irreducible polynomials over $\mathbb{C}$ and $\mathbb{R}$ only in the last moment. Therefore most part of our theory applies to any field.

Suppose L is a linear operator on a finite dimensional $\mathbb{F}$-vector space V. We consider the *annilator*

$$\text{Ann}L = \{g(t) \in \mathbb{F}[t] \colon g(L) = O\},$$

which is all the polynomials vanishing on L. The Cayley-Hamilton Theorem (Theorem 7.1.8) says such polynomials exist. Moreover, if $g_1(L) = O$ and $g_2(L) = O$, and $d(t)$ is the greatest common divisor of $g_1(t)$ and $g_2(t)$, then the discussion before Lemma 7.3.1 says $d(L) = O$. Therefore there is a unique monic polynomial $m(t)$ satisfying

$$\text{Ann}L = m(t)\mathbb{F}[t] = \{m(t)q(t) \colon q(t) \in \mathbb{F}[t]\}.$$

Definition 7.3.2. The *minimal polynomial* $m(t)$ of a linear operator L is the monic polynomial with the property that $g(L) = O$ if and only if $m(t)$ divides $g(t)$.

Suppose the characteristic polynomial of L is factorised into distinct irreducible polynomials

$$f(t) = \det(tI - L) = p_1(t)^{n_1} p_2(t)^{n_2} \cdots p_k(t)^{n_k}.$$

Then the minimal polynomial divides $f(t)$, and we have ($m_i \geq 1$ is proved in Exercise 7.75)

$$m(L) = p_1(t)^{m_1} p_2(t)^{m_2} \cdots p_k(t)^{m_k}, \quad 1 \leq m_i \leq n_i.$$

Example 7.3.1. The characteristic polynomial in the matrix in Example 7.1.15 is $(t-4)^2(t+2)$. The minimal polynomial is either $(t-4)(t+2)$ or $(t-4)^2(t+2)$. By

$$(A - 4I)(A + 2I) = \begin{pmatrix} -1 & 1 & -3 \\ -1 & 1 & -3 \\ -6 & 6 & -6 \end{pmatrix} \begin{pmatrix} 5 & 1 & -3 \\ -1 & 7 & -3 \\ -6 & 6 & 0 \end{pmatrix} = \begin{pmatrix} 12 & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \neq O,$$

the polynomial $(t-4)(t+2)$ is not minimal. The minimal polynomial is $(t-4)^2(t+2)$.

In fact, Example 7.1.15 shows that A is not diagonalisable. Then we may also use the subsequent Exercise 7.74 to see that the minimal polynomial cannot be $(t-4)(t+2)$.

Proposition 7.3.3. *Suppose L is a linear operator, and $f(t) = p_1(t)^{n_1} p_2(t)^{n_2} \cdots p_k(t)^{n_k}$ for distinct irreducible polynomials $p_1(t), p_2(t), \ldots, p_k(t)$. If $f(L) = O$, then*

$$V = \text{Ker}\, p_1(L)^{n_1} \oplus \text{Ker}\, p_2(L)^{n_2} \oplus \cdots \oplus \text{Ker}\, p_k(L)^{n_k}.$$

For the case $\mathbb{F} = \mathbb{C}$ is the complex numbers, we have $p_i(L) = L - \lambda_i I$. The kernel $\text{Ker}(L - \lambda_i I)^{n_i}$ contains the eigenspace, and is called a *generalised eigenspace*. Therefore the proposition generalises Proposition 7.1.4.

Proof. Let

$$f_i(t) = \frac{f(t)}{p_i(t)^{n_i}} = p_1(t)^{n_1} \cdots p_{i-1}(t)^{n_{i-1}} p_{i+1}(t)^{n_{i+1}} \cdots p_k(t)^{n_k} = \prod_{l \neq i} p_l(t)^{n_l}.$$

Then $f_1(t), f_2(t), \dots, f_k(t)$ are coprime. By Lemma 7.3.1, we have

$$V = \text{Ran} f_1(L) + \text{Ran} f_2(L) + \cdots + \text{Ran} f_k(L).$$

By $p_i(L)^{n_i} f_i(L)(\vec{v}) = f(L) = O$, we have $\text{Ran} f_i(L) \subset \text{Ker}\, p_i(L)^{n_i}$. Therefore

$$V = \text{Ker}\, p_1(L)^{n_1} + \text{Ker}\, p_2(L)^{n_2} + \cdots + \text{Ker}\, p_k(L)^{n_k}.$$

To show the sum is direct, we consider

$$\vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_k = \vec{0}, \quad \vec{v}_i \in \text{Ker}\, p_i(L)^{n_i}.$$

We will prove that $f_j(L)(\vec{v}_i) = \vec{0}$ for all i, j. Then by Lemma 7.3.1 and the fact that $f_1(t), f_2(t), \dots, f_k(t)$ are coprime, we get all $\vec{v}_i = \vec{0}$.

For $i \neq j$, let

$$f_{ij}(t) = \frac{f(t)}{p_i(t)^{n_i} p_j(t)^{n_j}} = \frac{f_i(t)}{p_j(t)^{n_j}} = \prod_{l \neq i,j} p_l(t)^{n_l}.$$

Then

$$f_j(L)(\vec{v}_i) = f_{ij}(L) p_i(L)^{n_i}(\vec{v}_i) = \vec{0}.$$

This proves $f_j(L)(\vec{v}_i) = \vec{0}$ for all $i \neq j$. Applying $f_i(L)$ to $\vec{0} = \vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_k$, we get

$$\vec{0} = f_i(L)(\vec{v}_1) + f_i(L)(\vec{v}_2) + \cdots + f_i(L)(\vec{v}_k) = f_i(L)(\vec{v}_i).$$

This proves that we also have $f_j(L)(\vec{v}_i) = \vec{0}$ for $i = j$. Therefore we indeed have $f_j(L)(\vec{v}_i) = \vec{0}$ for all i, j. $\square$

Example 7.3.2. For the matrix in Example 7.1.15, we have $\text{Ker}(A + 2I) = \mathbb{R}(1, 1, 2)$ and

$$(A - 4I)^2 = \begin{pmatrix} 1 & -1 & 3 \\ 1 & -1 & 3 \\ 6 & -6 & 6 \end{pmatrix}^2 = 18 \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{pmatrix},$$
$$\text{Ker}(A - 4I)^2 = \mathbb{R}(1, 1, 0) \oplus \mathbb{R}(1, 0, -1).$$

Then we have the direct sum decomposition

$$\mathbb{R}^3 = \text{Ker}(A - 4I)^2 \oplus \text{Ker}(A + 2I) = (\mathbb{R}(1, 1, 0) \oplus \mathbb{R}(1, 0, -1)) \oplus \mathbb{R}(1, 1, 2).$$

Exercise 7.74. Prove that a linear operator is diagonalisable if and only if the minimal polynomial completely factorises and has no repeated root: $m(t) = (t-\lambda_1)(t-\lambda_2)\cdots(t-\lambda_k)$, with $\lambda_1, \lambda_2, \dots, \lambda_k$ distinct.

Exercise 7.75. Suppose $\det(tI - L) = p_1(t)^{n_1} p_2(t)^{n_2} \cdots p_k(t)^{n_k}$, where $p_1(t)$, $p_2(t)$, $\dots$, $p_k(t)$ are distinct irreducible polynomials. Suppose $m(L) = p_1(t)^{m_1} p_2(t)^{m_2} \cdots p_k(t)^{m_k}$ is the minimal polynomial of L.

1. Prove that $\operatorname{Ker} p_i(t)^{n_i} \neq \{\vec{0}\}$. *Hint*: First use Exercise 7.15 to prove the case of cyclic subspace. Then induct.
2. Prove that $m_i > 0$.
3. Prove that eigenvalues are exactly roots of the minimal polynomial. In other words, the minimal polynomial and the characteristic polynomial have the same roots.

Exercise 7.76. Suppose $p_1(t)^{m_1} p_2(t)^{m_2} \cdots p_k(t)^{m_k}$ is the minimal polynomial of L. Prove that $\operatorname{Ker} p_i(t)^{m_i+1} = \operatorname{Ker} p_i(t)^{m_i} \supsetneq \operatorname{Ker} p_i(t)^{m_i-1}$.

Exercise 7.77. Prove that the minimal polynomial of the linear operator on the cyclic subspace in Example 7.1.5 is $t^k + a_{k-1}t^{k-1} + a_{k-2}t^{k-2} + \cdots + a_1 t + a_0$.

Exercise 7.78. Find minimal polynomial. Determine whether the matrix is diagonalisable.

1. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.
2. $\begin{pmatrix} 3 & -1 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 2 \end{pmatrix}$.
3. $\begin{pmatrix} 3 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix}$.
4. $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 3 & 0 \end{pmatrix}$.

Exercise 7.79. Describe all 3×3 matrices satisfying $A^2 = I$. What about $A^3 = A$?

Exercise 7.80. Find the minimal polynomial of the derivative operator on P_n.

Exercise 7.81. Find the minimal polynomial of the transpose of $n \times n$ matrix.

7.3.2 Nilpotent Operator

By Proposition 7.1.5, the subspace $H = \operatorname{Ker} p_i(L)^{n_i}$ in Proposition 7.3.3 is L-invariant. Therefore it remains to study the restriction $L|_H$ of L to H. Since $p_i(L|_H)^{n_i} = O$, the operator $T = p_i(L|_H)$ has the following property.

Definition 7.3.4. A linear operator T is *nilpotent* if $T^n = O$ for some n.

We may regard applying a linear operator as "hitting" vectors. A nilpotent operator means that every vector is killed by hitting sufficiently many times.

Exercise 7.82. Prove that the only eigenvalue of a nilpotent operator is 0.

Exercise 7.83. Consider the matrix that shifts the coordinates by one position

$$A=\begin{pmatrix}0 & & & & O\\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ O & & & 1 & 0\end{pmatrix}:\begin{pmatrix}x_1\\ x_2\\ x_3\\ \vdots\\ x_n\end{pmatrix}\to\begin{pmatrix}0\\ x_1\\ x_2\\ \vdots\\ x_{n-1}\end{pmatrix}.$$

Show that $A^n=O$ and $A^{n-1}\neq O$.

Exercise 7.84. Show that any matrix of the following form is nilpotent

$$\begin{pmatrix}0 & 0 & \cdots & 0 & 0\\ * & 0 & \cdots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ * & * & \cdots & 0 & 0\\ * & * & \cdots & * & 0\end{pmatrix},\quad \begin{pmatrix}0 & * & \cdots & * & *\\ 0 & 0 & \cdots & * & *\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & *\\ 0 & 0 & \cdots & 0 & 0\end{pmatrix}.$$

Let T be a nilpotent operator on V. Let m be the smallest number such that $T^m=O$. This means that all vectors are killed after hitting m times. We may then ask more refined question on how vectors are killed:

1. What is the exact number i of hits needed to kill a vector $\vec{v}$? This means $T^i(\vec{v})=\vec{0}$ and $T^{i-1}(\vec{v})\neq\vec{0}$.

2. A vector may be the result of prior hits. This means that the vector is $T^j(\vec{v})$. If $\vec{v}$ needs exactly i hits to get killed, then $T^j(\vec{v})$ needs exactly $i-j$ hits to get killed.

The second question leads to the search for "fresh" vectors that have no history of prior hits, i.e., not of the form $T(\vec{v})$. This means that we find a direct summand F (for fresh) of $T(V)$

$$V=F\oplus T(V).$$

We expect vectors in $T(V)$ to be obtained by applying T repeatedly to the fresh vectors in F.

Now we ask the first question on F, which is the exact number of hits needed to kill vectors. We start with vectors that must be hit maximal number of times. In other words, they cannot be killed by $m-1$ hits. This means we take F_m to be a direct summand of the subspace $\mathrm{Ker}T^{m-1}$ of vectors that are killed by fewer than m hits

$$V=F_m\oplus\mathrm{Ker}T^{m-1}.$$

Next we try to find fresh vectors that are killed by exactly $m-1$ hits. This means we exclude the subspace $\mathrm{Ker}T^{m-2}$ of vectors that are killed by fewer than $m-1$

hits. We also should exclude $T(F_m)$, which are killed by exactly $m-1$ hits but are not fresh. Therefore we take F_{m-1} to be a direct summand

$$\text{Ker}T^{m-1} = F_{m-1} \oplus (T(F_m) + \text{Ker}T^{m-2}).$$

Continuing with the similar idea, we inductively take F_i to be a direct summand

$$\text{Ker}T^i = F_i \oplus (T(F_{i+1}) + T^2(F_{i+2}) + \cdots + T^{m-i}(F_m) + \text{Ker}T^{i-1}). \tag{7.3.1}$$

We claim that the sum (7.3.1) is actually direct

$$\text{Ker}T^i = F_i \oplus T(F_{i+1}) \oplus T^2(F_{i+2}) \oplus \cdots \oplus T^{m-i}(F_m) \oplus \text{Ker}T^{i-1}. \tag{7.3.2}$$

The first such statement (for $i = m$) is $V = F_m \oplus \text{Ker}T^{m-1}$, which is direct by our construction. We inductively assume the direct sum (7.3.2) for $\text{Ker}T^{i+1}$, and try to prove the similar sum (7.3.1) for $\text{Ker}T^i$ is also direct. Here we note that the induction starts from $i = m$ and goes down to $i = 0$. By Proposition 3.3.2, for the sum (7.3.1) to be direct, we only need to prove $T(F_{i+1}) + T^2(F_{i+2}) + \cdots + T^{m-i}(F_m) + \text{Ker}T^{i-1}$ is direct. By Exercise 3.56, this means that, if

$$T(\vec{v}_{i+1}) + T^2(\vec{v}_{i+2}) + \cdots + T^{m-i}(\vec{v}_m) \in \text{Ker}T^{i-1}, \quad \vec{v}_j \in F_j \text{ for } i < j \le m,$$

then all the terms in the sum $T^{j-i}(\vec{v}_j) = \vec{0}$. Note that the above is the same as

$$\vec{v}_{i+1} + T(\vec{v}_{i+2}) + \cdots + T^{m-i-1}(\vec{v}_m) \in \text{Ker}T^i.$$

By the inductively assumed direct sum (7.3.2) for $\text{Ker}T^{i+1}$, and Exercise 3.56, we get

$$\vec{v}_{i+1} = T(\vec{v}_{i+2}) = \cdots = T^{m-i-1}(\vec{v}_m) = \vec{0}.$$

This implies

$$T(\vec{v}_{i+1}) = T^2(\vec{v}_{i+2}) = \cdots = T^{m-i}(\vec{v}_m) = \vec{0}.$$

Combining all direct sums (7.3.2), and using Proposition 3.3.2, we get

$$\begin{aligned} V &= F_m \oplus \text{Ker}T^{m-1} \\ &= F_m \oplus F_{m-1} \oplus T(F_m) \oplus \text{Ker}T^{m-2} \\ &= F_m \oplus F_{m-1} \oplus T(F_m) \oplus F_{m-2} \oplus T(F_{m-1}) \oplus T^2(F_m) \oplus \text{Ker}T^{m-3} \\ &= \cdots \\ &= \oplus_{0 \le j < i \le m} T^j(F_i). \end{aligned}$$

We summarise the direct sum decomposition into the following.

$$\begin{array}{c|ccccccc} K_m & F_m & & & & & \\ K_{m-1} & F_{m-1} & T(F_m) & & & & \\ K_{m-2} & F_{m-2} & T(F_{m-1}) & T^2(F_m) & & & \\ \vdots & \vdots & \vdots & \vdots & & & \\ K_2 & F_2 & T(F_3) & T^2(F_4) & \cdots & T^{m-2}(F_m) & \\ K_1 & F_1 & T(F_2) & T^2(F_3) & \cdots & T^{m-2}(F_{m-1}) & T^{m-1}(F_m) \\ \hline V & F & T(F) & T^2(F) & \cdots & T^{m-2}(F) & T^{m-1}(F) \end{array}$$

We claim that all the diagonal subspaces are isomorphic. This means applying T repeatedly gives isomorphisms

$$F_i \cong T(F_i) \cong T^2(F_i) \cong \cdots \cong T^{i-1}(F_i).$$

Since the map $T^j|_{F_i}\colon F_i \to T^j(F_i)$ is automatically onto, we only need to show $\mathrm{Ker}(T^j|_{F_i}) = F_i \cap \mathrm{Ker}T^j = \{\vec{0}\}$ for $j < i$. This is a consequence of the direct sum $F_i \oplus \mathrm{Ker}T^{i-1}$ as part of (7.3.2), and $\mathrm{Ker}T^j \subset \mathrm{Ker}T^{i-1}$ for $j < i$.

The isomorphisms between diagonal subspaces imply that the direct sum of diagonal subspaces is a direct sum of copies of the leading fresh subspace

$$G_i = F_i \oplus T(F_i) \oplus T^2(F_i) \oplus \cdots \oplus T^{i-1}(F_i) \cong F_i \oplus F_i \oplus F_i \oplus \cdots \oplus F_i. \tag{7.3.3}$$

The subspace G_i is T-invariant, and the restriction $T|_{G_i}$ is simply the "right shift"

$$T(\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_i) = (\vec{0}, \vec{v}_1, \vec{v}_2, \dots, \vec{v}_{i-1}), \quad \vec{v}_j \in F_i.$$

In block form, we have $V = G_1 \oplus G_2 \oplus \cdots \oplus G_m$, and

$$T = T|_{G_1} \oplus T|_{G_2} \oplus \cdots \oplus T|_{G_m}, \quad T|_{G_i} = \begin{pmatrix} O & & & & O \\ I & O & & & \\ & I & \ddots & & \\ & & \ddots & O & \\ O & & & I & O \end{pmatrix}.$$

The block matrix is $i \times i$, and each entry in the block matrix is $\dim G_i \times \dim G_i$. However, it is possible to have some $G_i = \{\vec{0}\}$. In this case, the block $T|_{G_i}$ is not needed (or does not appear) in the decomposition for T.

Instead of taking the diagonal sum above for the decomposition $V = \oplus_{0 \le j < i \le m} T^j(F_i)$, we may also take the column sum or row sum. The first column adds up to become the subspace

$$F = F_m \oplus F_{m-1} \oplus \cdots \oplus F_1$$

of all fresh vectors. The sum of subspaces in the i-th column is then the subspace

$$T^{i-1}(F) = T^{i-1}(F_m) \oplus T^{i-1}(F_{m-1}) \oplus \cdots \oplus T^{i-1}(F_i)$$

obtained by hitting fresh vectors $i-1$ times, and we get

$$V = F \oplus T(F) \oplus T^2(F) \oplus \cdots \oplus T^{m-1}(F).$$

In other words, V consists of fresh vectors and the various hits of the fresh vectors.

If we add the subspaces in the k-th row, we get the subspace of vectors killed by exactly k hits

$$K_k = F_k \oplus T(F_{k+1}) \oplus T^2(F_{k+2}) \oplus \cdots \oplus T^{m-k}(F_m).$$

The whole space is then decomposed by counting the exact hits

$$V = K_m \oplus K_{m-1} \oplus K_{m-2} \oplus \cdots \oplus K_1.$$

In particular, we know $\mathrm{Ker}T^k$ consists of vectors killed by exactly i hits, for all $i \leq k$

$$\mathrm{Ker}T^k = K_k \oplus K_{k-1} \oplus \cdots \oplus K_1.$$

Example 7.3.3. Continuing the discussion of the matrix in Example 7.1.15. We have the nilpotent operator $T = A - 4I$ on $V = \mathrm{Ker}(A-4I)^2 = \mathbb{R}(1,1,0) \oplus \mathbb{R}(1,0,-1)$. In Example 7.3.2, we get $\mathrm{Ker}(A-4I) = \mathbb{R}(1,1,0)$ and the filtration

$$V = \mathrm{Ker}(A-4I)^2 = \mathbb{R}(1,1,0) \oplus \mathbb{R}(1,0,-1) \supset \mathrm{Ker}(A-4I) = \mathbb{R}(1,1,0) \supset \{\vec{0}\}.$$

We may choose the fresh $F_2 = \mathbb{R}(1,0,-1)$ between the two kernels. Then $T(F_2)$ is the span of

$$T(1,0,-1) = (A-4I)(1,0,-1) = (2,2,0).$$

This suggests us to revise the basis of V to

$$\mathrm{Ker}(A-4I)^2 = \mathbb{R}(1,0,-1) \oplus \mathbb{R}((A-4I)(1,0,-1)) = \mathbb{R}(1,0,-1) \oplus \mathbb{R}(2,2,0).$$

The matrices of $T|_{\mathrm{Ker}(A-4I)^2}$ and $A|_{\mathrm{Ker}(A-4I)^2}$ with respect to the basis are

$$[T|_{\mathrm{Ker}(A-4I)^2}] = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad [A|_{\mathrm{Ker}(A-4I)^2}] = [T] + 4I = \begin{pmatrix} 4 & 0 \\ 1 & 4 \end{pmatrix}.$$

We also have $\mathrm{Ker}(A+2I) = \mathbb{R}(1,1,2)$. With respect to the basis

$$\alpha = \{(1,0,-1), (2,2,0), (1,1,2)\},$$

we have

$$[A]_{\alpha\alpha} = \begin{pmatrix} [A|_{\mathrm{Ker}(A-4I)^2}] & O \\ O & [A|_{\mathrm{Ker}(A+2I)}] \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

and

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ -1 & 0 & 2 \end{pmatrix}^{-1}.$$

We remark that there is no need to introduce F_1 because $T(F_2)$ already fills up $\mathrm{Ker}(A-4I)$. If $\mathrm{Ker}(A-4I) \neq T(F_2)$, then we need to find the direct summand F_1 of $T(F_2)$ in $\mathrm{Ker}(A-4I)$.

Example 7.3.4. The derivative operator $D(f) = f' \colon P_n \to P_n$ satisfies $D^{n+1} = O$. In fact, we know $\mathrm{Ker}D^i = P_{i-1}$ is the subspace of polynomials of degree $< i$.

Note that F_{n+1} is the direct summand of $\mathrm{Ker}D^n = P_{n-1}$ in $\mathrm{Ker}D^{n+1} = P_n$. This means $F_{n+1} = \mathbb{R}f(t)$, for a polynomial $f(t)$ of degree n. For example, we may choose $f = \frac{t^n}{n!}$ (another good choice is $f = \frac{(t-a)^n}{n!}$). Then we have $D(f) = \frac{t^{n-1}}{(n-1)!}$, and

$$P_{n-1} = \mathbb{R}D(f) \oplus P_{n-2}, \text{ or } \mathrm{Ker}D^n = D(F_{n+1}) \oplus \mathrm{Ker}D^{n-1}.$$

Therefore $F_n = \{\vec{0}\}$. Next, we have $D^2(f) = \frac{t^{n-2}}{(n-2)!}$, and (note $D(F_n) = \{\vec{0}\}$)

$$P_{n-2} = \mathbb{R}D^2(f) \oplus P_{n-3}, \text{ or } \mathrm{Ker}D^{n-1} = D(F_n) \oplus D^2(F_{n+1}) \oplus \mathrm{Ker}D^{n-2}.$$

Therefore $F_{n-1} = \{\vec{0}\}$. In general, we have

$$F_{n+1} = \mathbb{R}\tfrac{t^n}{n!}, \quad D^{n-i}(F_{n+1}) = \mathbb{R}\tfrac{t^i}{i!}, \quad F_n = F_{n-1} = \cdots = F_1 = \{\vec{0}\},$$

and

$$P_n = F_{n+1} \oplus D(F_{n+1}) \oplus D^2(F_{n+1}) \oplus \cdots \oplus D^n(F_{n+1}).$$

The equality means that a polynomial of degree $\le n$ equals its n-th order Taylor expansion at 0. If we use $f = \frac{(t-a)^n}{n!}$, then the equality means that the polynomial equals its n-th order Taylor expansion at a.

If we use the basis $f, D(f), D^2(f), \dots, D^n(f)$, then the matrix of D with respect to the basis is

$$[D] = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

7.3.3 Jordan Canonical Form

In this section, we assume the characteristic polynomial $f(t) = \det(tI - L)$ completely factorises. This means $p_l(t) = t - \lambda_l$ in the earlier discussions, and the direct sum in Proposition 7.3.3 becomes

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_k, \quad V_l = \mathrm{Ker}(L - \lambda_l I)^{n_l}.$$

Correspondingly, we have

$$L = (T_1 + \lambda_1 I) \oplus (T_2 + \lambda_2 I) \oplus \cdots \oplus (T_k + \lambda_k I),$$

where $T_l = L|_{V_l} - \lambda_l I$ satisfies $T_l^{n_l} = O$ and is therefore a nilpotent operator on V_l.

Suppose m_l is the smallest number such that $T_l^{m_l} = O$. Then we have further direct sum decomposition

$$V_l = \oplus_{0 \le j < i \le m_l} T_l^j(F_{il}) \cong \oplus_{1 \le i \le m_l} F_{il}^{\oplus i}.$$

For each i, $F_{il}^{\oplus i}$ is the direct sum of i copies of F_{il}, and restriction of T_l is the "right shift". This means

$$L|_{V_l} = T_l + \lambda_l I = \oplus_{1\le i\le m_l} \begin{pmatrix} \lambda_l I & & & & O \\ I & \lambda_l I & & & \\ & I & \ddots & & \\ & & \ddots & \lambda_l I & \\ O & & & I & \lambda_l I \end{pmatrix}_{i\times i}.$$

The identity I in the block is on the subspace F_{il}.

To explicitly write down the matrix of the linear operator, we may take a basis α_{il} of the subspace $F_{il} \subset V_l \subset V$. Then the disjoint union

$$\alpha = \cup_{1\le l\le k} \cup_{0\le j<i\le m_l} T_l^j(\alpha_{il})$$

is a basis of V. The fresh part of the basis is α_0 below

$$\alpha_0 = \cup_{1\le l\le k} \cup_{1\le i\le m_l} \alpha_{il}, \quad \alpha = \alpha_0 \cup T(\alpha_0) \cup T^2(\alpha_0) \cup \cdots .$$

Each fresh basis vector $\vec{v} \in \alpha_0$ belongs to some α_{il}. The fresh basis vector generates an L-invariant subspace (actually T_l-cyclic subspace, see Example 7.1.5)

$$\mathbb{R}\vec{v} \oplus \mathbb{R}T_l(\vec{v}) \oplus \cdots \oplus \mathbb{R}T_l^{i-1}(\vec{v}).$$

The matrix of the restriction of L with respect to the basis is called a *Jordan block*

$$J = \begin{pmatrix} \lambda_l & & & & O \\ 1 & \lambda_l & & & \\ & 1 & \ddots & & \\ & & \ddots & \lambda_l & \\ O & & & 1 & \lambda_l \end{pmatrix}_{i\times i}.$$

The matrix of the whole L is a direct sum of the Jordan blocks, one for each fresh basis vector.

Theorem 7.3.5 (Jordan Canonical Form). *Suppose the characteristic polynomial of a linear operator completely factorises. Then there is a basis, such that the matrix of the linear operator is a direct sum of Jordan blocks.*

Exercise 7.85. Find the Jordan canonical form of the matrix in Example 7.78.

Exercise 7.86. In terms of Jordan canonical form, what is the condition for diagonalisability?

Exercise 7.87. Prove that for complex matrices, A and A^T are similar.

Exercise 7.88. Compute the powers of a Jordan block. Then compute the exponential of a Jordan block.

Exercise 7.89. Prove that the *geometric multiplicity* $\dim \operatorname{Ker}(L - \lambda_l I)$ is the number of Jordan blocks with eigenvalue λ_l.

Exercise 7.90. Prove that the minimal polynomial $m(t) = (t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2}\cdots(t-\lambda_k)^{m_k}$, where m_i is the smallest number such that $\operatorname{Ker}(L-\lambda_l I)^{m_l} = \operatorname{Ker}(L-\lambda_l I)^{m_l+1}$.

Exercise 7.91. By applying the complex Jordan canonical form, show that any real linear operator is a direct sum of following two kinds of real Jordan canonical forms

$$\begin{pmatrix} d & & & & O \\ 1 & d & & & \\ & 1 & \ddots & & \\ & & \ddots & d & \\ O & & & 1 & d \end{pmatrix}, \quad \begin{pmatrix} a & -b & & & & & & & & O \\ b & a & 0 & & & & & & & \\ 1 & 0 & a & -b & & & & & & \\ & 1 & b & a & \ddots & & & & & \\ & & 1 & 0 & \ddots & \ddots & & & & \\ & & & 1 & \ddots & \ddots & \ddots & & & \\ & & & & \ddots & \ddots & a & -b & & \\ & & & & & \ddots & b & a & 0 & \\ & & & & & & 1 & 0 & a & -b \\ O & & & & & & & 1 & b & a \end{pmatrix}.$$

7.3.4 Rational Canonical Form

In general, the characteristic polynomial $f(t) = \det(tI - L)$ may not completely factorise. We need to study the restriction of L to the invariant subspace $\operatorname{Ker} p_l(L)^{n_l}$.

We assume L is a linear operator on V, and

$$p(t) = t^p + a_{n-1}t^{n-1} + \cdots + a_2t^2 + a_1t + a_0 \tag{7.3.4}$$

is an (monic) irreducible polynomial of degree d, and $T = p(L)$ is nilpotent. Let m be the smallest number satisfying $T^m = p(L)^m = O$. In Section 7.3.2, we construct fresh subspaces $F_m, F_{m-1}, \dots, F_1$ inductively by (7.3.2), which we rewrite as

$$\operatorname{Ker}T^i = F_i \oplus W_i \oplus \operatorname{Ker}T^{i-1}, \tag{7.3.5}$$

with

$$W_i = T(F_{i+1}) \oplus T^2(F_{i+2}) \oplus \cdots \oplus T^{m-i}(F_m). \tag{7.3.6}$$

Moreover, for each i, we have direct sum and isomorphism (7.3.3).

For any polynomial $f(t)$, we divide by $p(t)$, and then divide the quotient by $p(t)$, and then repeat the process. We get

$$f(t) = r_0(t) + r_1(t)p(t) + r_2(t)p(t)^2 + \cdots, \quad \deg r_j(t) < \deg p(t) = d.$$

By $T = p(L)$ and $T^i = O$ on F_i, for any $\vec{v} \in F_i$, we have

$$f(L)(\vec{v}) = r_0(L)(\vec{v}) + T(r_1(L)(\vec{v})) + T^2(r_2(L)(\vec{v})) + \cdots + T^{i-1}(r_{i-1}(L)(\vec{v})).$$

For this to be compatible with the direct sum (7.3.3), we wish to have all $r_j(L)(\vec{v}) \in F_i$. Then by the isomorphisms in (7.3.3), we may regard

$$f(L)(\vec{v}) = (r_0(L)(\vec{v}), r_1(L)(\vec{v}), r_2(L)(\vec{v}), \dots, r_{i-1}(L)(\vec{v})) \in F_i^{\oplus i}.$$

By $\deg r_i(t) < d$, we need to consider

$$r(L)(\vec{v}) = a_0\vec{v} + a_1 L(\vec{v}) + a_2 L^2(\vec{v}) + \cdots + a_{d-1} L^{d-1}(\vec{v}).$$

Then our wish can be interpreted as finding a suitable "home" E_i for $\vec{v}$.

Lemma 7.3.6. *There are subspaces E_i that give direct sums*

$$\begin{aligned} F_i &= \{r(L)(\vec{v}) \colon \vec{v} \in E_i,\ \deg r(t) < d\} \\ &= E_i \oplus L(E_i) \oplus L^2(E_i) \oplus \cdots \oplus L^{d-1}(E_i), \end{aligned}$$

and they satisfy (7.3.5).

Recall F_i is inductively constructed as direct summand of $W_i \oplus \text{Ker}T^{i-1}$ in (7.3.5). The direct summand is obtained by picking vectors $\vec{v}_1, \vec{v}_2, \cdots$ between $W_i \oplus \text{Ker}T^{i-1}$ and $\text{Ker}T^i$ one by one, to make sure they are always linearly independent. Then these vectors span F_i. In fact, we should regard the picking to be 1-dimensional subspaces $\mathbb{R}\vec{v}_1, \mathbb{R}\vec{v}_2, \cdots$ that give a direct sum $F_i = \mathbb{R}\vec{v}_1 \oplus \mathbb{R}\vec{v}_2 \oplus \cdots$.

The refinement we wish to make in the lemma is the following. When we pick $\vec{v}_1$, we obtain not just one vector, but d vectors $\vec{v}_1, L(\vec{v}_1), L^2(\vec{v}_1), \dots, L^{d-1}(\vec{v}_1)$. Therefore instead of picking $\mathbb{R}\vec{v}_1$, we actually pick a subspace $C(\vec{v}_1)$, where

$$C(\vec{v}) = \mathbb{R}\vec{v} + \mathbb{R}L(\vec{v}) + \mathbb{R}L^2(\vec{v}) + \cdots + \mathbb{R}L^{d-1}(\vec{v}) = \{r(L)(\vec{v}) \colon \deg r(t) < d\}. \quad (7.3.7)$$

Then the next pick $\vec{v}_2$ is between $C(\vec{v}_1) \oplus W_i \oplus \text{Ker}T^{i-1}$ and $\text{Ker}T^i$, such that the subspace $C(\vec{v}_2)$ we obtain is independent of $C(\vec{v}_1)$. In other words, we need keep the sum $C(\vec{v}_1) + C(\vec{v}_2)$ to be direct. The process continues, and F_i is a direct sum of $C(\vec{v}_i)$ instead of $\mathbb{R}\vec{v}_i$ in the original construction.

Finally, we note that $E_i = \mathbb{R}\vec{v}_1 \oplus \mathbb{R}\vec{v}_2 \oplus \cdots$ in the lemma. Then the direct sum in the lemma means the sum (7.3.7) that defines $C(\vec{v})$ is direct.

Proof. We construct E_i inductively. First we assume $E_m, E_{m-1}, \dots, E_{i+1}$ have been constructed. Then we have the subspaces $F_m, F_{m-1}, \dots, F_{i+1}$ as given by the lemma, and the subspace W_i as given by (7.3.6). Next we pick $\vec{v}_1 \in \text{Ker}T^i - (W_i \oplus \text{Ker}T^{i-1})$, and ask whether $C(\vec{v}_1) + W_i \oplus \text{Ker}T^{i-1}$ is the whole $\text{Ker}T^i$. If the answer is no, then we pick $\vec{v}_2 \in \text{Ker}T^i - (C(\vec{v}_1) + W_i \oplus \text{Ker}T^{i-1})$, and ask whether $C(\vec{v}_1) + C(\vec{v}_2) + W_i \oplus \text{Ker}T^{i-1}$ is the whole $\text{Ker}T^i$. The process continues until we get

$$\text{Ker}T^i = C(\vec{v}_1) \oplus C(\vec{v}_2) \oplus \cdots \oplus C(\vec{v}_{l_i}) \oplus W_i \oplus \text{Ker}T^{i-1}. \tag{7.3.8}$$

We need to show the sum above is direct, and the sum in (7.3.7) is also direct.

We show the direct sum (7.3.9) by induction. Suppose we already have the direct sum

$$H = C(\vec{v}_1) \oplus C(\vec{v}_2) \oplus \cdots \oplus C(\vec{v}_{l-1}) \oplus W_i \oplus \text{Ker}T^{i-1}. \tag{7.3.9}$$

Then for the next pick $\vec{w} = \vec{v}_l \in \text{Ker}T^i - H$, we need to argue that $C(\vec{w}) + H$ is direct.

The key here is that H is L-invariant. By $TL = p(L)L = Lp(L) = LT$, we know $T^{i-1}(\vec{x}) = \vec{0} \implies T^{i-1}(L(\vec{x})) = L(T^{i-1}(\vec{x})) = L(\vec{0}) = \vec{0}$. This shows $\text{Ker}T^{i-1}$ is L-invariant. For W_i, by (7.3.6), we consider applying L to $T^{k-i}(F_k)$, $i < k \le m$. By the inductive assumption, we have

$$\begin{aligned}
F_k &= E_k + L(E_k) + L^2(E_k) + \cdots + L^{d-1}(E_k),\\
L(F_k) &= L(E_k) + L^2(E_k) + \cdots + L^{d-1}(E_k) + L^d(E_k) \subset F_k + L^d(E_k),\\
L(T^{k-i}(F_k)) &= T^{k-i}(L(F_k)) \subset T^{k-i}(F_k) + T^{k-i}(L^d(E_k)).
\end{aligned}$$

By $T^{k-i}(F_k) \subset W_i \subset H$, the problem $L(T^{k-i}(F_k)) \subset H$ is reduced to $T^{k-i}(L^d(E_k)) \subset H$. Since $p(t)$ is a monic polynomial of degree d, we have $p(t) = r(t) + t^d$ with $\deg r(t) < d$. Then $L^d = -r(L) + p(L) = -r(L) + T$, and for any $\vec{x} \in E_k$, we have

$$\begin{aligned}
L^d(\vec{x}) &= -r(L)(\vec{x}) + T(\vec{x}) \in F_k + T(E_k) \subset F_k + T(\text{Ker}T^k),\\
T^{k-i}(L^d(\vec{x})) &\in T^{k-i}(F_k) + T^{k-i+1}(\text{Ker}T^k) \subset W_i + \text{Ker}T^{i-1} \subset H.
\end{aligned}$$

For $C(\vec{v}_k)$, $1 \le k < l$, we have the similar argument

$$\begin{aligned}
L(C(\vec{v}_k)) &= \mathbb{R}L(\vec{v}_k) + \mathbb{R}L^2(\vec{v}_k) + \cdots + \mathbb{R}L^{d-1}(\vec{v}_k) + \mathbb{R}L^d(\vec{v}_k) \subset C(\vec{v}_k) + \mathbb{R}L^d(\vec{v}_k),\\
L^d(\vec{v}_k) &= -r(L)(\vec{v}_k) + T(\vec{v}_k) \in C(\vec{v}_k) + T(\text{Ker}T^i) \subset C(\vec{v}_k) + \text{Ker}T^{i-1} \subset H.
\end{aligned}$$

Next we argue the direct sum $C(\vec{w}) \oplus H$. By Exercise 3.56, this means

$$r(L)(\vec{w}) \in H,\ \deg r(t) < d \implies r(L)(\vec{w}) = \vec{0}.$$

If $r(t) \neq 0$, then by $\deg r(t) < \deg p(t)$ and $p(t)$ irreducible, we know $r(t)$ and $p(t)^m$ are coprime. Therefore we have $s(t)r(t) + q(t)p(t)^m = 1$ for some polynomials $s(t)$ and $q(t)$. Then by $p(L)^m = T^m = O$, we get

$$\vec{w} = s(L)r(L)(\vec{w}) + q(L)p(L)^m(\vec{w}) = s(L)r(L)(\vec{w}).$$

Since $r(L)(\vec{w}) \in H$, and H is L-invariant, we get $s(L)r(L)(\vec{w}) \in H$. This contradicts $\vec{w} \in \text{Ker}T^i - H$. Therefore $r(t) = 0$, and $r(L)(\vec{w}) = \vec{0}$. This proves the direct sum $C(\vec{w}) \oplus H$.

Next, we argue the sum in (7.3.7) is also direct. This means

$$\begin{aligned} r(L)(\vec{v}) = c_0\vec{v} + c_1L(\vec{v}) + c_2L^2(\vec{v}) + \cdots + c_{d-1}L^{d-1}(\vec{v}) = \vec{0} \\ \implies c_0\vec{v} = c_1L(\vec{v}) = c_2L^2(\vec{v}) = \cdots = c_{d-1}L^{d-1}(\vec{v}) = \vec{0}. \end{aligned}$$

Again we use $s(t)r(t) + q(t)p(t)^m = 1$ in the argument for $C(\vec{w}) \oplus H$ to get

$$\vec{v} = s(L)r(L)(\vec{v}) + q(L)T^m(\vec{w}) = s(L)\vec{0} + q(L)\vec{0} = \vec{0}.$$

This gives the implication we want.

Having argued that all sums are direct, we let

$$\begin{aligned} E_i &= \mathbb{R}\vec{v}_1 \oplus \mathbb{R}\vec{v}_2 \oplus \cdots \oplus \mathbb{R}\vec{v}_{l_i}, \\ F_i &= E_i \oplus L(E_i) \oplus L^2(E_i) \oplus \cdots \oplus L^{d-1}(E_i) = C(\vec{v}_1) \oplus C(\vec{v}_2) \oplus \cdots \oplus C(\vec{v}_{l_i}). \end{aligned}$$

Then (7.3.9) becomes (7.3.5). □

By Lemma 7.3.6, the whole space is a direct sum

$$V = \oplus_{0\le j<i\le m} T^j(F_i) = \oplus_{0\le j<i\le m,\, 0\le k<d} T^jL^k(E_i) \cong \oplus_{1\le i\le m} E_i^{\oplus di}.$$

The last isomorphism is given by $T^jL^k(E_i) = p(L)^jL^k(E_i) \cong E_i$ for all $0 \le j < i$ and $0 \le k < d$. In other words, any $\mathbb{R}\vec{v} \subset E_i$ gives di copies of $\mathbb{R}\vec{v}$ in the decomposition above

$$\begin{aligned} &\vec{v},\ L(\vec{v}),\ \ldots,\ L^{d-1}(\vec{v}), \\ &T(\vec{v}),\ TL(\vec{v}),\ \ldots,\ TL^{d-1}(\vec{v}), \\ &\ldots\ldots, \\ &T^{i-1}(\vec{v}),\ T^{i-1}L(\vec{v}),\ \ldots,\ T^{i-1}L^{d-1}(\vec{v}). \end{aligned}$$

The operator L shifts each row to the right. By $p(L) = O$ for the irreducible polynomial p in (7.3.4), the application of L to the last vector of each row is the following

$$\begin{aligned} L(T^jL^{d-1}(\vec{v})) &= T^jL^d(\vec{v}) = T^j(p(L)(\vec{v}) - a_0\vec{v} - a_1L(\vec{v}) - \cdots - a_{k-1}L^{d-1}(\vec{v})) \\ &= T^{j+1}(\vec{v}) - a_0T^j(\vec{v}) - a_1T^jL(\vec{v}) - \cdots - a_{d-1}T^jL^{d-1}(\vec{v}). \end{aligned}$$

Therefore for each i, we get the block form of L on $\oplus_{0\le j<i,\,0\le k<d} T^j L^k(E_i) \cong E_i^{\oplus di}$

$$L|_{E_i^{\oplus di}} = \begin{pmatrix} O & & & -a_0 I & & & & & & & & & & \\ I & O & & -a_1 I & & & & & & & & & & \\ & \ddots & \ddots & \vdots & & & & & & & & & & \\ & & I & -a_{k-1} I & & & & & & & & & & \\ & & & I & O & & & -a_0 I & & & & & & \\ & & & & I & O & & -a_1 I & & & & & & \\ & & & & & \ddots & \ddots & \vdots & & & & & & \\ & & & & & & I & -a_{k-1} I & & & & & & \\ & & & & & & & \ddots & \ddots & & & & & \\ & & & & & & & & \ddots & \ddots & & & & \\ & & & & & & & & & I & O & & & -a_0 I \\ & & & & & & & & & & I & O & & -a_1 I \\ & & & & & & & & & & & \ddots & \ddots & \vdots \\ & & & & & & & & & & & & I & -a_{k-1} I \end{pmatrix}.$$

The identity is over E_i, and the whole L has decomposition

$$L = L|_{E_m^{\oplus dm}} \oplus L|_{E_{m-1}^{\oplus d(m-1)}} \oplus \cdots \oplus L|_{E_1^{\oplus d}}.$$

Finally, the discussion so far is only about one irreducible factor in the minimal polynomial of a general linear operator. In general, the minimal polynomial of a linear operator L is $p_1(t)^{m_1} p_2(t)^{m_2} \cdots p_k(t)^{m_k}$ for distinct monic irreducible polynomials $p_1(t), p_2(t), \ldots, p_k(t)$. Then we have a direct sum

$$V = \operatorname{Ker} p_1(L)^{m_1} \oplus \operatorname{Ker} p_2(L)^{m_2} \oplus \cdots \oplus \operatorname{Ker} p_k(L)^{m_k},$$

and the corresponding decomposition

$$L = L|_{\operatorname{Ker} p_1(L)^{m_1}} \oplus L|_{\operatorname{Ker} p_2(L)^{m_2}} \oplus \cdots \oplus L|_{\operatorname{Ker} p_k(L)^{m_k}}.$$

Then $T_i = p_i(L)$ satisfies $T_i^{m_i} = O$ on $\operatorname{Ker} p_i(L)^{m_i}$, and we have

$$L|_{\operatorname{Ker} p_i(L)^{m_i}} = L|_{E_{im_i}^{\oplus d_i m_i}} \oplus L|_{E_{i(m_i-1)}^{\oplus d_i(m_i-1)}} \oplus \cdots \oplus L|_{E_{i1}^{\oplus d_i}}, \quad d_i = \deg p_i(t).$$

Each $L|_{E_{ij}^{\oplus d_i j}}$ is given by the block matrix above.

Chapter 8

Tensor

8.1 Bilinear

8.1.1 Bilinear Map

Let U, V, W be vector spaces. A map $B\colon V \times W \to U$ is *bilinear* if it is linear in U and linear in V

$$B(x_1\vec{v}_1 + x_2\vec{v}_2, \vec{w}) = x_1B(\vec{v}_1, \vec{w}) + x_2B(\vec{v}_2, \vec{w}),$$
$$B(\vec{v}, y_1\vec{w}_1 + y_2\vec{w}_2) = y_1B(\vec{v}, \vec{w}_1) + y_2B(\vec{v}, \vec{w}_2).$$

In case $U = \mathbb{F}$, the bilinear map is a *bilinear function* $b(\vec{x}, \vec{y})$.

The bilinear property extends to

$$B(x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m,\ y_1\vec{w}_1 + y_2\vec{w}_2 + \cdots + y_n\vec{w}_n) = \sum_{\substack{1\le i\le m\\ 1\le j\le n}} x_iy_jB(\vec{v}_i, \vec{w}_j).$$

If $\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ and $\beta = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n\}$ are bases of V and W, then the formula above shows that bilinear maps B are in one-to-one correspondence with the collection of values $B(\vec{v}_i, \vec{w}_j)$ on bases. These values form an $m \times n$ "matrix"

$$[B]_{\alpha\beta} = (B_{ij}), \quad B_{ij} = B(\vec{v}_i, \vec{w}_j).$$

In case $U = \mathbb{F}$, we denote $b_{ij} = b(\vec{v}_i, \vec{w}_j)$, and get

$$b(\vec{x}, \vec{y}) = \sum_{i,j} b_{ij}x_iy_j = [\vec{x}]_\alpha^T[b]_{\alpha\beta}[\vec{y}]_\beta.$$

We may define linear combination of bilinear maps $V \times W \to U$ is obvious way. This makes all the bilinear maps into a vector space $\mathrm{Bilinear}(V \times W, U)$. On the other hand, we may regard a bilinear map as a linear transformation

$$\vec{v} \mapsto B(\vec{v}, \cdot)\colon V \to \mathrm{Hom}(W, U).$$

We may also regard a bilinear map as another linear transformation

$$\vec{w} \mapsto B(\cdot, \vec{w}) \colon W \to \mathrm{Hom}(V, U).$$

The three viewpoints are equivalent. This means we have an isomorphism of vector spaces

$$\mathrm{Hom}(V, \mathrm{Hom}(W, U)) \cong \mathrm{Bilinear}(V \times W, U) \cong \mathrm{Hom}(W, \mathrm{Hom}(V, U)).$$

Example 8.1.1. In a real inner product space, the inner product is a bilinear function $\langle \cdot, \cdot \rangle \colon V \times V \to \mathbb{R}$. The complex inner product is not bilinear because it is conjugate linear in the second vector. The matrix of the inner product with respect to an orthonormal basis α is $[\langle \cdot, \cdot \rangle]_{\alpha\alpha} = I$.

Example 8.1.2. Recall the dual space $V^* = \mathrm{Hom}(V, \mathbb{F})$ of linear functionals on an $\mathbb{F}$-vector space V. The *evaluation pairing*

$$b(\vec{x}, l) = l(\vec{x}) \colon V \times V^* \to \mathbb{F}$$

is a bilinear function. The corresponding map for the second vector $V^* \to \mathrm{Hom}(V, \mathbb{F})$ is the identity. The corresponding map for the first vector $V \to \mathrm{Hom}(V^*, \mathbb{F}) = V^{**}$ is $\vec{v} \mapsto \vec{v}^{**}$, $\vec{v}^{**}(l) = l(\vec{v})$.

Example 8.1.3. The *cross product* in $\mathbb{R}^3$

$$(x_1, x_2, x_3) \times (y_1, y_2, y_3) = (x_2y_3 - x_3y_2,\ x_3y_1 - x_1y_3,\ x_1y_2 - x_2y_1)$$

is a bilinear map, with $\vec{e}_i \times \vec{e}_j) = \pm\vec{e}_k$ when i, j, k are distinct, and $\vec{e}_i \times \vec{e}_i = \vec{0}$. The sign $\pm$ is given by the orientation of the basis $\{\vec{e}_i, \vec{e}_j, \vec{e}_k\}$. The cross product also has the alternating property $\vec{x} \times \vec{y} = -\vec{y} \times \vec{x}$.

Example 8.1.4. Composiiton of linear transform. Product of matrix.

Exercise 8.1. For matching linear transformations L, show that the compositions $B(L(\vec{v}), \vec{w})$, $B(\vec{v}, L(\vec{w}))$, $L(B(\vec{v}, \vec{w}))$ are still linear transformations.

Exercise 8.2. Show that a bilinear map $B \colon V \times W \to U_1 \oplus U_2$ is given by two bilinear maps $B_1 \colon V \times W \to U_1$ and $B_2 \colon V \times W \to U_2$. What if V or W is a direct sum?

Exercise 8.3. Show that a map $B \colon V \times W \to \mathbb{F}^n$ is bilinear if and only if each coordinate $b_i \colon V \times W \to \mathbb{F}$ of B is a bilinear function.

Exercise 8.4. For a linear functional $l(x_1, x_2, x_3) = a_1x_1 + a_2x_2 + a_3x_3$, what is the bilinear function $l(\vec{x} \times \vec{y})$?

8.1.2 Bilinear Function

A bilinear function $b\colon V \times W \to \mathbb{F}$ is determined by its values $b_{ij} = b(\vec{v}_i, \vec{w}_j)$ on bases $\alpha = \{\vec{v}_1, \dots, \vec{v}_m\}$ and $\beta = \{\vec{w}_1, \dots, \vec{w}_n\}$ of V and W

$$b(\vec{x}, \vec{y}) = \sum_{i,j} b_{ij} x_i y_j = [\vec{x}]_\alpha^T B [\vec{y}]_\beta, \quad [\vec{x}]_\alpha = (x_1, \dots, x_m), \quad [\vec{y}]_\beta = (y_1, \dots, y_n).$$

Here the *matrix of bilinear function* is

$$B = [b]_{\alpha\beta} = (b(\vec{v}_i, \vec{w}_j))_{i,j=1}^{m,n}.$$

By $b(\vec{x}, \vec{y}) = [\vec{x}]_\alpha^T [b]_{\alpha\beta} [\vec{y}]_\beta$ and $[\vec{x}]_{\alpha'} = [I]_{\alpha'\alpha} [\vec{x}]_\alpha$, we get the change of matrix caused by the change of bases

$$[b]_{\alpha'\beta'} = [I]_{\alpha\alpha'}^T [b]_{\alpha\beta} [I]_{\beta\beta'}.$$

A bilinear function induces a linear transformation

$$L(\vec{v}) = b(\vec{v}, \cdot)\colon V \to W^*.$$

Conversely, a linear transformation $L\colon V \to W^*$ gives a bilinear function $b(\vec{v}, \vec{w}) = L(\vec{v})(\vec{w})$.

If W is a real inner product space, then we have the induced isomorphism $W^* \cong W$ by Proposition 4.3.1. Combined with the linear transformation above, we get a linear transformation, still denoted L

$$L\colon V \to W^* \cong W, \quad \vec{v} \mapsto b(\vec{v}, \cdot) = \langle L(\vec{v}), \cdot \rangle.$$

This means

$$b(\vec{v}, \vec{w}) = \langle L(\vec{v}), \vec{w} \rangle \quad \text{for all } \vec{v} \in V,\ \vec{w} \in W.$$

Therefore real bilinear functions $b\colon V \times W \to \mathbb{R}$ are in one-to-one correspondence with linear transformations $L\colon V \to W$.

Similarly, the bilinear function also corresponds to a linear transformation

$$L^*(\vec{w}) = b(\cdot, \vec{w})\colon W \to V^*.$$

The reason for the notation L^* is that the linear transformation is $W \cong W^{**} \xrightarrow{L^*} V^*$, or the dual linear transformation up to the natural double dual isomorphism of W. If we additionally know that V is a real inner product space, then we may combine $W \to V^*$ with the isomorphism $V^* \cong V$ by Proposition 4.3.1, to get the following linear transformation, still denoted L^*

$$L^*\colon W \to V^* \cong V, \quad \vec{w} \mapsto b(\cdot, \vec{w}) = \langle \cdot, L^*(\vec{w}) \rangle.$$

This means

$$b(\vec{v}, \vec{w}) = \langle \vec{v}, L^*(\vec{w}) \rangle \quad \text{for all } \vec{v} \in V,\ \vec{w} \in W.$$

If both V and W are real inner product spaces, then we have

$$b(\vec{v}, \vec{w}) = \langle L(\vec{v}), \vec{w} \rangle = \langle \vec{v}, L^*(\vec{w}) \rangle.$$

Exercise 8.5. What are the matrices of bilinear functions $b(L(\vec{v}),\vec{w})$, $b(\vec{v},L(\vec{w}))$, $l(B(\vec{v},\vec{w}))$?

Example 8.1.5. An inner product $\langle\cdot,\cdot\rangle\colon V\times V\to\mathbb{R}$ is a dual pairing. The induced isomorphism

$$V\cong V^*\colon \vec{a}\mapsto\langle\vec{a},\cdot\rangle$$

makes V into a *self dual* vector space. In a self dual vector space, it makes sense to ask for self dual basis. For inner product, a basis α is self dual if

$$\langle\vec{v}_i,\vec{v}_j\rangle=\delta_{ij}.$$

This means exactly that the basis is orthonormal.

Example 8.1.6. The *evaluation pairing* in Example 8.1.2 is a dual pairing. For a basis α of V, the dual basis with respect to the evaluation pairing is the dual basis α^* in Section 2.4.1.

Example 8.1.7. The function space is infinite dimensional. Its dual space needs to consider the topology, which means that the dual space consists of only *continuous* linear functionals. For the vector space of power p integrable functions

$$L^p[a,b]=\{f(t)\colon \int_a^b|f(t)|^pdt<\infty\},\quad p\ge 1,$$

the continuous linear functionals on $L^p[a,b]$ are of the form

$$l(f)=\int_a^b f(t)g(t)dt,$$

where the function $g(t)$ satisfies

$$\int_a^b|g(t)|^qdt<\infty,\quad \frac{1}{p}+\frac{1}{q}=1.$$

This shows that the dual space $L^p[a,b]^*$ of all continuous linear functionals on $L^p[a,b]$ is $L^q[a,b]$. In particular, the *Hilbert space* $L^2[a,b]$ of square integrable functions is self dual.

8.1.3 Quadratic Form

A *quadratic form* on a vector space V is $q(\vec{x})=b(\vec{x},\vec{x})$, where b is some bilinear function on $V\times V$. Since replacing b with the *symmetric* bilinear function $\frac{1}{2}(b(\vec{x},\vec{y})+b(\vec{y},\vec{x}))$ gives the same q, we will always take b to be symmetric

$$b(\vec{x},\vec{y})=b(\vec{y},\vec{x}).$$

Then (the symmetric) b can be recovered from q by *polarisation*

$$b(\vec{x},\vec{y}) = \frac{1}{4}(q(\vec{x}+\vec{y}) - q(\vec{x}-\vec{y})) = \frac{1}{2}(q(\vec{x}+\vec{y}) - q(\vec{x}) - q(\vec{y})).$$

The discussion assumes that 2 is invertible in the base field $\mathbb{F}$. If this is not the case, then there is a subtle difference between quadratic forms and *symmetric forms*. The subsequent discussion always assumes that 2 is invertible.

The matrix of q with respect to a basis $\alpha = \{\vec{v}_1, \dots, \vec{v}_n\}$ is the matrix of b with respect to the basis, and is symmetric

$$[q]_\alpha = [b]_{\alpha\alpha} = B = (b(\vec{v}_i, \vec{v}_j)).$$

Then the quadratic form can be expressed in terms of the α-coordinate $[\vec{x}]_\alpha = (x_1, \dots, x_n)$

$$q(\vec{x}) = [\vec{x}]_\alpha^T [q]_\alpha [\vec{x}]_\alpha = \sum_{1\le i\le n} b_{ii}x_i^2 + 2\sum_{1\le i<j\le n} b_{ij}x_ix_j.$$

For another basis β, we have

$$[q]_\beta = [I]_{\alpha\beta}^T [q]_\alpha [I]_{\alpha\beta} = P^T BP, \quad P = [I]_{\alpha\beta}.$$

In particular, the *rank* of a quadratic form is well defined

$$\operatorname{rank} q = \operatorname{rank}[q]_\alpha.$$

Exercise 8.6. Prove that a function is a quadratic form if and only if it is homogeneous of second order

$$q(c\vec{x}) = c^2 q(\vec{x}),$$

and satisfies the *parallelogram identity*

$$q(\vec{x}+\vec{y}) + q(\vec{x}-\vec{y}) = 2q(\vec{x}) + 2q(\vec{y}).$$

Similar to the diagonalisation of linear operators, we may ask about the *canonical forms* of quadratic forms. The goal is to eliminate the *cross terms* $b_{ij}x_ix_j$, $i \ne j$, by choosing a different basis. Then the quadratic form consists of only the square terms

$$q(\vec{x}) = b_1x_1^2 + \cdots + b_nx_n^2.$$

We may get the canonical form by the method of *completing the square*. The method is a version of Gaussian elimination, and can be applied to any base field $\mathbb{F}$ in which 2 is invertible.

In terms of matrix, this means that we want to express a symmetric matrix B as $P^T DP$ for diagonal D and invertible P.

Example 8.1.8. For $q(x,y,z) = x^2 + 13y^2 + 14z^2 + 6xy + 2xz + 18yz$, we gather together all the terms involving x and complete the square

$$\begin{aligned} q &= x^2 + 6xy + 2xz + 13y^2 + 14z^2 + 18yz \\ &= [x^2 + 2x(3y+z) + (3y+z)^2] + 13y^2 + 14z^2 + 18yz - (3y+z)^2 \\ &= (x+3y+z)^2 + 4y^2 + 13z^2 + 12yz. \end{aligned}$$

The remaining terms involve only y and z. Gathering all the terms involving y and completing the square, we get $4y^2 + 13z^2 + 12yz = (2y+3z)^2 + 4z^2$ and

$$q = (x+3y+z)^2 + (2y+3z)^2 + (2z)^2 = u^2 + v^2 + w^2.$$

In terms of matrix, the process gives

$$\begin{pmatrix} 1 & 3 & 1 \\ 3 & 13 & 9 \\ 1 & 9 & 14 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix}.$$

Geometrically, the original variables x, y, z are the coordinates with respect to the standard basis ϵ. The new variables u, v, w are the coordinates with respect to a new basis α. The two coordinates are related by

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} x+3y+z \\ 2y+3z \\ 2z \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad [I]_{\alpha\epsilon} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then the basis α is the columns of the matrix

$$[\alpha]_\epsilon = [I]_{\epsilon\alpha} = [I]_{\alpha\epsilon}^{-1} = \begin{pmatrix} 1 & -\frac{3}{2} & \frac{7}{4} \\ 0 & \frac{1}{2} & -\frac{3}{4} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Example 8.1.9. The cross terms in the quadratic form $q(x,y,z) = x^2 + 4y^2 + z^2 - 4xy - 8xz - 4yz$ can be eliminated as follows

$$\begin{aligned} q &= [x^2 - 2x(2y+4z) + (2y+4z)^2] + 4y^2 + z^2 - 4yz - (2y+4z)^2 \\ &= (x-2y-4z)^2 - 15z^2 - 20yz \\ &= (x-2y-4z)^2 - 15[z^2 + \tfrac{4}{3}yz + \left(\tfrac{2}{3}y\right)^2] + 15\left(\tfrac{2}{3}y\right)^2 \\ &= (x-2y-4z)^2 + \tfrac{20}{3}y^2 - 15(z + \tfrac{2}{3}y)^2. \end{aligned}$$

Since we do not have y^2 term after the first step, we use z^2 term to complete the square in the second step.

We also note that the division by 3 is used. If we cannot divide by 3, then this means $3 = 0$ in the field $\mathbb{F}$. This implies $-2 = 1$, $-4 = -4$, $-15 = 0$, $-20 = 1$ in $\mathbb{F}$, and we get

$$q = (x-2y-4z)^2 - 15z^2 - 20yz = (x+y-z)^2 + yz.$$

We may further eliminate the cross terms by introducing $y = u + v$, $z = u - v$, so that

$$q = (x + y - z)^2 + u^2 - v^2 = (x + y - z)^2 + \tfrac{1}{4}(y + z)^2 - \tfrac{1}{4}(y - z)^2.$$

Example 8.1.10. The quadratic form $q = xy + yz$ has no square term. We may eliminate the cross terms by introducing $x = u+v$, $y = u-v$, so that $q = u^2 - v^2 + uz - vz$. Then we complete the square and get

$$q = \left(u - \tfrac{1}{2}z\right)^2 - \left(v + \tfrac{1}{2}z\right)^2 = \tfrac{1}{4}(x + y - z)^2 - \tfrac{1}{4}(x - y + z)^2.$$

Example 8.1.11. The cross terms in the quadratic form

$$\begin{aligned} q = {} & 4x_1^2 + 19x_2^2 - 4x_4^2 \\ & - 4x_1x_2 + 4x_1x_3 - 8x_1x_4 + 10x_2x_3 + 16x_2x_4 + 12x_3x_4 \end{aligned}$$

can be eliminated as follows

$$\begin{aligned} q &= 4[x_1^2 - x_1x_2 + x_1x_3 - 2x_1x_4] + 19x_2^2 - 4x_4^2 + 10x_2x_3 + 16x_2x_4 + 12x_3x_4 \\ &= 4\left[x_1^2 + 2x_1\left(-\tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 - x_4\right) + \left(-\tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 - x_4\right)^2\right] \\ &\quad + 19x_2^2 - 4x_4^2 + 10x_2x_3 + 16x_2x_4 + 12x_3x_4 - 4\left(-\tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 - x_4\right)^2 \\ &= 4\left(x_1 - \tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 - x_4\right)^2 + 18\left[x_2^2 + \tfrac{2}{3}x_2x_3 + \tfrac{2}{3}x_2x_4\right] - x_3^2 - 8x_4^2 + 16x_3x_4 \\ &= (2x_1 - x_2 + x_3 - 2x_4)^2 + 18\left[x_2^2 + 2x_2\left(\tfrac{1}{3}x_3 + \tfrac{1}{3}x_4\right) + \left(\tfrac{1}{3}x_3 + \tfrac{1}{3}x_4\right)^2\right] \\ &\quad - x_3^2 - 8x_4^2 + 16x_3x_4 - 18\left(\tfrac{1}{3}x_3 + \tfrac{1}{3}x_4\right)^2 \\ &= (2x_1 - x_2 + x_3 - 2x_4)^2 + 18\left(x_2 + \tfrac{1}{3}x_3 + \tfrac{1}{3}x_4\right)^2 - 3(x_3^2 - 4x_3x_4) - 10x_4^2 \\ &= (2x_1 - x_2 + x_3 - 2x_4)^2 + 2(3x_2 + x_3 + x_4)^2 \\ &\quad - 3[x_3^2 + 2x_3(-2x_4) + (-2x_4)^2] - 10x_4^2 + 3(-2x_4)^2 \\ &= (2x_1 - x_2 + x_3 - 2x_4)^2 + 2(3x_2 + x_3 + x_4)^2 - 3(x_3 - 2x_4)^2 + 2x_4^2 \\ &= y_1^2 + 2y_2^2 - 3y_3^3 + 2y_4^2. \end{aligned}$$

The new variables y_1, y_2, y_3, y_4 are the coordinates with respect to the basis of the columns of

$$\begin{pmatrix} 2 & -1 & 1 & -2 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & -\frac{2}{3} & -\frac{1}{2} \\ 0 & \frac{1}{3} & -\frac{1}{3} & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Example 8.1.12. For the quadratic form $q(x, y, z) = x^2 + iy^2 + 3z^2 + 2(1+i)xy + 4yz$

over complex numbers $\mathbb{C}$, the following eliminates the cross terms

$$\begin{aligned}q &= [x^2+2(1+i)xy+((i+1)y)^2]+iy^2+3z^2+4yz-(i+1)^2y^2\\ &= (x+(1+i)y)^2-iy^2+3z^2+4yz\\ &= (x+(1+i)y)^2-i\left[y^2+4iyz+(2iz)^2\right]+3z^2+i(2i)^2z^2\\ &= (x+(1+i)y)^2-i(y+2iz)^2+(3-4i)z^2.\end{aligned}$$

We may further use (pick one of two possible complex square roots)

$$\sqrt{-i}=\sqrt{e^{-i\frac{\pi}{2}}}=e^{-i\frac{\pi}{4}}=\tfrac{1-i}{\sqrt{2}},\quad \sqrt{3-4i}=\sqrt{(2-i)^2}=2-i,$$

to get

$$q=(x+(1+i)y)^2+\left(\tfrac{1-i}{\sqrt{2}}y+\sqrt{2}(1+i)z\right)^2+((2-i)z)^2.$$

In terms of matrix, the process gives

$$\begin{pmatrix}1&1+i&0\\1+i&i&2\\0&2&3\end{pmatrix}=\begin{pmatrix}1&1+i&0\\0&\frac{1-i}{\sqrt{2}}&\sqrt{2}(1+i)\\0&0&2-i\end{pmatrix}^T\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&1+i&0\\0&\frac{1-i}{\sqrt{2}}&\sqrt{2}(1+i)\\0&0&2-i\end{pmatrix}.$$

The new variables $x+(1+i)y, \frac{1-i}{\sqrt{2}}y+\sqrt{2}(1+i)z, (2-i)z$ are the coordinates with respect to some new basis. By the row operation

$$\begin{pmatrix}1&1+i&0&1&0&0\\0&\frac{1-i}{\sqrt{2}}&\sqrt{2}(1+i)&0&1&0\\0&0&2-i&0&0&1\end{pmatrix}\xrightarrow[\frac{2+i}{5}R_3]{\frac{1+i}{\sqrt{2}}R_2}\begin{pmatrix}1&1+i&0&1&0&0\\0&1&2i&0&\frac{1+i}{\sqrt{2}}&0\\0&0&1&0&0&\frac{2+i}{5}\end{pmatrix}$$

$$\xrightarrow{R_2-2iR_3}\begin{pmatrix}1&1+i&0&1&0&0\\0&1&0&0&\frac{1+i}{\sqrt{2}}&\frac{-2+4i}{5}\\0&0&1&0&0&\frac{2+i}{5}\end{pmatrix}\xrightarrow{R_1-(1+i)R_2}\begin{pmatrix}1&0&0&1&\sqrt{2}i&\frac{-6+2i}{5}\\0&1&0&0&\frac{1+i}{\sqrt{2}}&\frac{-2+4i}{5}\\0&0&1&0&0&\frac{2+i}{5}\end{pmatrix},$$

the new basis is the last three columns of last matrix above.

Exercise 8.7. Eliminate the cross terms.

1. $x^2+4xy-5y^2$.
2. $2x^2+4xy$.
3. $4x_1^2+4x_1x_2+5x_2^2$.
4. $x^2+2y^2+z^2+2xy-2xz$.
5. $-2u^2-v^2-6w^2-4uw+2vw$.
6. $x_1^2+x_3^2+2x_1x_2+2x_1x_3+2x_1x_4+2x_3x_4$.

Exercise 8.8. Eliminate the cross terms in the quadratic form $x^2+2y^2+z^2+2xy-2xz$ by first completing a square for terms involving z, then completing for terms involving y.

Next we study the process of completing the square in general. Let $q(\vec{x})=\vec{x}^TB\vec{x}$ for $\vec{x}\in\mathbb{R}^n$ and a symmetric $n\times n$ matrix B. The *leading principal minors* of B are the determinants of the square submatrices formed by the entries in the first k rows and first k columns of B

$$d_1=b_{11},\ d_2=\det\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix},\ d_3=\det\begin{pmatrix}b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{pmatrix},\ \dots,\ d_n=\det B.$$

If $d_1\neq 0$ (this means b_{11} is invertible in $\mathbb{F}$), then eliminating all the cross terms involving x_1 gives

$$\begin{aligned}q(\vec{x})&=b_{11}\left(x_1^2+2x_1\tfrac{1}{b_{11}}(b_{12}x_2+\cdots+b_{1n}x_n)+\tfrac{1}{b_{11}^2}(b_{12}x_2+\cdots+b_{1n}x_n)^2\right)\\&\quad+b_{22}x_2^2+\cdots+b_{nn}x_n^2+2b_{23}x_2x_3+2b_{24}x_2x_4+\cdots+2b_{(n-1)n}x_{n-1}x_n\\&\quad-\tfrac{1}{b_{11}}(b_{12}x_2+\cdots+b_{1n}x_n)^2\\&=d_1\left(x_1+\tfrac{b_{12}}{d_1}x_2+\cdots+\tfrac{b_{1n}}{d_1}x_n\right)^2+q_2(\vec{x}_2).\end{aligned}$$

Here q_2 is a quadratic form not involving x_1. In other words, it is a quadratic form of the truncated vector $\vec{x}_2=(x_2,\dots,x_n)$. The symmetric coefficient matrix B_2 for q_2 is obtained as follows. For all $2\le i\le n$, we apply the column operation $R_i-\frac{b_{1i}}{b_{11}}R_1$ to B to eliminate all the entries in the first column below the first entry b_{11}. The result is a matrix $\begin{pmatrix}d_1 & *\\ \vec{0} & B_2\end{pmatrix}$. In fact, we may further apply the similar row operations $C_i-\frac{b_{1i}}{b_{11}}C_1$ and get a symmetric matrix $\begin{pmatrix}d_1 & 0\\ \vec{0} & B_2\end{pmatrix}$. Since the operations do not change the determinant of the matrix (and all the leading principal minors), the principal minors $d_1^{(2)},\dots,d_{n-1}^{(2)}$ of B_2 are related to the principal minors of B by $d_{i+1}=d_1d_i^{(2)}$.

The discussion sets up an inductive argument. To facilitate the argument, we denote $B_1=B$, $d_i^{(1)}=d_i$, and get $d_i^{(2)}=\frac{d_{i+1}^{(1)}}{d_1^{(1)}}$. If $d_1,\dots,d_k$ are all nonzero, then we may complete the squares in k steps and obtain

$$\begin{aligned}q(\vec{x})&=d_1^{(1)}(x_1+c_{12}x_2+\cdots+c_{1n}x_n)^2+d_1^{(2)}(x_2+c_{23}x_3+\cdots+c_{2n}x_n)^2\\&\quad+\cdots+d_1^{(k)}(x_k+c_{k(k+1)}x_{k+1}+\cdots+c_{kn}x_n)^2+q_{k+1}(\vec{x}_{k+1}).\end{aligned}$$

The calculation of the coefficients is inspired by $d_i^{(2)}=\frac{d_{i+1}^{(1)}}{d_1^{(1)}}$

$$d_1^{(i)}=\frac{d_2^{(i-1)}}{d_1^{(i-1)}}=\frac{d_3^{(i-2)}}{d_2^{(i-2)}}=\cdots=\frac{d_i^{(1)}}{d_{i-1}^{(1)}}=\frac{d_i}{d_{i-1}}.$$

Moreover, the coefficient of x_{k+1}^2 in q_{k+1} is $d_1^{(k+1)} = \dfrac{d_{k+1}}{d_k}$.

Proposition 8.1.1 (Lagrange-Beltrami Identity). *Suppose $q(\vec{x}) = \vec{x}^T B\vec{x}$ is a quadratic form of rank r, over a field in which $2 \neq 0$. If all the leading principal minors $d_1, \dots, d_r$ of the symmetric coefficient matrix B are nonzero, then there is an upper triangular change of variables*

$$y_i = x_i + c_{i(i+1)}x_{i+1} + \cdots + c_{in}x_n, \quad i = 1, \dots, r,$$

such that

$$q = d_1y_1^2 + \frac{d_2}{d_1}y_2^2 + \cdots + \frac{d_r}{d_{r-1}}y_r^2.$$

Examples 8.1.9 and 8.1.10 shows that the nonzero condition on leading principal minors may not always be satisfied. Still, the examples show that it is always possible to eliminate cross terms after a suitable change of variable.

Proposition 8.1.2. *Any quadratic form of rank r and over a field in which $2 \neq 0$ can be expressed as*

$$q = b_1x_1^2 + \cdots + b_rx_r^2$$

after a suitable change of variable.

In terms of matrix, this means that any symmetric matrix B can be written as $B = P^T DP$, where P is invertible and B is a diagonal matrix with exactly r nonzero entries. For $\mathbb{F} = \mathbb{C}$, we may further get the unique canonical form by replacing x_i with $\sqrt{b_i}x_i$

$$q = x_1^2 + \cdots + x_r^2, \quad r = \operatorname{rank} q.$$

Two quadratic forms q and q' are *equivalent* if $q'(\vec{x}) = q(L(\vec{x}))$ for some invertible linear transformation. In terms of symmetric matrices, this means that S and P^TSP are equivalent. The unique canonical form above implies the following.

Theorem 8.1.3. *Two complex quadratic forms are equivalent if and only if they have the same rank.*

For $\mathbb{F} = \mathbb{R}$, we may replace x_i with $\sqrt{b_i}x_i$ in case $b_i > 0$ and with $\sqrt{-b_i}x_i$ in case $b_i < 0$. The canonical form we get is (after rearranging the order of x_i if needed)

$$q = x_1^2 + \cdots + x_s^2 - x_{s+1}^2 - \cdots - x_r^2.$$

Theorem 8.2.4 discusses this canonical form.

8.2 Hermitian

8.2.1 Sesquilinear Function

The complex inner product is not bilinear. It is a *sesquilinear function* (defined for complex vector spaces V and W) in the following sense

$$s(x_1\vec{v}_1 + x_2\vec{v}_2, \vec{w}) = x_1 s(\vec{v}_1, \vec{w}) + x_2 s(\vec{v}_2, \vec{w}),$$
$$s(\vec{v}, y_1\vec{w}_1 + y_2\vec{w}_2) = \bar{y}_1 s(\vec{v}, \vec{w}_1) + \bar{y}_2 s(\vec{v}, \vec{w}_2).$$

The sesquilinear function can be regarded as a bilinear function $s\colon V \times \bar{W} \to \mathbb{C}$, where $\bar{W}$ is the conjugate vector space of W.

A sesquilinear function is determined by its values $s_{ij} = s(\vec{v}_i, \vec{w}_j)$ on bases $\alpha = \{\vec{v}_1, \dots, \vec{v}_m\}$ and $\beta = \{\vec{w}_1, \dots, \vec{w}_n\}$ of V and W

$$s(\vec{x}, \vec{y}) = \sum_{i,j} s_{ij} x_i \bar{y}_j = [\vec{x}]_\alpha^T S \overline{[\vec{y}]}_\beta, \quad [\vec{x}]_\alpha = (x_1, \dots, x_m), \quad [\vec{y}]_\beta = (y_1, \dots, y_n).$$

Here the *matrix of sesquilinear function* is

$$S = [s]_{\alpha\beta} = (s(\vec{v}_i, \vec{w}_j))_{i,j=1}^{m,n}.$$

By $s(\vec{x}, \vec{y}) = [\vec{x}]_\alpha^T [s]_{\alpha\beta} \overline{[\vec{y}]}_\beta$ and $[\vec{x}]_{\alpha'} = [I]_{\alpha'\alpha} [\vec{x}]_\alpha$, we get the change of matrix caused by the change of bases

$$[s]_{\alpha'\beta'} = [I]_{\alpha\alpha'}^T [s]_{\alpha\beta} \overline{[I]}_{\beta\beta'}.$$

A sesquilinear function $s\colon V \times W \to \mathbb{C}$ induces a linear transformation

$$L(\vec{v}) = s(\vec{v}, \cdot)\colon V \to \bar{W}^* = \overline{\mathrm{Hom}}(W, \mathbb{C}) = \mathrm{Hom}(\bar{W}, \mathbb{C}).$$

Here $\bar{W}^*$ is the *conjugate dual space* consisting of conjugate linear functionals. We have a one-to-one correspondence between sesquilinear functions s and linear transformations L.

If we also know that W is a complex (Hermitian) inner product space, then we have a linear isomorphism $\vec{w} \mapsto \langle \vec{w}, \cdot \rangle\colon W \cong \bar{W}^*$, similar to Proposition 4.3.1. Combined with the linear transformation above, we get a linear transformation, still denoted L

$$L\colon V \to \bar{W}^* \cong W, \quad \vec{v} \mapsto s(\vec{v}, \cdot) = \langle L(\vec{v}), \cdot \rangle.$$

This means

$$s(\vec{v}, \vec{w}) = \langle L(\vec{v}), \vec{w} \rangle \quad \text{for all } \vec{v} \in V,\ \vec{w} \in W.$$

Therefore sesquilinear functions $s\colon V \times W \to \mathbb{C}$ are in one-to-one correspondence with linear transformations $L\colon V \to W$.

The sesquilinear function also induces a conjugate linear transformation

$$L^*(\vec{w}) = s(\cdot, \vec{w})\colon W \to V^* = \mathrm{Hom}(V, \mathbb{C}),$$

The correspondence $s \mapsto L^*$ is again one-to-one if V and W have finite dimensions. If we also know V is a complex inner product space, then we have a conjugate linear isomorphism $\vec{v} \mapsto \langle \cdot, \vec{v} \rangle \colon V \cong V^*$, similar to Proposition 4.3.1. Combined with the conjugate linear transformation above, we get a linear transformation, still denoted L^*

$$L^* \colon W \to V^* \cong V, \quad \vec{w} \mapsto s(\cdot, \vec{w}) = \langle \cdot, L(\vec{w}) \rangle.$$

This means

$$s(\vec{v}, \vec{w}) = \langle \vec{v}, L^*(\vec{w}) \rangle \quad \text{for all } \vec{v} \in V,\ \vec{w} \in W.$$

Therefore sesquilinear functions $s \colon V \times W \to \mathbb{C}$ are in one-to-one correspondence with linear transformations $L^* \colon W \to V$.

If both V and W are complex inner product spaces, then we have

$$s(\vec{v}, \vec{w}) = \langle L(\vec{v}), \vec{w} \rangle = \langle \vec{v}, L^*(\vec{w}) \rangle.$$

Exercise 8.9. Suppose $s(\vec{x}, \vec{y})$ is sesquilinear. Prove that $\overline{s(\vec{y}, \vec{x})}$ is also sesquilinear. What are the linear transformations induced by $\overline{s(\vec{y}, \vec{x})}$? How are the matrices of the two sesquilinear functions related?

A sesquilinear function is a conjugate dual pairing if both $L \colon V \to \bar{W}^*$ and $L^* \colon W \to V^*$ are isomorphisms. In fact, one isomorphism implies the other. This is also the same as both are one-to-one, or both are onto.

The basic examples of conjugate dual pairing are the complex inner product and the evaluation pairing

$$s(\vec{x}, l) = l(\vec{x}) \colon V \times \bar{V}^* \to \mathbb{C}.$$

A basis $\alpha = \{\vec{v}_1, \dots, \vec{v}_n\}$ of V and a basis $\beta = \{\vec{w}_1, \dots, \vec{w}_n\}$ of W are *dual bases* with respect to the dual pairing if

$$s(\vec{v}_i, \vec{w}_j) = \delta_{ij}, \text{ or } [s]_{\alpha\beta} = I.$$

We may express any $\vec{x} \in V$ in terms of α

$$\vec{x} = s(\vec{x}, \vec{w}_1)\vec{v}_1 + s(\vec{x}, \vec{w}_2)\vec{v}_2 + \dots + s(\vec{x}, \vec{w}_n)\vec{v}_n,$$

and express any $\vec{y} \in W$ in terms of β

$$\vec{y} = \overline{s(\vec{v}_1, \vec{y})}\vec{w}_1 + \overline{s(\vec{v}_2, \vec{y})}\vec{w}_2 + \dots + \overline{s(\vec{v}_n, \vec{y})}\vec{w}_n,$$

For the inner product, a basis is self dual if and only if it is an orthonormal basis. For the evaluation pairing, the dual basis $\alpha^* = \{\vec{v}_1^*, \dots, \vec{v}_n^*\} \subset \bar{V}^*$ of $\alpha = \{\vec{v}_1, \dots, \vec{v}_n\} \subset V$ is given by

$$\vec{v}_i^*(\vec{x}) = \bar{x}_i, \quad [\vec{x}]_\alpha = (x_1, \dots, x_n).$$

Exercise 8.10. Suppose a sesquilinear form $s \colon V \times W \to \mathbb{C}$ is a dual pairing. Suppose α and β are bases of V and W. Prove that the following are equivalent.

1. α and β are dual bases with respect to s.

2. L takes α to β^* (conjugate dual basis of $\bar{W}^*$).

3. L^* takes β to α^* (dual basis of V^*).

Exercise 8.11. Prove that if α and β are dual bases with respect to dual pairing $s(\vec{x}, \vec{y})$, then β and α are dual bases with respect to $\overline{s(\vec{y}, \vec{x})}$.

Exercise 8.12. Suppose α and β are dual bases with respect to a sesquilinear dual pairing $s\colon V \times W \to \mathbb{C}$. What is the relation between matrices $[s]_{\alpha\beta}$, $[L]_{\beta^*\alpha}$, $[L^*]_{\alpha^*\beta}$?

8.2.2 Hermitian Form

A sesquilinear function $s\colon V \times V \to \mathbb{C}$ is *Hermitian* if it satisfies

$$s(\vec{x}, \vec{y}) = \overline{s(\vec{y}, \vec{x})}.$$

A typical example is the complex inner product. If we take $\vec{x} = \vec{y}$, then we get a *Hermitian form*

$$q(\vec{x}) = s(\vec{x}, \vec{x}).$$

Conversely, the Hermitian sesquilinear function s can be recovered from q by *polarisation* (see Exercise 6.21)

$$s(\vec{x}, \vec{y}) = \tfrac{1}{4}(q(\vec{x}+\vec{y}) - q(\vec{x}-\vec{y}) + iq(\vec{x}+i\vec{y}) - iq(\vec{x}-i\vec{y})).$$

Proposition 8.2.1. *A sesquilinear function $s(\vec{x}, \vec{y})$ is Hermitian if and only if $s(\vec{x}, \vec{x})$ is always a real number.*

Exercise 6.25 is a similar result.

Proof. If s is Hermitian, then $s(\vec{x}, \vec{y}) = \overline{s(\vec{y}, \vec{x})}$ implies $s(\vec{x}, \vec{x}) = \overline{s(\vec{x}, \vec{x})}$, which means $s(\vec{x}, \vec{x})$ is a real number.

Conversely, suppose $s(\vec{x}, \vec{x})$ is always a real number. We want to show $s(\vec{x}, \vec{y}) = \overline{s(\vec{y}, \vec{x})}$, which is the same as $s(\vec{x}, \vec{y}) + s(\vec{y}, \vec{x})$ is real and $s(\vec{x}, \vec{y}) - s(\vec{y}, \vec{x})$ is imaginary. This follows from

$$\begin{aligned} s(\vec{x}, \vec{y}) + s(\vec{y}, \vec{x}) &= s(\vec{x}+\vec{y}, \vec{x}+\vec{y}) - s(\vec{x}, \vec{x}) - s(\vec{y}, \vec{y}) \\ is(\vec{x}, \vec{y}) - is(\vec{y}, \vec{x}) &= s(i\vec{x}+\vec{y}, i\vec{x}+\vec{y}) - s(i\vec{x}, i\vec{x}) - s(\vec{y}, \vec{y}). \end{aligned}$$ □

In terms of the matrix $S = [q]_\alpha = [s]_{\alpha\alpha}$ with respect to a basis α of V, the Hermitian property means that $S^* = S$, or S is a Hermitian matrix. For another basis β, we have

$$[q]_\beta = [I]_{\alpha\beta}^T [q]_\alpha \overline{[I]}_{\alpha\beta} = P^* S P, \quad P = \overline{[I]}_{\alpha\beta}.$$

Suppose V is an inner product space. Then the Hermitian form is given by a linear operator L by $q(\vec{x}) = \langle L(\vec{x}), \vec{x}\rangle$. By Proposition 8.2.1, we know $q(\vec{x})$ is a real number, and therefore $q(\vec{x}) = \overline{\langle L(\vec{x}), \vec{x}\rangle} = \langle \vec{x}, L(\vec{x})\rangle$. By polarisation, we get

$$\langle L(\vec{x}), \vec{y}\rangle = \langle \vec{x}, L(\vec{y})\rangle \quad \text{for all } \vec{x}, \vec{y}.$$

This means that Hermitian forms are in one-to-one correspondence with self-adjoint operators.

By Propositions 7.2.6 and 7.2.7, L has an orthonormal basis $\alpha = \{\vec{v}_1, \dots, \vec{v}_n\}$ of eigenvectors, with eigenvalues $d_1, \dots, d_n$. Then we have $\langle L(\vec{v}_i), \vec{v}_j\rangle = \langle d_i\vec{v}_i, \vec{v}_j\rangle = d_i\delta_{ij}$. This means

$$[q]_\alpha = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}$$

is diagonal, and the expression of q in α-coordinate has no cross terms

$$q(\vec{x}) = d_1x_1\bar{x}_1 + \cdots + d_nx_n\bar{x}_n = d_1|x_1|^2 + \cdots + d_n|x_n|^2.$$

Let $\lambda_{\max} = \max d_i$ and $\lambda_{\min} = \min d_i$ be the maximal and minimal eigenvalues of L. Then the formula above implies ($\|\vec{x}\|^2 = |x_1|^2 + \cdots + |x_n|^2$ by orthonormal basis)

$$\lambda_{\min}\|\vec{x}\|^2 \le q(\vec{x}) = \langle L(\vec{x}), \vec{x}\rangle \le \lambda_{\max}\|\vec{x}\|^2.$$

Moreover, the right equality is reached if and only if $x_i = 0$ whenever $d_i \ne \lambda_{\max}$. Since $\oplus_{d_i=\lambda}\mathbb{R}\vec{v}_i = \operatorname{Ker}(L-\lambda I)$, the right equality is reached exactly on the eigenspace $\operatorname{Ker}(L - \lambda_{\max}I)$. Similarly, the left equality is reached exactly on the eigenspace $\operatorname{Ker}(L - \lambda_{\min}I)$.

Proposition 8.2.2. *For self-adjoint operator L on an inner product space, the maximal and minimal eigenvalues of L are $\max_{\|\vec{x}\|=1}\langle L(\vec{x}), \vec{x}\rangle$ and $\min_{\|\vec{x}\|=1}\langle L(\vec{x}), \vec{x}\rangle$.*

Proof. We provide an alternative proof by using the Lagrange multiplier. We try to find the maximum of the function $q(\vec{x}) = \langle L(\vec{x}), \vec{x}\rangle = \langle \vec{x}, L(\vec{x})\rangle$ subject to the constraint $g(\vec{x}) = \langle \vec{x}, \vec{x}\rangle = \|\vec{x}\|^2 = 1$. By

$$\begin{aligned} q(\vec{x}_0 + \Delta\vec{x}) &= \langle L(\vec{x}_0 + \Delta\vec{x}), \vec{x}_0 + \Delta\vec{x}\rangle \\ &= \langle L(\vec{x}_0), \vec{x}_0\rangle + \langle L(\vec{x}_0), \Delta\vec{x}\rangle + \langle L(\Delta\vec{x}), \vec{x}_0\rangle + \langle L(\Delta\vec{x}), \Delta\vec{x}\rangle \\ &= q(\vec{x}_0) + \langle L(\vec{x}_0), \Delta\vec{x}\rangle + \langle \Delta\vec{x}, L(\vec{x}_0)\rangle + o(\|\Delta\vec{x}\|), \end{aligned}$$

we get (the "multi-derivative" is a linear functional)

$$q'(\vec{x}_0) = \langle L(\vec{x}_0), \cdot\rangle + \langle \cdot, L(\vec{x}_0)\rangle.$$

By the similar reason, we have

$$g'(\vec{x}_0) = \langle \vec{x}_0, \cdot\rangle + \langle \cdot, \vec{x}_0\rangle.$$

If the maximum of q subject to the constraint $g = 1$ happens at $\vec{x}_0$, then we have

$$q'(\vec{x}_0) = \langle L(\vec{x}_0), \cdot\rangle + \langle \cdot, L(\vec{x}_0)\rangle = \langle \lambda\vec{x}_0, \cdot\rangle + \langle \cdot, \lambda\vec{x}_0\rangle = \lambda g'(\vec{x}_0)$$

for a real number λ. Let $\vec{v} = L(\vec{x}_0) - \lambda\vec{x}_0$. Then the equality means $\langle \vec{v}, \cdot\rangle + \langle \cdot, \vec{v}\rangle = 0$. Taking the variable $\cdot$ to be $\vec{v}$, we get $2\|\vec{v}\|^2 = 0$, or $\vec{v} = \vec{0}$. This proves that $L(\vec{x}_0) = \lambda\vec{x}_0$. Moreover, the maximum is

$$\max_{\|\vec{x}\|=1} q(\vec{x}) = q(\vec{x}_0) = \langle L(\vec{x}_0), \vec{x}_0\rangle = \langle \lambda\vec{x}_0, \vec{x}_0\rangle = \lambda g(\vec{x}_0) = \lambda.$$

For any other eigenvalue μ, we have $L(\vec{x}) = \mu\vec{x}$ for a unit length vector $\vec{x}$ and get

$$\mu = \mu\langle \vec{x}, \vec{x}\rangle = \langle \mu\vec{x}, \vec{x}\rangle = \langle L(\vec{x}), \vec{x}\rangle = q(\vec{x}) \le q(\vec{x}_0) = \lambda.$$

This proves that λ is the maximum eigenvalue. □

Example 8.2.1. In Example 7.2.1, we have the orthogonal diagonalisation of a Hermitian matrix

$$\begin{aligned}\begin{pmatrix} 2 & 1+i \\ 1-i & 3 \end{pmatrix} &= \begin{pmatrix} \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \frac{1-i}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1+i}{\sqrt{3}} \end{pmatrix}^{*} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} \frac{1-i}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1+i}{\sqrt{3}} \end{pmatrix}.\end{aligned}$$

In terms of Hermitian form, this means

$$2x\bar{x} + (1+i)x\bar{y} + (1-i)y\bar{x} + 3y\bar{y} = \left|\tfrac{1}{\sqrt{3}}((1-i)x - y)\right|^2 + 4\left|\tfrac{1}{\sqrt{3}}(x + (1+i)y)\right|^2.$$

Example 8.2.2. In Example 7.2.4, we found an orthogonal diagonalisation of the symmetric matrix in Example 7.1.14

$$\begin{pmatrix} 1 & -2 & -4 \\ -2 & 4 & -2 \\ -4 & -2 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ -\frac{4}{3\sqrt{5}} & -\frac{2}{3\sqrt{5}} & \frac{\sqrt{5}}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix}^{T} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ -\frac{4}{3\sqrt{5}} & -\frac{2}{3\sqrt{5}} & \frac{\sqrt{5}}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

Here we use $U^{-1} = U^T$ for orthogonal matrix. In terms of Hermitian form, this means

$$\begin{aligned}&x\bar{x} + 4y\bar{y} + z\bar{z} - 2x\bar{y} - 2y\bar{x} - 4x\bar{z} - 4z\bar{x} - 2y\bar{z} - 2z\bar{y} \\ &= 5\left|-\tfrac{1}{\sqrt{5}}x + \tfrac{2}{\sqrt{5}}y\right|^2 + 5\left|-\tfrac{4}{3\sqrt{5}}x - \tfrac{2}{3\sqrt{5}}y + \tfrac{\sqrt{5}}{3}z\right|^2 - 4\left|\tfrac{2}{3}x + \tfrac{1}{3}y + \tfrac{2}{3}z\right|^2.\end{aligned}$$

In terms of quadratic form, this means

$$\begin{aligned}&x^2+4y^2+z^2-4xy-8xz-4yz\\&=5\left(-\tfrac{1}{\sqrt{5}}x+\tfrac{2}{\sqrt{5}}y\right)^2+5\left(-\tfrac{4}{3\sqrt{5}}x-\tfrac{2}{3\sqrt{5}}y+\tfrac{\sqrt{5}}{3}z\right)^2-4\left(\tfrac{2}{3}x+\tfrac{1}{3}y+\tfrac{2}{3}z\right)^2.\end{aligned}$$

We remark that the quadratic form is also diagonalised by completing the square in Example 8.1.9.

Exercise 8.13. Prove that a linear operator on a complex inner product space is self-adjoint if and only if $\langle L(\vec{x}),\vec{x}\rangle$ is always a real number. Compare with Proposition 7.2.6.

8.2.3 Completing the Square

In Examples 8.2.1 and 8.2.2, we use orthogonal diagonalisation of Hermitian operators to eliminate the cross terms in Hermitian forms. We may also use the method of completing the square similar to the quadratic forms.

Example 8.2.3. For the Hermitian form in Example 8.2.1, we have

$$\begin{aligned}&2x\bar{x}+(1+i)x\bar{y}+(1-i)y\bar{x}+3y\bar{y}\\&=2\left[x\bar{x}+x\overline{\tfrac{1-i}{2}y}+\tfrac{1-i}{2}y\bar{x}+\tfrac{1-i}{2}y\overline{\tfrac{1-i}{2}y}\right]-\tfrac{|1-i|^2}{2}y\bar{y}+3y\bar{y}\\&=\left|x+\tfrac{1-i}{2}y\right|^2+2|y|^2.\end{aligned}$$

Example 8.2.4. We complete squares

$$\begin{aligned}&x\bar{x}-y\bar{y}+2z\bar{z}+(1+i)x\bar{y}+(1-i)y\bar{x}+3iy\bar{z}-3iz\bar{y}\\&=\left[x\bar{x}+(1+i)x\bar{y}+(1-i)y\bar{x}+(1-i)y\overline{(1-i)y}\right]\\&\quad-y\bar{y}+2z\bar{z}+3iy\bar{z}-3iz\bar{y}-|1-i|^2y\bar{y}\\&=|x+(1-i)y|^2-3y\bar{y}+2z\bar{z}+3iy\bar{z}-3iz\bar{y}\\&=|x+(1-i)y|^2-3\left[y\bar{y}-iy\bar{z}+iz\bar{y}+iz\overline{iz}\right]+2z\bar{z}+3|i|^2z\bar{z}\\&=|x+(1-i)y|^2-3|y+iz|^2+5|z|^2.\end{aligned}$$

In terms of Hermitian matrix, this means

$$\begin{pmatrix}1&1+i&0\\1-i&-1&3i\\0&-3i&2\end{pmatrix}=\begin{pmatrix}1&1-i&0\\0&1&i\\0&0&1\end{pmatrix}^*\begin{pmatrix}1&0&0\\0&-3&0\\0&0&5\end{pmatrix}\begin{pmatrix}1&1-i&0\\0&1&i\\0&0&1\end{pmatrix}.$$

The new variables $x+(1-i)y, y+iz, z$ are the coordinates with respect to the basis of the columns of the inverse matrix

$$\begin{pmatrix}1&1-i&0\\0&1&i\\0&0&1\end{pmatrix}^{-1}=\begin{pmatrix}1&-1+i&1+i\\0&1&-i\\0&0&1\end{pmatrix}.$$

The Lagrange-Beltrami Identity (Proposition 8.1.1) remains valid for Hermitian forms. We note that, by Exercise 7.61, all the principal minors of a Hermitian matrix are real.

Proposition 8.2.3 (Lagrange-Beltrami Identity)**.** *Suppose $q(\vec{x}) = \vec{x}^T S \bar{\vec{x}}$ is a Hermitian form of rank r. If all the leading principal minors $d_1, \dots, d_r$ of the Hermitian coefficient matrix S are nonzero, then there is an upper triangular change of variables*

$$y_i = x_i + c_{i(i+1)}x_{i+1} + \cdots + c_{in}x_n, \quad i = 1, \dots, r,$$

such that

$$q = d_1|y_1|^2 + \frac{d_2}{d_1}|y_2|^2 + \cdots + \frac{d_r}{d_{r-1}}|y_r|^2.$$

8.2.4 Signature

For Hermitian forms (including real quadratic forms), we may use either orthogonal diagonalisation or completing the square to reduce the form to

$$q(\vec{x}) = d_1|x_1|^2 + \cdots + d_r|x_r|^2.$$

Here d_i are real numbers, and r is the rank of q. By replacing x_i with $\sqrt{|d_i|}x_i$ and rearranging the orders of the variables, we may further get the *canonical form* of the Hermitian form

$$q(\vec{x}) = |x_1|^2 + \cdots + |x_s|^2 - |x_{s+1}|^2 - \cdots - |x_r|^2.$$

Here s is the number of $d_i > 0$, and $r-s$ is the number of $d_i < 0$. From the viewpoint of orthogonal diagonalisation, we have $q(\vec{x}) = \langle L(\vec{x}), \vec{x}\rangle$ for a self-adjoint operator L. Then s is the number of positive eigenvalues of L, and $r - s$ is the number of negative eigenvalues of L.

We know the rank r is unique. The following says that s is also unique. Therefore the canonical form of the Hermitian form is unique.

Theorem 8.2.4 (Sylvester's Law)**.** *After eliminating the cross terms in a quadratic form, the number of positive coefficients and the number of negative coefficients are independent of the elimination process.*

Proof. Suppose

$$\begin{aligned} q(\vec{x}) &= |x_1|^2 + \cdots + |x_s|^2 - |x_{s+1}|^2 - \cdots - |x_r|^2 \\ &= |y_1|^2 + \cdots + |y_t|^2 - |y_{t+1}|^2 - \cdots - |y_r|^2 \end{aligned}$$

in terms of the coordinates with respect to bases $\alpha = \{\vec{v}_1, \dots, \vec{v}_n\}$ and basis $\beta = \{\vec{w}_1, \dots, \vec{w}_n\}$.

We claim that $\vec{v}_1, \dots, \vec{v}_s, \vec{w}_{t+1}, \dots, \vec{w}_n$ are linearly independent. Suppose

$$x_1\vec{v}_1 + \cdots + x_s\vec{v}_s + y_{t+1}\vec{w}_{t+1} + \cdots + y_n\vec{w}_n = \vec{0}.$$

Then

$$x_1\vec{v}_1 + \cdots + x_s\vec{v}_s = -y_{t+1}\vec{w}_{t+1} - \cdots - y_n\vec{w}_n.$$

Applying q to both sides, we get

$$|x_1|^2 + \cdots + |x_s|^2 = -|y_{t+1}|^2 - \cdots - |y_r|^2 \leq 0.$$

This implies $x_1 = \cdots = x_s = 0$ and $y_{t+1}\vec{w}_{t+1} + \cdots + y_n\vec{w}_n = \vec{0}$. Since β is a basis, we get $y_{t+1} = \cdots = y_n = 0$. This completes the proof of the claim.

The claim implies $s+(n-t) \leq n$, or $s \leq t$. By symmetry, we also have $t \leq s$. □

The number $s - t$ is called the *signature* of the quadratic form. The Hermitian (quadratic) forms in Examples 8.1.8, 8.1.9, 8.1.10, 8.1.11, 8.2.1 have signatures $3, 1, 0, 2, 2$.

If all the leading principal minors are nonzero, then the Lagrange-Beltrami Identity gives a way of calculating the signature by comparing the signs of leading principal minors.

An immediate consequence of Sylvester's Law is the following analogue of Proposition 8.1.3.

Theorem 8.2.5. *Two Hermitian quadratic forms are equivalent if and only if they have the same rank and signature.*

8.2.5 Positive Definite

Definition 8.2.6. Let $q(\vec{x})$ be a Hermitian form.

1. q is *positive definite* if $q(\vec{x}) > 0$ for any $\vec{x} \neq 0$.
2. q is *negative definite* if $q(\vec{x}) < 0$ for any $\vec{x} \neq 0$.
3. q is *positive semi-definite* if $q(\vec{x}) \geq 0$ for any $\vec{x} \neq 0$.
4. q is *negative semi-definite* if $q(\vec{x}) \leq 0$ for any $\vec{x} \neq 0$.
5. q is *indefinite* if the values of q can be positive and can also be negative.

The type of Hermitian form can be easily determined by its canonical form. Let s, r, n be the signature, rank, and dimension. Then we have the following correspondence

$+$	$-$	semi $+$	semi $-$	indef
$s = r = n$	$-s = r = n$	$s = r$	$-s = r$	$s \neq r,\ -s \neq r$

The Lagrange-Beltrami Identity (Proposition 8.2.3) gives another criterion.

Proposition 8.2.7 (Sylvester's Criterion). *Suppose $q(\vec{x}) = \vec{x}^T S \bar{\vec{x}}$ is a Hermitian form of rank r and dimension n. Suppose all the leading principal minors $d_1, \dots, d_r$ of S are nonzero.*

1. *q is positive semi-definite if and only if $d_1, d_2, \dots, d_r$ are all positive. If we further have $r = n$, then q is positive definite.*

2. *q is negative semi-definite if and only if $d_1, -d_2, \dots, (-1)^r d_r$ are all positive. If we further have $r = n$, then q is negative definite.*

3. *Otherwise q is indefinite.*

If some $d_1, \dots, d_r$ are zero, then q cannot be positive or negative definite. The criterion for the other possibilities is a little more complicated[1].

Exercise 8.14. Suppose a Hermitian form has not square term $|x_1|^2$ (i.e., the coefficient $s_{11} = 0$). Prove that the form is indefinite.

Exercise 8.15. Prove that if a quadratic form $q(\vec{x})$ is positive definite, then $q(\vec{x}) \geq c\|\vec{x}\|^2$ for any $\vec{x}$ and a constant $c > 0$. What is the maximum of such c?

Exercise 8.16. Suppose q and q' are positive definite, and $a, b > 0$. Prove that $aq + bq'$ is positive definite.

The types of quadratic forms can be applied to self-adjoint operators L, by considering the Hermitian form $\langle L(\vec{x}), \vec{x} \rangle$. For example, L is positive definite if $\langle L(\vec{x}), \vec{x} \rangle > 0$ for all $\vec{x} \neq \vec{0}$.

Using the orthogonal diagonalisation to eliminate cross terms in $\langle L(\vec{x}), \vec{x} \rangle$, the type of L can be determined by its eigenvalues λ_i.

$+$	$-$	semi $+$	semi $-$	indef
all $\lambda_i > 0$	all $\lambda_i < 0$	all $\lambda_i \geq 0$	all $\lambda_i \leq 0$	some $\lambda_i > 0$, some $\lambda_j < 0$

Exercise 8.17. Prove that positive definite and negative operators are invertible.

Exercise 8.18. Suppose L and K are positive definite, and $a, b > 0$. Prove that $aL + bL$ is positive definite.

Exercise 8.19. Suppose L is positive definite. Prove that L^n is positive definite.

[1]Sylvester's Minorant Criterion, Lagrange-Beltrami Identity, and Nonnegative Definiteness, by Sudhir R. Ghorpade, Balmohan V. Limaye, in The Mathematics Student. Special Centenary Volume (2007), 123-130

Exercise 8.20. For any self-adjoint operator L, prove that $L^2 - L + I$ is positive definite. What is the condition on a, b such that $L^2 + aL + bI$ is always positive definite?

Exercise 8.21. Prove that for any linear operator L, L^*L is self-adjoint and positive semi-definite. Moreover, if L is one-to-one, then L^*L is positive definite.

A positive semi-definite operator has decomposition

$$L = \lambda_1 I \perp \cdots \perp \lambda_k I, \quad \lambda_i \geq 0.$$

Then we may construct the operator $\sqrt{\lambda_1} I \perp \cdots \perp \sqrt{\lambda_k} I$, which satisfies the following definition.

Definition 8.2.8. Suppose L is a positive semi-definite operator. The *square root* operator $\sqrt{L}$ is the positive semi-definite operator K satisfying $K^2 = L$ and $KL = LK$.

By applying Theorem 7.2.4 to the commutative $*$-algebra (the $*$-operation is trivial) generated by L and K, we find that L and K have simultaneous orthogonal decompositions

$$L = \lambda_1 I \perp \cdots \perp \lambda_k I, \quad K = \mu_1 I \perp \cdots \perp \mu_k I.$$

Then $K^2 = L$ means $\mu_i^2 = \lambda_i$. Therefore $\sqrt{\lambda_1} I \perp \cdots \perp \sqrt{\lambda_k} I$ is the unique operator satisfying the definition.

Suppose $L\colon V \to W$ is an one-to-one linear transformation between inner product spaces. Then L^*L is a positive definite operator, and $A = \sqrt{L^*L}$ is also a positive definite operator. Since positive definite operators are invertible, we may introduce a linear transformation $U = LA^{-1}\colon V \to W$. Then

$$U^*U = A^{-1}L^*LA^{-1} = A^{-1}A^2A^{-1} = I.$$

Therefore U is an isometry. The decomposition $L = UA$ is comparable to the polar decomposition of complex numbers

$$z = e^{i\theta} r, \quad r = |z| = \sqrt{\bar{z}z},$$

and is therefore called the *polar decomposition* of L.

8.3 Multilinear

tensor of two

tensor of many

exterior algebra

8.4 Invariant of Linear Operator

The quantities we use to describe the structure of linear operators should depend only on the linear operator itself. Suppose we define such a quantity $f(A)$ by the matrix A of the linear operator with respect to a basis. Since a change of basis changes A to $P^{-1}AP$, we need to require $f(PAP^{-1}) = f(A)$. In other words, f is a *similarity invariant*. Examples of invariants are the rank, the determinant and the *trace* (see Exercise 8.22)

$$\mathrm{tr}\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = a_{11} + a_{22} + \cdots + a_{nn}.$$

Exercise 8.22. Prove that $\mathrm{tr}AB = \mathrm{tr}BA$. Then use this to show that $\mathrm{tr}PAP^{-1} = \mathrm{tr}A$.

The characteristic polynomial $\det(tI - L)$ is an invariant of L. It is a monic polynomial of degree $n = \dim V$ and can be completely decomposed using complex roots

$$\begin{aligned}\det(tI - L) &= t^n - \sigma_1 t^{n-1} + \sigma_2 t^{n-2} - \cdots + (-1)^{n-1}\sigma_{n-1}t + (-1)^n\sigma_n \\ &= (t-\lambda_1)^{n_1}(t-\lambda_2)^{n_1}\cdots(t-\lambda_k)^{n_k}, \quad n_1 + n_2 + \cdots + n_k = n \\ &= (t-d_1)(t-d_2)\cdots(t-d_n).\end{aligned}$$

Here $\lambda_1, \lambda_2, \ldots, \lambda_k$ are all the distinct eigenvalues, and n_j is the *algebraic multiplicity* of λ_j. Moreover, $d_1, d_2, \ldots, d_n$ are all the eigenvalues repeated in their multiplicities (i.e., λ_1 repeated n_1 times, λ_2 repeated n_2 times, etc.). The (unordered) set $\{d_1, d_2, \ldots, d_n\}$ of all roots of the characteristic polynomial is the *spectrum* of L.

The polynomial is the same as the coefficients $\sigma_1, \sigma_2, \ldots, \sigma_n$. The "polynomial" $\sigma_1, \sigma_2, \ldots, \sigma_n$ determines the spectrum $\{d_1, d_2, \ldots, d_n\}$ by "finding the roots". Conversely, the spectrum determines the polynomial by *Vieta's formula*

$$\sigma_k = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} d_{i_1} d_{i_2} \cdots d_{i_k}.$$

Two special cases are the trace

$$\sigma_1 = d_1 + d_2 + \cdots + d_n = \mathrm{tr}L,$$

and the determinant

$$\sigma_n = d_1 d_2 \cdots d_n = \det L.$$

The general formula of σ_j in terms of L is given in Exercise 7.33.

8.4.1 Symmetric Function

Suppose a function $f(A)$ of square matrices A is an invariant of linear operators. If A is diagonalisable, then

$$A = PDP^{-1}, \quad D = \begin{pmatrix} d_1 & & O \\ & \ddots & \\ O & & d_n \end{pmatrix}, \quad f(A) = f(PDP^{-1}) = f(D).$$

Therefore $f(A)$ is actually a function of the spectrum $\{d_1, d_2, \dots, d_n\}$. Note that the order does not affect the value because exchanging the order of d_j is the same of exchanging columns of P. This shows that the invariant is really a function $f(d_1, d_2, \dots, d_n)$ satisfying

$$f(d_{i_1}, d_{i_2}, \dots, d_{i_n}) = f(d_1, d_2, \dots, d_n)$$

for any permutation $(i_1, i_2, \dots, i_n)$ of $(1, 2, \dots, n)$. These are *symmetric* functions.

The functions $\sigma_1, \sigma_2, \dots, \sigma_n$ given by Vieta's formula are symmetric. Since the spectrum (i.e., unordered set of possibly repeated numbers) $\{d_1, d_2, \dots, d_n\}$ is the same as the "polynomial" $\sigma_1, \sigma_2, \dots, \sigma_n$, symmetric functions are the same as functions of $\sigma_1, \sigma_2, \dots, \sigma_n$

$$f(d_1, d_2, \dots, d_n) = g(\sigma_1, \sigma_2, \dots, \sigma_n).$$

For this reason, we call $\sigma_1, \sigma_2, \dots, \sigma_n$ *elementary symmetric functions*.

For example, $d_1^3 + d_2^3 + d_3^3$ is clearly symmetric, and we have

$$\begin{aligned} d_1^3 + d_2^3 + d_3^3 &= (d_1 + d_2 + d_3)^3 - 3(d_1 + d_2 + d_3)(d_1 d_2 + d_1 d_3 + d_2 d_3) + 3 d_1 d_2 d_3 \\ &= \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3. \end{aligned}$$

We note that $\sigma_1, \sigma_2, \dots, \sigma_n$ are defined for n variables, and

$$\sigma_k(d_1, d_2, \dots, d_n) = \sigma_k(d_1, d_2, \dots, d_n, 0, \dots, 0), \quad k \le n.$$

Theorem 8.4.1. *Any symmetric polynomial $f(d_1, d_2, \dots, d_n)$ is a unique polynomial $g(\sigma_1, \sigma_2, \dots, \sigma_n)$ of the elementary symmetric polynomials.*

Proof. We prove by induction on n. If $n = 1$, then $f(d_1)$ is a polynomial of the only symmetric polynomial $\sigma_1 = d_1$. Suppose the theorem is proved for $n - 1$. Then $\tilde{f}(d_1, d_2, \dots, d_{n-1}) = f(d_1, d_2, \dots, d_{n-1}, 0)$ is a symmetric polynomial of $n - 1$ variables. By induction, we have $\tilde{f}(d_1, d_2 \dots, d_{n-1}) = \tilde{g}(\tilde{\sigma}_1, \tilde{\sigma}_2, \dots, \tilde{\sigma}_{n-1})$ for a polynomial $\tilde{g}$ and elementary symmetric polynomials $\tilde{\sigma}_1, \tilde{\sigma}_2, \dots, \tilde{\sigma}_{n-1}$ of $d_1, d_2, \dots, d_{n-1}$. Now consider

$$h(d_1, d_2 \dots, d_n) = f(d_1, d_2 \dots, d_n) - \tilde{g}(\sigma_1, \sigma_2, \dots, \sigma_{n-1}),$$

where $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ are the elementary symmetric polynomials of $d_1, d_2, \dots, d_n$. The polynomial $h(d_1, d_2, \dots, d_n)$ is still symmetric. By $\sigma_k(d_1, d_2, \dots, d_{n-1}, 0) = \tilde{\sigma}_k(d_1, d_2, \dots, d_{n-1})$, we have

$$h(d_1, d_2, \dots, d_{n-1}, 0) = f(d_1, d_2, \dots, d_{n-1}, 0) - \tilde{g}(\tilde{\sigma}_1, \tilde{\sigma}_2, \dots, \tilde{\sigma}_{n-1}) = 0.$$

This means that all the monomial terms of $h(d_1, d_2, \dots, d_n)$ have a d_n factor. By symmetry, all the monomial terms of $h(d_1, d_2, \dots, d_n)$ also have a d_j factor for every j. Therefore all the monomial terms of $h(d_1, d_2, \dots, d_n)$ have $\sigma_n = d_1 d_2 \cdots d_n$ factor. This implies that

$$h(d_1, d_2, \dots, d_n) = \sigma_n k(d_1, d_2, \dots, d_n),$$

for a symmetric polynomial $k(d_1, d_2, \dots, d_n)$. Since $k(d_1, d_2, \dots, d_n)$ has strictly lower total degree than f, a double induction on the total degree of h can be used. This means that we may assume $k(d_1, d_2, \dots, d_n) = \tilde{k}(\sigma_1, \sigma_2, \dots, \sigma_n)$ for a polynomial $\tilde{k}$. Then we get

$$f(d_1, d_2, \dots, d_n) = \tilde{g}(\sigma_1, \sigma_2, \dots, \sigma_{n-1}) + \sigma_n \tilde{k}(\sigma_1, \sigma_2, \dots, \sigma_n).$$

For the uniqueness, we need to prove that, if $g(\sigma_1, \sigma_2, \dots, \sigma_n) = 0$ as a polynomial of $d_1, d_2, \dots, d_n$, then $g = 0$ as a polynomial of $\sigma_1, \sigma_2, \dots, \sigma_n$. Again we induct on n. By taking $d_n = 0$, we have $\tilde{\sigma}_n = \sigma_n(d_1, d_2 \dots, d_{n-1}, 0) = 0$ and $g(\tilde{\sigma}_1, \tilde{\sigma}_2, \dots, \tilde{\sigma}_{n-1}, 0) = 0$ as a polynomial of $d_1, d_2, \dots, d_{n-1}$. Then by induction, we get $g(\tilde{\sigma}_1, \tilde{\sigma}_2, \dots, \tilde{\sigma}_{n-1}, 0) = 0$ as a polynomial of $\tilde{\sigma}_1, \tilde{\sigma}_2, \dots, \tilde{\sigma}_{n-1}$. This implies that $g(\sigma_1, \sigma_2, \dots, \sigma_{n-1}, 0) = 0$ as a polynomial of $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ and further implies that $g(\sigma_1, \sigma_2, \dots, \sigma_{n-1}, \sigma_n) = \sigma_n h(\sigma_1, \sigma_2, \dots, \sigma_{n-1}, \sigma_n)$ for some polynomial h. Since $\sigma_n \neq 0$ as a polynomial of $d_1, d_2, \dots, d_n$, the assumption $g(\sigma_1, \sigma_2, \dots, \sigma_n) = 0$ as a polynomial of $d_1, d_1 \dots, d_n$ implies that $h(\sigma_1, \sigma_2, \dots, \sigma_n) = 0$ as a polynomial of $d_1, d_1 \dots, d_n$. Since h has strictly lower degree than g, a further double induction on the degree of g implies that $h = 0$ as a polynomial of $\sigma_1, \sigma_2, \dots, \sigma_n$. □

Exercise 8.23 (Newton's Identity). Consider the symmetric polynomial

$$s_k = d_1^k + d_2^k + \cdots + d_n^k.$$

Explain that for $i = 1, 2, \dots, n$, we have

$$d_i^n - \sigma_1 d_i^{n-1} + \sigma_2 d_i^{n-2} - \cdots + (-1)^{n-1} \sigma_{n-1} d_i + (-1)^n \sigma_n = 0.$$

Then use the equalities to derive

$$s_n - \sigma_1 s_{n-1} + \sigma_2 s_{n-2} - \cdots + (-1)^{n-1} \sigma_{n-1} s_1 + (-1)^n n \sigma_n = 0.$$

This gives a recursive relation for expressing s_n as a polynomial of $\sigma_1, \sigma_2, \dots, \sigma_n$.

The discussion so far assumes the linear operator is diagonalisable. To extend the result to general not necessarily diagonalisable linear operators, we establish the fact that any linear operator is a limit of diagonalisable linear operators.

First we note that if $\det(\lambda I - L)$ has no repeated roots, and $n = \dim V$, then we have n distinct eigenvalues and corresponding eigenvectors. By Proposition 7.1.4, these eigenvectors are linearly independent and therefore must be a basis of V. We conclude the following.

Proposition 8.4.2. *Any complex linear operator is the limit of a sequence of diagonalisable linear operators.*

Proof. By Proposition 7.1.10, the proposition is the consequence of the claim that any complex linear operator is approximated by a linear operator such that the characteristic polynomial has no repeated root. We will prove the claim by inducting on the dimension of the vector space. The claim is clearly true for linear operators on 1-dimensional vector space.

By the fundamental theorem of algebra (Theorem 6.1.1), any linear operator L has an eigenvalue λ. Let $H = \mathrm{Ker}(\lambda I - L)$ be the corresponding eigenspace. Then we have $V = H \oplus H'$ for some subspace H'. In the blocked form, we have

$$L = \begin{pmatrix} \lambda I & * \\ O & K \end{pmatrix}, \quad I \colon H \to H,\ K \colon H' \to H'.$$

By induction on dimension, K is approximated by an operator K', such that $\det(tI - K')$ has no repeated root. Moreover, we may approximate λI by the diagonal matrix

$$T = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_r \end{pmatrix}, \quad r = \dim H,$$

such that λ_i are very close to λ, are distinct, and are not roots of $\det(tI - K')$. Then

$$L' = \begin{pmatrix} T & * \\ O & K' \end{pmatrix}$$

approximates L, and by Exercise 7.27, has $\det(tI - L') = (t - \lambda_1) \cdots (t - \lambda_r) \det(tI - K')$. By our setup, the characteristic polynomial of L' has no repeated root. □

Since polynomials are continuous, by using Proposition 8.4.2 and taking the limit, we may extend Theorem 8.4.1 to all linear operators.

Theorem 8.4.3. *Polynomial invariants of linear operators are exactly polynomial functions of the coefficients of the characteristic polynomial.*

A key ingredient in the proof of the theorem is the continuity. The theorem cannot be applied to invariants such as the rank because it is not continuous

$$\operatorname{rank}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{cases} 1 & \text{if } a \neq 0, \\ 0 & \text{if } a = 0. \end{cases}$$

Exercise 8.24. Identify the traces of powers L^k of linear operators with the symmetric functions s_n in Exercises 8.23. Then use Theorem 8.4.3 and Newton's identity to show that polynomial invariants of linear operators are exactly polynomials of $\mathrm{tr} L^k$, $k = 1, 2, \dots$.