Pricing options on discrete realized variance with partially exact and bounded approximations

Yue Kuen Kwok

Department of Mathematics
Hong Kong University of Science and Technology

This is a joint work with Wendong Zheng.
Discrete realized variance

The discrete realized variance (DRV) of the risky asset price process $S_t$ with respect to a given tenor is defined to be

$$I_T^{(N)} = \sum_{k=1}^{N} \left( \ln \frac{S_{tk}}{S_{tk-1}} \right)^2 = \sum_{k=1}^{N} (X_{tk} - X_{tk-1})^2,$$

where $X_t = \ln S_t$ is the log asset price process.

The terminal payoff functions of options on DRV take the form: $\max(I_T^{(N)} - K, 0)$ and $\max(K - I_T^{(N)}, 0)$, where $K$ is the strike price.

Analytic approximation of the prices of options on DRV poses mathematical challenges due to the exotic path dependence of sum of the quadratic terms involving $(X_{tk} - X_{tk-1})^2$. 
Literature on analytic approximation methods for pricing options on DRV

- Keller-Ressel and Muhle-Karbe (2013) adjust the prices of options on continuous realized variance (CRV) by the asymptotic formula for the short-time limit of the *discretization gap* between the CRV and DRV.

- Sepp (2012) considers an *analytic approximation for the characteristic function* of the DRV when the price dynamics is governed by the Heston model via the combination of the distribution of the CRV under the Heston model with that of the DRV under the Black-Scholes model.

- Zheng and Kwok (2014) develop the *saddlepoint approximation formulas* for pricing options on DRV and volatility derivatives under both Lévy models and stochastic volatility models with jumps.
• Drimus and Farkas (2013) derive the *discretization adjustment* term added to the price of an option on CRV that serves to adjust the discretization effect in the discrete sampling of the realized variance. Conditional on the realization of the instantaneous variance process, the residual randomness arising from discrete sampling can be approximated by a normal random variable.

• Drimus (2012) adopts the conditioning variable approach to derive the lower bound for the prices of options on CRV under log-OU models.
Scope of the present work

- We derive efficient and accurate analytic approximation formulas for pricing options on DRV under affine stochastic volatility models with jumps using an enhanced version of the conditioning variable approach.

- We use the PEB approximation method (Lord, 2006) that adds adjustment terms by finding an analytic approximation to the residual component in the conditioning variable method.

- The application of the PEB method in pricing options on DRV relies on the adoption of either the normal distribution approximation (Drimus and Farkas, 2013) or gamma distribution approximation (Keller-Ressel and Mulhe-Karbe, 2013) that is based on some asymptotic behavior of the DRV of the underlying asset price process.
Lower bound of the call price based on conditioning variable

Let $A$ be an event of the form $\{I_T > c\}$ with $c > 0$ such that $A \in \mathcal{F}_T$.

For the undiscounted price of the call option on the DRV with strike price $K$, we have

\[
\mathbb{E}[(I_T^{(N)} - K)^+] = \mathbb{E}[(I_T^{(N)} - K)^+1_{A^c}] + \mathbb{E}[(I_T^{(N)} - K)^+1_A]
\]

\[
= \mathbb{E}[(I_T^{(N)} - K)^+1_{A^c}] + \mathbb{E}[(K - I_T^{(N)})^+1_A] + \mathbb{E}[(I_T^{(N)} - K)^1_A]
\geq \mathbb{E}[(I_T^{(N)} - K)1_{\{I_T > c\}}].
\]

The last term gives a lower bound for the undiscounted call option price and the corresponding maximum value among all choices of nonnegative values of $c$ that provides the best lower bound.
Optimal lower bound based on conditioning on $I_T = [\ln S, \ln S]_T$

1. Derive the lower bound of the undiscounted price of the option on DRV based on the conditioning variable approach, where the conditioning variable is chosen to be $I_T = [\ln S, \ln S]_T$.

2. Derive the adjustment terms that approximate the residual component.

We compute the lower bound when the characteristic function of $I_T$ is available in an analytic form.

Note that $I_T$ is chosen to be the conditioning variable since $I_T^{(N)}$ and $I_T$ are highly correlated and we take the advantage that $I_T$ is tractable.
For convenience, we write
\[ g(c) = \mathbb{E}[(I_T^{(N)} - K)1_{\{I_T>c\}}] = \int_c^\infty \mathbb{E}[I_T^{(N)} - K|I_T = y] f_I(y) \, dy, \]
where \( f_I \) is the density function of \( I_T \).

We observe that the conditional expectation \( g(c) \) increases in value with increasing \( c \) when \( c \) is small and eventually drops to zero as \( c \) becomes sufficiently large due to the rapid decay of \( f_I \).

We expect that \( g(c) \) achieves its maximum value at some finite value \( c^* \) that is close to \( K \). The critical value \( c^* \) satisfies the first order condition:
\[ g'(c) = -\mathbb{E}[I_T^{(N)} - K|I_T = c] f_I(c) = 0. \]
An illustrative plot of $g'(c)$. As deduced from the analytic property of $g'(c)$, there always exists a unique positive root of $g'(c)$. 
Evaluation of the lower bound with known characteristic function via Fourier inversion method

For a typical quadratic term \((X_{tk} - X_{tk-1})^2\) in \(I_T^{(N)}\), \(k = 1, 2, \ldots, N\), the conditional expectation can be evaluated via the following transformation:

\[
\mathbb{E}[(X_{tk} - X_{tk-1})^2 | I_T = c] = -\frac{\partial^2}{\partial \phi^2} \mathbb{E}[e^{i\phi(X_{tk} - X_{tk-1})} | I_T = c] \bigg|_{\phi=0}.
\]

We write \(\Delta_k = X_{tk} - X_{tk-1}\) and let \(f_{\Delta_k, I}\) and \(\Phi_{\Delta_k, I}\) be the joint density function and joint characteristic function of \(\Delta_k\) and \(I_T\), respectively.
By virtue of the Parseval identity and interchanging order of integration, we obtain

\[
E[e^{i\phi\Delta_k} | I_T = c] = \frac{1}{f_I(c)} \int_{-\infty}^{\infty} e^{i\phi x} f_{\Delta_k, I}(x, c) \, dx
\]

\[
= \frac{1}{f_I(c)} \int_{-\infty}^{\infty} e^{i\phi x} \frac{1}{4\pi^2} \int_{i\beta_i - \infty}^{i\beta_i + \infty} \int_{i\alpha_i - \infty}^{i\alpha_i + \infty} \Phi_{\Delta_k, I}(\alpha, \beta) e^{-i\alpha x - i\beta c} \, d\alpha \, d\beta \, dx
\]

\[
= \frac{1}{f_I(c)} \int_{i\beta_i - \infty}^{i\beta_i + \infty} \frac{1}{2\pi} \Phi_{\Delta_k, I}(\phi, \beta) e^{-i\beta c} \, d\beta,
\]

where \( \alpha = \alpha_r + i\alpha_i \) and \( \beta = \beta_r + i\beta_i \) are complex Fourier transform variables.

The respective imaginary part \( \alpha_i \) and \( \beta_i \) of the pair of transform variables \( \alpha \) and \( \beta \) are chosen to be some appropriate fixed constants to ensure convergence of the generalized Fourier transform integral.
Summing all the individual expectations of $e^{i\phi \Delta_k}$ conditional on $I_T = c$, where $k = 1, 2, \cdots, N$, the first order condition can be expressed as

$$g'(c) = \int_0^{\infty} \frac{1}{\pi} \sum_{k=1}^{N} \Re \left[ \frac{\partial^2}{\partial \phi^2} \Phi_{\Delta_k, I} (\phi, \beta_r + i \beta_i) \bigg|_{\phi=0} e^{-i(\beta_r+i\beta_i)c} \right] d\beta_r + K f_I (c) = 0,$$

where $\Re(\cdot)$ stands for the real part.

The characteristic function $\Phi_I$ of $I$ exist in closed form for most affine jump-diffusion models, including the Heston stochastic volatility model with jumps.
Heston stochastic volatility model with jumps

The joint dynamics of the asset price $S_t$ and the instantaneous variance $V_t$ are specified by

$$ \frac{dS_t}{S_t} = (r - q) \, dt + \sqrt{V_t} \left( \rho \, dW_t^2 + \sqrt{1 - \rho^2} \, dW_t^1 \right) + (e^J - 1) \, dP_t, $$

$$ dV_t = \kappa(\theta - V_t) \, dt + \varepsilon \sqrt{V_t} \, dW_t^2, $$

where $W_t^1$ and $W_t^2$ are two independent Brownian motions, $P_t$ is a Poisson process with intensity $\lambda$, the jump size $J$ is assumed to have a normal distribution with mean $\nu$ and variance $\delta^2$, $\rho$ is the correlation coefficient, $r$ and $q$ are the constant riskfree rate and dividend yield, respectively.

Thanks to the affine structure of the Heston stochastic volatility model with jumps, we are able to express $\Phi_{\Delta_k,I}(\alpha, \beta)$ in an exponential affine form.
The joint characteristic function of the triplet is given by

\[ \mathbb{E}_t[e^{uX_T+wI_T+bV_T+c}] = \exp(uX_t + wI_t + B(\tau, q)V_t + D(\tau, q)), \]

where the parameter functions $B$ and $D$ are determined by solving a Riccati system of ordinary differential equations. Here, $q = (u, w, b, c)^T$ denotes the initial values of the transform variables. It then follows that

\[
\Phi_{\Delta_k, I}(\alpha, \beta) = \mathbb{E}\left[ \mathbb{E}_{t_k}[e^{i\beta I_T}]e^{i\alpha \Delta_k} \right]
\]
\[
= \mathbb{E}\left[ e^{i\alpha \Delta_k + i\beta I_{t_k} + B(T-t_k, q_1)V_{t_k} + D(T-t_k, q_1)} \right]
\]
\[
= \mathbb{E}\left[ e^{i\beta I_{t_k-1} + B(\Delta t_k, q_2)V_{t_k-1} + D(\Delta t_k, q_2)} \right]
\]
\[
= e^{B(t_k-1, q_3)V_0 + D(\Delta t_k, q_3)},
\]

where

\[
q_1 = (0, 0, i\beta, 0)^T,
\]
\[
q_2 = (i\alpha, B(T-t_k, q_1), i\beta, D(T-t_k, q_1))^T,
\]
\[
q_3 = (0, B(\Delta t_k, q_2), i\beta, D(\Delta t_k, q_2))^T.
\]
It is convenient to express \( f_I(c) \) as a Fourier inversion integral of \( \Phi_I \) such that

\[
f_I(c) = \frac{1}{\pi} \int_0^\infty \Re \left[ \Phi_I(\beta_r + i\beta_i) e^{-i(\beta_r+i\beta_i)c} \right] d\beta_r.
\]

As a result, the first order condition can be expressed in the following compact form:

\[
g'(c) = \frac{1}{\pi} \int_0^\infty \Re \left[ \Psi(\beta_r + i\beta_i) e^{-i(\beta_r+i\beta_i)c} \right] d\beta_r = 0,
\]

where

\[
\Psi(\beta) = \sum_{k=1}^{N} \frac{\partial^2}{\partial \phi^2} \Phi_{\Delta_k, I}(\phi, \beta) \bigg|_{\phi=0} + K \Phi_I(\beta).
\]
Since the integral representation of $g''(c)$ is readily available and it takes the form

$$g''(c) = \frac{1}{\pi} \int_0^\infty \Re \left[ -i(\beta_r + i\beta_i)\psi(\beta_r + i\beta_i)e^{-i(\beta_r + i\beta_i)c} \right] \, d\beta_r,$$

one may solve $g'(c) = 0$ for the critical value $c^*$ via Newton’s iteration method with the initial guess $c = K$. One may check for $g''(c) < 0$ to ensure that $c^*$ is a maximizer of $g(c)$. 
Finally, we can calculate the optimal lower bound for the option price as follows:

$$g(c^*) = \sum_{k=1}^{N} \mathbb{E} [\Delta_k^2 1_{I_T > c^*}] - K \mathbb{P}(I_T > c^*)$$

$$= -\frac{1}{\pi} \int_{0}^{\infty} \Re \left[ \sum_{k=1}^{N} \frac{\partial^2}{\partial \phi^2} \Phi_{\Delta_k, I}(\phi, \beta_r + i\beta_i) \left|_{\phi=0} \frac{e^{-ic^* (\beta_r + i\beta_i)}}{i(\beta_r + i\beta_i)} \right] \right] d\beta_r$$

$$- K \frac{1}{\pi} \int_{0}^{\infty} \Re \left[ \Phi_{I}(\beta_r + i\beta_i) \frac{e^{-ic^* (\beta_r + i\beta_i)}}{i(\beta_r + i\beta_i)} \right] d\beta_r$$

$$= -\frac{1}{\pi} \int_{0}^{\infty} \Re \left[ \Psi(\beta_r + i\beta_i) \frac{e^{-ic^* (\beta_r + i\beta_i)}}{i(\beta_r + i\beta_i)} \right] d\beta_r.$$

Note that it is necessary to restrict $\beta_i$ to be negative.
Deficiencies of the lower bound approximation for DRV

- The lower bound derived from the conditioning variable approach works quite well for arithmetic Asian options based on conditioning on the geometric average counterpart, whereas the lower bound \( g(c^*) \) defined in the above fails to provide sufficiently accurate approximation formulas for short-maturity options on DRV.

- One major difference is that while we observe dominance of arithmetic average over geometric average, there is a lack of strict dominance of the DRV over the continuous counterpart or vice versa.

- The discrepancy between the DRV and CRV becomes more profound when maturity or sampling period becomes shorter.
Partially exact and bounded approximation

To provide a better approximation, it is natural to consider an analytic approximation to the residual terms

$$\mathbb{E}[(I^{(N)} - K)^+1_{\{I \leq c^*\}}] + \mathbb{E}[(K - I^{(N)})^+1_{\{I > c^*\}}]$$

in the decomposition of the option price. This approach is termed the partially exact and bounded (PEB) approximation. We consider an approximation to the conditional distribution of $I^{(N)}|I$ so that evaluation of the two residual terms can be performed efficiently.

In the second step of the PEB approximation for the call option on DRV, we propose two analytic approximation methods based on the normal distribution and gamma distribution approximations.
Conditional normal distribution approximation

Based on the generalized Central Limit Theorem and asymptotic analysis of the DRV of an asset price process under stochastic volatility, one may approximate $I^{(N)}|I$ by $\hat{I}^{(N)}|I$ for a sufficiently large value of $N$, where

$$\hat{I}^{(N)}|I \sim \mathcal{N}(I, \frac{2}{N}I^2).$$

The result works well under stochastic volatility with jumps.
Let $\Phi_{\hat{I}(N), I}(\alpha, \beta)$ denote the joint characteristic function of $\hat{I}(N)$ and $I$. Since $\hat{I}(N)|I$ is approximated by $\mathcal{N}\left(I, \frac{2}{N}I^2\right)$, by introducing the approximation: $e^{-\alpha^2I^2/N} \approx 1 - \frac{\alpha^2I^2}{N}$ under $O(N^{-2})$ approximation, we have

$$
\Phi_{\hat{I}(N), I}(\alpha, \beta) = \mathbb{E}[e^{i\alpha\hat{I}(N)+i\beta I}] = \mathbb{E}[\mathbb{E}[e^{i\alpha\hat{I}(N)+i\beta I}|I]] = \mathbb{E}[e^{i\alpha I - \frac{\alpha^2I^2}{N}e^{i\beta I}}]
$$

$$
\approx \mathbb{E}\left[e^{i(\alpha+\beta)I(1 - \frac{\alpha^2I^2}{N})}\right] = \Phi_I(\alpha + \beta) + \frac{\alpha^2}{N}\Phi_I^{(2)}(\alpha + \beta),
$$

where $\Phi_I$ denotes the characteristic function of $I$ and $\Phi_I^{(2)}$ refers to the second order derivative of $\Phi_I$. 
We derive an analytic approximation of the two residual terms by writing them as Fourier integrals via the Parseval Theorem. For the first residual term, we propose

\[
\mathbb{E}\left[ (I(N) - K) + 1\{I \leq c^*\} \right]
\approx \mathbb{E}\left[ (\hat{I}(N) - K) + 1\{I \leq c^*\} \right] = \frac{1}{4\pi^2} \int_{ib}^{ib+\infty} \int_{ia}^{ia+\infty} e^{-i\alpha K - i\beta c^*} \frac{\Phi_{\hat{I}(N), I(\alpha, \beta)}}{i\beta \alpha^2} \, d\alpha \, d\beta,
\]

where \(a < 0\) and \(b > 0\) are chosen such that the integration contours are within the domain of convergence of the two-dimensional generalized Fourier transform.
We apply an analytic approximation of the joint characteristic function \( \Phi_{\bar{I}(N), I}(\alpha, \beta) \). For convenience, we write \( z = \alpha + \beta \) so that

\[
\mathbb{E}[(I^{(N)} - K)^+ 1\{I \leq c^*\}] \\
\approx \frac{1}{2\pi} \int_{iu-\infty}^{iu+\infty} e^{-iz c^*} \Phi_I(z) \frac{1}{2\pi} \int_{ia-\infty}^{ia+\infty} \frac{e^{-i\alpha(K-c^*)}}{i(z-\alpha)\alpha^2} \, d\alpha \, dz \\
+ \frac{1}{2\pi} \int_{iu-\infty}^{iu+\infty} e^{-iz c^*} \Phi^{(2)}_I(z) \frac{1}{N} \frac{1}{2\pi} \int_{ia-\infty}^{ia+\infty} \frac{e^{-i\alpha(K-c^*)}}{i(z-\alpha)} \, d\alpha \, dz,
\]

where \( u = a + b > a \) specifies the horizontal contour of the complex integral with respect to \( z \).
In a similar manner, we approximate the second residual term by

\[
\mathbb{E}[(K - I^{(N)}) + 1_{\{I>c^*\}}] \\
\approx \mathbb{E}[(K - \hat{I}^{(N)}) + 1_{\{I>c^*\}}] \\
= \frac{1}{4\pi^2} \int_{i\hat{b} - \infty}^{i\hat{b} + \infty} \int_{i\hat{a} - \infty}^{i\hat{a} + \infty} e^{-i\alpha K - i\beta c^*} \frac{\Phi_{\hat{I}(N),\hat{I}(\alpha, \beta)}}{-i\beta \alpha^2} \, d\alpha \, d\beta,
\]

where \( \hat{a} > 0 \) and \( \hat{b} < 0 \) are chosen to ensure that the integration contours are within the domain of convergence of the two-dimensional generalized Fourier transform. By letting \( z = \alpha + \beta \), we obtain

\[
\mathbb{E}[(K - I^{(N)}) + 1_{\{I>c^*\}}] \\
\approx -\frac{1}{2\pi} \int_{i\hat{u} - \infty}^{i\hat{u} + \infty} e^{-icz^*} \Phi_{I}(z) \frac{1}{2\pi} \int_{i\hat{a} - \infty}^{i\hat{a} + \infty} \frac{e^{-i\alpha(K-c^*)}}{i(z - \alpha)\alpha^2} \, d\alpha \, dz \\
- \frac{1}{2\pi} \int_{i\hat{u} - \infty}^{i\hat{u} + \infty} e^{-icz^*} \Phi_{I}^{(2)}(z) \frac{1}{2\pi} \int_{i\hat{a} - \infty}^{i\hat{a} + \infty} \frac{e^{-i\alpha(K-c^*)}}{i(z - \alpha)} \, d\alpha \, dz,
\]

where \( \hat{u} = \hat{a} + \hat{b} < \hat{a} \) specifies the horizontal contour of the complex integral with respect to \( z \).
The corresponding integrands in the Fourier integrals in eqs. (i) and (ii) are identical. The two Fourier integrals differ only in the choices of the contours, where one is along a horizontal contour below the real axis oriented in the positive direction while the other is along a horizontal contour above the real axis oriented in the negative direction.

By combining the respective first terms in eqs. (i) and (ii), we obtain

\[
A = \frac{1}{2\pi} \int_{i\tilde{u}-\infty}^{i\tilde{u}+\infty} e^{-icz^*} \, \Phi_I(z) \, \frac{1}{2\pi} \oint_C \frac{e^{-i\alpha(K-c^*)}}{i(z-\alpha)\alpha^2} \, d\alpha \, dz
\]

\[
= \frac{1}{2\pi} \int_{i\tilde{u}-\infty}^{i\tilde{u}+\infty} e^{-icz^*} \, \Phi_I(z) \, \frac{1 + iz(c^* - K) - e^{iz(c^*-K)}}{z^2} \, dz.
\]
By combining the respective second terms in eqs. (i) and (ii), we obtain

\[ B = \frac{1}{2\pi} \int_{i\tilde{u}+\infty}^{i\tilde{u}-\infty} \frac{e^{-izc^*} \Phi_I^{(2)}(z)}{N} 2 \pi \int_C \frac{e^{-i\alpha(K-c^*)}}{i(z - \alpha)} \, d\alpha \, dz \]

\[ = \frac{1}{2\pi} \int_{i\tilde{u}+\infty}^{i\tilde{u}-\infty} e^{-izc^*} \Phi_I^{(2)}(z) \left[ -e^{-iz(K-c^*)} \right] \, dz, \]

\[ = \frac{K^2}{2\pi N} \int_{i\tilde{u}-\infty}^{i\tilde{u}+\infty} e^{-izK} \Phi_I(z) \, dz \quad \text{(applying integration by parts twice)} \]

\[ = \frac{K^2}{N} f_I(K). \]

We manage to express the approximation of the two residual terms as the sum of an one-dimensional integral and an explicitly known term.
Financial interpretation of the two terms

The term $A$ is simply equal to the following quantity:

$$\mathbb{E}[(I - K)^{+}1_{I \leq c^*}] + \mathbb{E}[(K - I)^{+}1_{I > c^*}] .$$

Keeping the single term $A$ alone in the analytic approximation would be equivalent to approximating the two residual terms by simply replacing $I^{(N)}$ by $I$.

Since the optimal solution $c^* \approx K$, we expect that both $\{K < I \leq c^*\}$ and $\{K \geq I > c^*\}$ are small probability events. Therefore, the correction contributed by $A$ would be small and secondary.
The second term $B$ is seen to be identical to the discretization adjustment term that accounts for the discrete sampling effect of DRV in the approximation of $\mathbb{E}[(I^{(N)} - K)^+]$ by $\mathbb{E}[(I - K)^+]$.

It is interesting to observe that $B$ has dependence on $N$ but no dependence on $c^*$ while $A$ has the reverse properties of functional dependence. The term $B$ provides the discretization gap between $I^{(N)}$ and $I$ that is not captured by the optimal lower bound. In general, the contribution of $B$ as an adjustment term added to the optimal lower bound is more significant compared to that of $A$. 
Conditional gamma distribution approximation

The conditional normal distribution is based on the asymptotic behavior of $I^{(N)}$ as $N \to \infty$. When we consider pricing of short-maturity options on DRV, the asymptotic behavior of the DRV as $T \to 0$ is more relevant. The annualized CRV tends to $V_0$ as $T \to 0$ while the DRV converges in distribution to a gamma distribution with shape parameter $N/2$ and scale parameter $2V_0/N$, where $V_0$ is the initial value of the instantaneous variance.

We propose to approximate $I^{(N)}$ by $\hat{I}^{(N)}$, which has a gamma distribution with shape parameter $N/2$ and scale parameter $2I/N$ conditional on $I$, where

$$\hat{I}^{(N)}|I \sim \text{gamma}(N/2, 2I/N).$$
The above gamma approximation has the same conditional mean and variance as the earlier normal approximation.

The gamma approximation is advantageous over the normal distribution in the following two aspects. Firstly, it becomes exact in asymptotic limit as $T \to 0$. Secondly, the gamma approximation retains nonnegativity of $I^{(N)}|I$. 
We express the residual terms as nested conditional expectation:

\[ \mathbb{E}[\mathbb{E}[(K - I^{(N)})^+ | I] 1_{\{I > c^*\}}] + \mathbb{E}[\mathbb{E}[(I^{(N)} - K)^+ | I] 1_{\{I \leq c^*\}}]. \]

Substituting the explicit form of the gamma density function and applying the put-call parity relation, the inner expectation can be evaluated as follows:

\[
\mathbb{E}[(K - I^{(N)})^+ | I] \\
\approx \int_0^K (K - y) \frac{y^{N/2-1}e^{-Ny/2I}}{\Gamma(N/2)(2I/N)^{N/2}} \, dy \\
= \frac{1}{\Gamma(N/2)} \left[ (K - I) \gamma \left( \frac{N}{2}, \frac{KN}{2I} \right) + K \exp \left( \left( \frac{N}{2} - 1 \right) \ln \frac{KN}{2I} - \frac{KN}{2I} \right) \right]
\]

\[
\mathbb{E}[(I^{(N)} - K)^+ | I] \\
= E[I^{(N)} | I] - K + E[(K - I^{(N)})^+ | I] \\
\approx I - K + \frac{1}{\Gamma(N/2)} \left[ (K - I) \gamma \left( \frac{N}{2}, \frac{KN}{2I} \right) + K \exp \left( \left( \frac{N}{2} - 1 \right) \ln \frac{KN}{2I} - \frac{KN}{2I} \right) \right],
\]

where \( \Gamma(\cdot) \) is the gamma function and \( \gamma(s, x) = \int_0^x z^{s-1}e^{-t} \, dz \) is the lower incomplete gamma function.
Putting the above results together, the correction term $C_g$ that is added to the optimal lower bound based on the conditional gamma distribution approximation is given by

$$C_g = \int_{0}^{\infty} G(y) f_I(y) \, dy + \int_{0}^{c^*} (y - K) f_I(y) \, dy,$$

where

$$G(y) = \frac{1}{\Gamma\left(\frac{N}{2}\right)} \left[ (K - y) \gamma\left(\frac{N}{2}, \frac{KN}{2y}\right) + K \exp\left((\frac{N}{2} - 1) \ln\frac{KN}{2y} - \frac{KN}{2y}\right) \right].$$

Unlike the earlier derivation of the conditional normal distribution approximation, we do not use the method of double Fourier transform in the above derivation of the conditional gamma distribution approximation. Since the joint characteristic function of $I^{(N)}$ and $I$ is intractable, the double Fourier transform method cannot be applied.
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<td>67.630</td>
<td>11.696(0.88%)</td>
<td>11.671(1.10%)</td>
<td>11.788(0.10%)</td>
<td>11.791(0.09%)</td>
<td>11.801(0.011)</td>
</tr>
<tr>
<td></td>
<td>N=252</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>90.836</td>
<td>34.210(0.62%)</td>
<td>34.160(0.76%)</td>
<td>34.382(0.12%)</td>
<td>34.379(0.13%)</td>
<td>34.423(0.011)</td>
</tr>
<tr>
<td></td>
<td>113.545</td>
<td>23.131(0.89%)</td>
<td>23.088(1.07%)</td>
<td>23.328(0.04%)</td>
<td>23.341(0.01%)</td>
<td>23.338(0.011)</td>
</tr>
<tr>
<td></td>
<td>136.254</td>
<td>14.652(2.28%)</td>
<td>14.642(2.35%)</td>
<td>15.077(0.55%)</td>
<td>15.059(0.43%)</td>
<td>14.994(0.011)</td>
</tr>
</tbody>
</table>

All the option prices are interpreted as basis points, where the calculated results have been multiplied by $10^4$. “Cont” refers to the prices of the call options on the CRV, “LB” means the lower bound approximation, “PEBn” means the PEB approximation with normal distribution, “PEBg” means the PEB approximation with gamma distribution, and “MC” refers to the Monte Carlo simulation results using the Euler scheme. The relative errors (RE) uses the Monte Carlo simulation results as the benchmark for comparison.
• The performance of the lower bound approximation is quite similar to the crude approximation using the price of the call option on the CRV. For short-maturity options, though numerical accuracy is not quite satisfactory in general, the lower bound approximation slightly outperforms the “Cont” approximation.

• Both the PEB approximation methods with the normal or gamma distribution approximation have shown significant improvement over the lower bound approximation.
• The PEB method with the normal distribution approximation fails to deliver a consistent accurate approximation for the one-month call options.

• The PEB method with the gamma distribution approximation provides very accurate results for the short-maturity options. This is expected since the gamma distribution approximation is exact in the asymptotic limit when $T \to 0$. The gamma distribution approximation remains to perform equally well for relatively long maturities.
Plot of percentage error in numerical pricing of short-dated ($N = 20$) and long-dated ($N = 126$) call options on daily sampled realized variance of the three approximation methods against moneyness. The volatility of variance parameter $\varepsilon$ is set to be 0.9.
• For short-maturity options \((N = 20)\), the normal and gamma distribution approximations are seen to exhibit comparable performance, while the LB approximation remains to be inferior.

• When the maturity of the option is lengthened to be half a year \((N = 126)\), the percentage errors in all three approximations are within 1%.

• In general, we find that it is reliable to use the gamma distribution approximation for short-maturity options and the normal distribution approximation for long-maturity options.
Conclusion

- We propose an extension of the PEB approximation for pricing options on DRV. Our numerical tests demonstrate that the PEB approximation formulas provide very good performance for pricing options on DRV under the Heston stochastic volatility model with jumps, without the shortcoming exhibited in other analytic approximation methods where accuracy may deteriorate substantially in pricing options with short maturities.

- The high level of numerical accuracy is attributed to the adoption of either the normal or gamma approximation of the distribution of DRV conditional on the quadratic variation.
Thanks to the affine structure of the Heston model with jumps, the PEB approximation is seen to be particularly effective for pricing options on DRV under the Heston model with jumps.

Since the gamma distribution approximation is exact in asymptotic limit as maturity tends to zero, the PEB method with the gamma distribution approximation is more reliable when pricing short-maturity options on DRV.

For options with medium to long maturities, the gamma distribution is closely connected to the normal distribution. The approximation using either distribution is seen to be highly reliable and provide numerical accuracy within 1% error for most reasonable ranges of model parameter values.