Mathematical Excalibur

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Olympiad Corner

Below are the problems of the Team Selection Test 1 for the Dutch IMO team held in June, 2014.

Problem 1. Determine all pairs (a,b) of positive integers satisfying

$$a^2+b \mid a^2b+a \text{ and } b^2-a \mid ab^2+b.$$

Problem 2. Let $\triangle ABC$ be a triangle. Let M be the midpoint of BC and let D be a point on the interior of side AB. The intersection of AM and CD is called E. Suppose that $|AD|=|DE|$. Prove that $|AB|=|CE|$.

Problem 3. Let a, b and c be rational numbers for which $a+bc$, $b+ac$ and $a+b$ are all non-zero and for which we have

$$\frac{1}{a+bc}+\frac{1}{b+ac}=\frac{1}{a+b}.$$

Prove that $\sqrt{(c-3)(c+1)}$ is rational.

Problem 4. Let $\triangle ABC$ be a triangle with $|AC|=2|AB|$ and let O be its circumcenter. Let D be the intersection of the angle bisector of $\angle A$ and BC. Let E be the orthogonal projection of O on AD and let $F\neq D$ be a point on AD satisfying $|CD|=|CF|$. Prove that $\angle EBF=\angle ECF$.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***April 10, 2015***.

For individual subscription for the next five issues for the 14-15 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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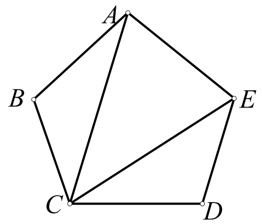
Polygonal Problems

Kin Yin Li

In geometry textbooks, we often come across problems about triangles and quadrilaterals. In this article we will present some problems about n-sided polygons with $n > 4$. This type of problem appears every few years in math olympiads of many countries.

Example 1. Prove that if $ABCDE$ is a convex pentagon with all sides equal and $\angle A\geq\angle B\geq\angle C\geq\angle D\geq\angle E$, then it is a regular pentagon.

Solution.



Since

$$AC=2AB\sin\frac{\angle B}{2}\geq 2CD\sin\frac{\angle D}{2}=CE,$$

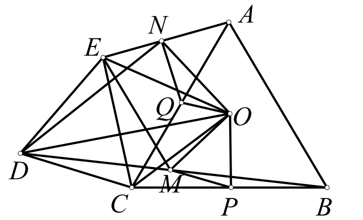
we get $\angle AEC\geq\angle EAC$. Next,

$$\angle EAC=\angle A-\frac{180^\circ-\angle B}{2}=\angle A+\frac{\angle B}{2}-90^\circ$$
$$\geq\angle E+\frac{\angle D}{2}-90^\circ=\angle E-\frac{180^\circ-\angle D}{2}$$
$$=\angle AEC$$

Hence, $\angle EAC=\angle AEC$. Then equality holds everywhere above so that $\angle A=\angle E$ and we are done.

Example 2. (*Bulgaria*, 1979) In convex pentagon $ABCDE$, $\triangle ABC$ and $\triangle CDE$ are equilateral. Prove that if O is the center of $\triangle ABC$ and M, N are midpoints of BD, AE respectively, then $\triangle OME\sim\triangle OND$.

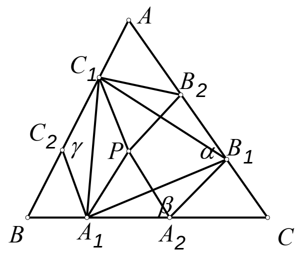
Solution.



Let P, Q be the midpoints of BC, AC respectively. Observe that $\angle COP=60^\circ$, $OC=2OP$, $PM\|CD$, $\angle DCE=60^\circ$ and $EC = DC = 2MP$. Then rotating about O by 60° clockwise and follow by doubling distance from O, we see $\triangle OPM$ goes to $\triangle OCE$. Hence $\angle EOM=\angle COP=60^\circ$ and $OE=2OM$. Similarly we can rotate about O by 60° counterclockwise and double distance from O to bring $\triangle OQN$ to $\triangle OCD$. Then $\angle DON = 60^\circ$, $OD = 2ON$ and so $\triangle OME\sim\triangle OND$.

Example 3. (*IMO* 2005) Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC, B_1, B_2 on CA and C_1, C_2 on AB, so that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths. Prove that A_1B_2, B_1C_2 and C_1A_2 are concurrent.

Solution.



Let P be the point inside $\triangle ABC$ such that $\triangle A_1A_2P$ is equilateral. Observe that $A_1P\|C_1C_2$ and $A_1P=C_1C_2$. So $A_1PC_1C_2$ is a rhombus. Similarly, $B_1PB_2B_1$ is a rhombus. So $\triangle C_1B_2P$ is equilateral. Let $\alpha=\angle B_2B_1A_2$, $\beta=\angle B_1A_2A_1$ and $\gamma=\angle C_1C_2A_1$. Then α and β are external angles of $\triangle CB_1A_2$ with $\angle C=60^\circ$. So $\alpha+\beta=240^\circ$. Now $\angle B_2PA_2=\alpha$ and $\angle C_1PA_1=\gamma$. So $\alpha+\gamma=360^\circ-(\angle C_1PB_2+\angle A_1PA_2)=240^\circ$. So $\beta=\gamma$. Similarly, $\angle C_1B_2B_1=\beta$. Hence, $\triangle A_1A_2B_1$, $\triangle B_1B_2C_1$ and $\triangle C_1C_2A_1$ are congruent, which implies $\triangle A_1B_1C_1$ is equilateral. Since sides of $A_1A_2B_1B_2C_1C_2$ have equal lengths, lines A_1B_2, B_1C_2 and C_1A_2 are the perpendicular bisectors of the sides of $\triangle A_1B_1C_1$ and the result follows.

(continued on page 2)

<u>*Example 4.*</u> (*Czechoslovakia*, 1974) Prove that if a circumscribed hexagon *ABCDEF* satisfies

$$AB=BC,\ CD=DE \text{ and } EF=FA,$$

then the area of $\triangle ACE$ is less than or equal to the area of $\triangle BDF$.

<u>*Solution.*</u> Let O be the circumcenter of hexagon *ABCDEF* and R be the radius of the circumcircle. Let

$$\alpha=\angle CAE, \beta=\angle AEC, \gamma=\angle ACE.$$

From the given conditions on the sides, we get

$$\angle AOB=\angle BOC=\beta,$$
$$\angle COD=\angle DOE=\alpha,$$
$$\angle EOF=\angle FOA=\gamma.$$

Let $[XYZ]$ denote the area of $\triangle XYZ$. We have

$$\begin{aligned}[ACE]&=\frac{EC\cdot CA\cdot AE}{4R}\\ &=\frac{2R\sin\alpha\cdot 2R\sin\beta\cdot 2R\sin\gamma}{4R}\\ &=2R^2\sin\alpha\sin\beta\sin\gamma.\end{aligned}$$

Similarly,

$$[BDF]=2R^2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\alpha}{2}.$$

Now for positive α, β, γ satisfying $\alpha+\beta+\gamma=180^\circ$, we have

$$\begin{aligned}&\sin^2\alpha\sin^2\beta\sin^2\gamma\\ &=(\sin\alpha\sin\beta)(\sin\gamma\sin\alpha)(\sin\beta\sin\gamma)\\ &=\prod_{cyc}\frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}\\ &\le\prod_{cyc}\frac{1-\cos(\alpha+\beta)}{2}\\ &=\sin^2\frac{\alpha+\beta}{2}\sin^2\frac{\beta+\gamma}{2}\sin^2\frac{\gamma+\alpha}{2}.\end{aligned}$$

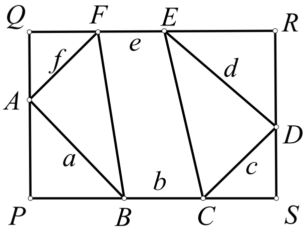
Therefore, $[ACE]\le[BDF]$.

<u>*Example 5.*</u> (*IMO* 1996) Let *ABCDEF* be a convex hexagon such that *AB* is parallel to *DE*, *BC* is parallel to *EF* and *CD* is parallel to *FA*. Let R_A, R_C, R_E be the circumradii of triangles *FAB*, *BCD*, *DEF* respectively, and let P denote the perimeter of the hexagon. Prove that

$$R_A+R_C+R_E\ge\frac{P}{2}.$$

<u>*Solution.*</u> Let a, b, c, d, e, f denote the lengths of the sides *AB*, *BC*, *CD*, *DE*, *EF*, *FA* respectively. By the parallel conditions, we have $\angle A=\angle D$, $\angle B=\angle E$, $\angle C=\angle F$.

Consider rectangle *PQRS* such that A is on *PQ*; *F*,*E* are on *QR*; D is on *RS* and *B*,*C* are on *SP*.



We have $BF\ge PQ=SR$. So $2BF\ge PQ+SR$, which is the same as

$$2BF\ge(a\sin B+f\sin C)+(c\sin C+d\sin B).$$

Similarly,

$$2BD\ge(c\sin A+b\sin B)+(c\sin B+f\sin A),$$
$$2DF\ge(c\sin C+d\sin A)+(a\sin A+b\sin C).$$

Next, by the extended sine law,

$$R_A=\frac{BF}{2\sin A}, R_C=\frac{BD}{2\sin C}, R_E=\frac{DE}{2\sin E}.$$

Then using the inequalities and equations above, we have

$$\begin{aligned}&R_A+R_C+R_E\\ &\ge\frac{a}{4}\left(\frac{\sin B}{\sin A}+\frac{\sin A}{\sin B}\right)+\cdots+\frac{f}{4}\left(\frac{\sin A}{\sin F}+\frac{\sin F}{\sin A}\right)\\ &\ge\frac{a+b+c+d+e+f}{2}=\frac{P}{2}.\end{aligned}$$

<u>*Example 6.*</u> (*Great Britain*, 1988) Let four consecutive vertices A, B, C, D of a regular polygon satisfy

$$\frac{1}{AB}=\frac{1}{AC}+\frac{1}{AD}.$$

Determine the number of sides of the polygon.

<u>*Solution.*</u> Let the circumcircle of the polygon have center O and radius R. Let $\alpha=\angle AOB$, then $0<3\alpha=\angle AOD<360^\circ$. So $0<\alpha<120^\circ$. Also, from

$$AB=2R\sin\frac{\alpha}{2},\qquad AC=2R\sin\alpha,$$
$$AD=2R\sin\frac{3\alpha}{2},$$

we get

$$\frac{1}{\sin\frac{\alpha}{2}}=\frac{1}{\sin\alpha}+\frac{1}{\sin\frac{3\alpha}{2}}.$$

Clearing denominators, we have

$$\begin{aligned}0&=\sin\alpha\sin\frac{3\alpha}{2}-\left(\sin\alpha+\sin\frac{3\alpha}{2}\right)\sin\frac{\alpha}{2}\\ &=\frac{1}{2}\left(\cos\frac{\alpha}{2}-\cos\frac{5\alpha}{2}\right)-\frac{1}{2}\left(\cos\frac{\alpha}{2}-\cos\frac{3\alpha}{2}\right)\\ &\qquad-\frac{1}{2}(\cos\alpha-\cos 2\alpha)\\ &=\frac{1}{2}\left(\left(\cos\frac{3\alpha}{2}+\cos 2\alpha\right)-\left(\cos\alpha+\cos\frac{5\alpha}{2}\right)\right)\\ &=\cos\frac{7\alpha}{4}\left(\cos\frac{\alpha}{4}-\cos\frac{3\alpha}{4}\right)\\ &=2\cos\frac{7\alpha}{4}\sin\frac{\alpha}{4}\sin\frac{\alpha}{2}.\end{aligned}$$

Then $7\alpha/4=90^\circ$, that is $\alpha=360^\circ/7$. So the polygon has 7 sides.

<u>*Example 7.*</u> (*Austria*, 1973) Prove that if the angles of a convex octagon are all equal and the ratio of all pairs of adjacent sides is rational, then each pair of opposite sides has equal length.

<u>*Solution.*</u> Without loss of generality, we may assume the sides of such a polygon $A_1A_2\ldots A_8$ are rational (since the conclusion is the same for octagons similar to such an octagon). Now the sum of all angles of the octagon is $6\times180^\circ$. Hence each angle is 45°.

Let v_n be the vector from A_n to A_{n+1} for $n=1,2,\ldots,8$ (with $A_9=A_1$). Then the angle between v_n and v_{n+1} at the origin is 45°. Observe that the sum of these vectors is zero since we start at A_1 and traverse the octagon once to return to A_1.

Let i and j be a pair of unit vectors perpendicular to each other at the origin. By rotation, we may assume v_1 is a vector in the i direction and v_3 is in the j direction. Then $v_1+v_5=xi$ and $v_3+v_7=yj$ for some rational x and y. Also,

$$v_2+v_4+v_6+v_8=r\sqrt{2}i\pm r\sqrt{2}j$$

for some rational r. Then

$$(x+r\sqrt{2})i+(y\pm r\sqrt{2})j=\sum_{n=1}^{8}v_n=0.$$

Since, x and r are rational, we must have $x=r=0$. That is, $v_5=-v_1$. By rotating the i, j vectors by 45°, similarly we get $v_6=-v_2$. Then also $v_7=-v_3$ and $v_8=-v_4$. The result follows.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***April 10, 2015.***

Problem 461. Inside rectangle $ABCD$, there is a circle. Points W, X, Y, Z are on the circle such that lines AW, BX, CY, DZ are tangent to the circle. If AW=3, BX=4, CY=5, then find DZ with proof.

Problem 462. For all $x_1, x_2, \ldots, x_n \ge 0$, let $x_{n+1} = x_1$, then prove that

$$\sum_{k=1}^{n}\sqrt{\frac{1}{(x_k+1)^2}+\frac{x_{k+1}^2}{(x_{k+1}+1)^2}} \ge \frac{n}{\sqrt{2}}.$$

Problem 463. Let S be a set with 20 elements. N 2-element subsets of S are chosen with no two of these subsets equal. Find the least number N such that among any 3 elements in S, there exist 2 of them belong to one of the N chosen subsets.

Problem 464. Determine all positive integers n such that for n, there exists an integer m with 2^n-1 divides m^2+289.

Problem 465. Points A, E, D, C, F, B lie on a circle Γ in clockwise order. Rays AD, BC, the tangents to Γ at E and at F pass through P. Chord EF meets chords AD and BC at M and N respectively. Prove that lines AB, CD, EF are concurrent.

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Solutions

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Problem 456. Suppose $x_1, x_2, \ldots, x_n$ are non-negative and their sum is 1. Prove that there exists a permutation σ of $\{1,2,\cdots,n\}$ such that

$x_{\sigma(1)}x_{\sigma(2)}+x_{\sigma(2)}x_{\sigma(3)}+\cdots+x_{\sigma(n)}x_{\sigma(1)} \le 1/n.$

Solution. **CHAN Long Tin** (Cambridge University, Year 3), **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania), **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4), **Samiron SADHUKHAN** (Kendriya Vidyalaya, India) and **WONG Yat** (G. T. (Ellen Yeung) College).

Assume the contrary is true. Let $\sigma(n+1) = \sigma(1)$ for all permutations σ. For $1 \le i < j \le n$, the terms x_ix_j and x_jx_i appear a total of $2n(n-2)!$ times in

$$\sum_{n\in S_n}\sum_{k=1}^{n} x_{\sigma(k)}x_{\sigma(k+1)}.$$

So, we have

$$\begin{aligned}\frac{n!}{n} &< \sum_{\sigma\in S_n}\sum_{k=1}^{n} x_{\sigma(k)}x_{\sigma(k+1)}\\ &= 2n(n-2)!\sum_{1\le i<j\le n} x_ix_j\\ &= n(n-2)!\left[\left(\sum_{i=1}^{n}x_i\right)^2-\sum_{i=1}^{n}x_i^2\right]\\ &= n(n-2)!\left(1-\sum_{i=1}^{n}x_i^2\right).\end{aligned}$$

This simplifies to (*) $\sum_{i=1}^{n}x_i^2 < \frac{1}{n}$. However, by the Cauchy-Schwarz inequality,

$$n\sum_{i=1}^{n}x_i^2 = \sum_{i=1}^{n}1^2\sum_{i=1}^{n}x_i^2 \ge \left(\sum_{i=1}^{n}x_i\right)^2 = 1,$$

which contradicts (*).

Problem 457. Prove that for each n = 1,2,3,…, there exist integers a, b such that if integers x, y are relatively prime, then $\sqrt{(a-x)^2+(b-y)^2} > n.$

Solution. **Samiron SADHUKHAN** (Kendriya Vidyalaya, India) and **WONG Yat** (G. T. (Ellen Yeung) College).

There are $(2n+1)^2$ ordered pairs (r,s) of integers satisfying $|r|, |s| \le n$. Assign a <u>distinct</u> prime number $p_{r,s}$ to each such (r,s). By the Chinese remainder theorem, there exist integers a,b such that for all integers r; s satisfying $|r|, |s| \le n$, we have $a\equiv r \pmod{p_{r,s}}$ and $b\equiv s \pmod{p_{r,s}}$.

Let integers x, y be relatively prime. Assume (x,y) has distance at most n from (a,b). Then $|a-x|\le n$ and $|b-y|\le n$. Let $a-x=r$ and $b-y=s$. Then $x=a-r$ and $y=b-s$ are multiples of $p_{r,s}$, contradicting $\gcd(x,y) = 1$. Therefore,

$$\sqrt{(a-x)^2+(b-y)^2} > n.$$

Problem 458. Nonempty sets A_1, A_2, A_3 form a partition of $\{1,2,\ldots,n\}$. If $x+y=z$ have no solution with x in A_i, y in A_j, z in A_k and $\{i,j,k\}=\{1,2,3\}$, then prove that A_1, A_2, A_3 cannot have the same number of elements.

Solution. **Oliver GEUPEL** (Brühl, NRW, Germany) and **John GLIMMS**.

Without loss of generality, say $1\in A_1$ and the smallest element in $A_2 \cup A_3$ is $b\in A_2$. Let the elements in A_3 be $c_1, c_2, \ldots, c_k$ in increasing order.

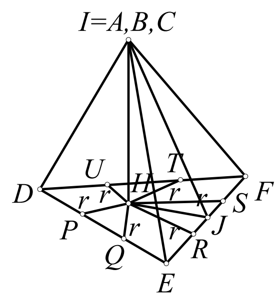
Assume $c_{i+1}-c_i=1$ for some i. Then take i to be the smallest possible. Since $b\in A_2$, the equations $(c_i-b)+b=c_i$ and $(c_i-b+1)+b=c_{i+1}$ imply c_i-b and c_i-b+1 are both not in A_1.

Since $1\in A_1$ and $(c_i-b)+1= c_i-b+1$, so either c_i-b+1 and c_i-b both are in A_2 or both are in A_3. Since i is smallest such that $c_{i+1}-c_i=1$, so c_i-b+1 and c_i-b cannot be in A_3. However, c_i-b+1 and c_i-b in A_2, $b-1$ in A_1 (by property of b) and $(b-1)+(c_i-b+1)=c_i$ lead to contradiction. So $c_{i+1}-c_i \ge 2$ for all i.

Finally, since $1+(c_i-1)=c_i$, we get $c_i-1\notin B$. Hence $c_i-1\in A$. Then A_1 contains 1, c_1-1, c_2-1, ... , c_k-1. Therefore, A_1 has more elements than A_3.

Problem 459. H is the orthocenter of acute $\triangle ABC$. D,E,F are midpoints of sides BC, CA, AB respectively. Inside $\triangle ABC$, a circle with center H meets DE at P,Q, EF at R,S, FD at T,U. Prove that $CP=CQ=AR=AS=BT=BU$.

Solution. **John GLIMMS.**



Let lines AH and FE meet at J. From $AH\perp BC$ and $BC \parallel FE$, we get FE is perpendicular to AJ and HJ. By folding along DE, EF and FD, we can make a tetrahedron having $\triangle DEF$ as the base and points A, B, C meet at a point I. Then FE is perpendicular to IJ and HJ. So FE is perpendicular to the plane through I,J,H. Then $FE\perp IH$. Similarly, $DE\perp IH$. Then the plane through D,E,F is perpendicular to IH. By Pythagoras' theorem, $IH^2+r^2 = CP^2 = CQ^2 = AR^2 = AS^2 = BT^2 = BV^2$, where r is the radius of the circle.

Other commended solvers: **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India), **Andrea FANCHINI**

(Cantú, Italy), **William FUNG**, **Oliver GEUPEL** (Brühl, NRW, Germany), **MANOLOUDIS Apostolis** (4 High School of Korydallos, Piraeus, Greece), **Samiron SADHUKHAN** (Kendriya Vidyalaya, India), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 460. If $x,y,z > 0$ and $x+y+z+2 = xyz$, then prove that

$$x+y+z+6 \ge 2\left(\sqrt{yz}+\sqrt{zx}+\sqrt{xy}\right) .$$

Solution. **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India), **CHAN Long Tin** (Cambridge University, Year 3), **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania), **Oliver GEUPEL** (Brühl, NRW, Germany), **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4), **Vijaya Prasad NULLARI** (Retired Principal, AP Educational Service, India), **Nicuşor ZLOTA** ("Traian Vuia" Technical College,Focşani, Romania) and **Titu ZVONARU** (Comăneşti, Romania).

Let

$$a=\frac{1}{1+x}, b=\frac{1}{1+y}, c=\frac{1}{1+z}.$$

Using $x+y+z+2 = xyz$, we get $a+b+c = 1$. Then $x = (1-a)/a = (b+c)/a$ and similarly $y=(c+a)/b$ and $z=(a+b)/c$. By the AM-GM inequality, we have

$$\begin{aligned} & x+y+z+6 \\ &= \sum_{cyc}\frac{b+c}{a}+6 \\ &= \sum_{cyc}\left(\frac{c+a}{c}+\frac{a+b}{b}\right) \\ &\ge 2\sum_{cyc}\sqrt{\frac{(c+a)(a+b)}{bc}} \\ &= 2\left(\sqrt{yz}+\sqrt{zx}+\sqrt{xy}\right). \end{aligned}$$

Other commended solvers: **Paolo PERFETTI** (Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy), **WONG Yat** (G. T. (Ellen Yeung) College).

Olympiad Corner

(*Continued from page 1*)

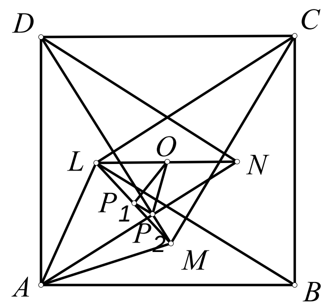
Problem 5. On each of the 2014^2 squares of a 2014×2014-board a light bulb is put. Light bulbs can be either on or off. In the starting situation a number of light bulbs are on. A move consists of choosing a row or column in which at least 1007 light bulbs are on and changing the state of all 2014 light bulbs in this row or column (from on to off or from off to on). Find the smallest non-negative integer k such that from each starting situation there is a finite sequence of moves to a situation in which at most k light bulbs are on.

Polygonal Problems

(*Continued from page 2*)

<u>*Example 8.*</u> (*IMO* 1997) Equilateral triangles *ABK, BCL, CDM, DAN* are constructed inside the square *ABCD*. Prove that the midpoints of the four segments *KL, LM, MN, NK* and the midpints of the eight segments *AK, BK, BL, CL, CM, DM, DN, AN* are the twelve vertices of a regular dodecagon.

<u>*Solution.*</u>



Let us denote the midpoints of segments *LM, AN, BL, MN, BK, CM, NK, CL, DN, KL, DM, AK* by P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}, respectively. To prove the dodecagon

$$P_1P_2P_3P_4P_5P_6P_7P_8P_9P_{10}P_{11}P_{12}$$

is regular, we observe that $BL=BA$ and $\angle ABL=30°$. Then $\angle BAL=75°$. Similarly $\angle DAM=75°$. So

$$\angle LAM = \angle BAL + \angle DAM - \angle BAD = 60°.$$

Along with $AL=AM$, we see triangle ALM is equilateral.

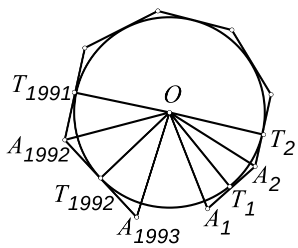
Looking at triangles OLM and ALN, we get $OP_1=½LM$, $OP_2=½AL$ and $OP_2 \parallel AL$. Hence, $OP_1=OP_2$, $\angle P_1OP_2=\angle P_1AL = 30°$, $\angle P_2OM = \angle DAL=15°$ and $\angle P_2OP_3 = 2\angle P_2OM = 30°$. By symmetry, we can conclude that the dodecagon is regular.

<u>*Example 9.*</u> (*IMO* 1992, *Shortlisted Problem from India*) Show that in the plane there exists a convex polygon of 1992 sides satisfying the following conditions:

(i) its sides lengths are 1,2,3,…,1992 in some order;

(ii) the polygon is circumscribable about a circle.

<u>*Solution.*</u> For a positive number r, let us draw a circle of radius r and let us draw a polygonal path $A_1A_2\ldots A_{1993}$ such that for i=1 to 1992, side A_iA_{i+1} is tangent to the circle at a point T_i and $T_{1992}A_{1993} = A_1T_1$, $T_1A_2 = A_2T_2$, … , $T_{1991}A_{1992} = A_{1992}T_{1992}$.



To achieve condition (i), we need A_1A_2, A_2A_3, …, $A_{1992}A_{1993}$ to be a permutation of 1, 2, …, 1992. This can be done as follow:

If $i\equiv 1 \pmod 4$, then let $A_iT_i=1/2$.
If $i\equiv 3 \pmod 4$, then let $A_iT_i=3/2$.
If $i\equiv 0,2 \pmod 4$, then let $A_iT_i=i-3/2$.

We can check that the lengths of A_iA_{i+1} for i=1 to 1992 are 1, 2, 4, 3, 5, 6, 8, 7,…, 1989,1990,1992,1991.

To achieve condition (ii), we define a function

$$f(r)=\sum_{i=1}^{1992}\angle A_iOA_{i+1} = 2\sum_{i=1}^{1992}\arctan\frac{A_iT_i}{r}.$$

Observe that $f(r)$ is a continuous function on $(0,\infty)$. As r tends to 0, $f(r)$ tends to infinity and as r tends to infinity, $f(r)$ tends to 0. By the intermediate value theorem, there exists r such that $f(r) = 2\pi$. Then $A_{1993}=A_1$ and $A_1A_2\ldots A_{1992}$ is a desired polygon.

We remark that if 1992 is replaced by other positive integers of the form $4k$, then there are such $4k$-sided polygon.