Olympiad Corner

The 41st International Mathematical Olympiad, July 2000:
Time allowed: 4 hours 30 minutes
Each problem is worth 7 points.

Problem 1. Two circles \( \Gamma_1 \) and \( \Gamma_2 \) intersect at \( M \) and \( N \). Let \( \ell \) be the common tangent to \( \Gamma_1 \) and \( \Gamma_2 \) so that \( M \) is closer to \( \ell \) than \( N \). Is \( \ell \) tangent to \( \Gamma_1 \) at \( A \) and \( \Gamma_2 \) at \( B \). Let the line through \( M \) parallel to \( \ell \) meets the circle \( \Gamma_1 \) again at \( C \) and the circle \( \Gamma_2 \) at \( D \). Lines \( CA \) and \( DB \) meet at \( E \); lines \( AN \) and \( CD \) meet at \( P \); lines \( BN \) and \( CD \) meet at \( Q \). Show that \( EP = EQ \).

Problem 2. Let \( a, b, c \) be positive real numbers such that \( abc = 1 \). Prove that
\[
(a - 1 + 1/b)(b - 1 + 1/c)(c - 1 + 1/a) \leq 1
\]

Problem 3. Let \( n \geq 2 \) be a positive integer. Initially, there are \( n \) fleas on a horizontal line, not all at the same point. For a positive real number \( \lambda \), define a move as follows:
Choose any two fleas, at points \( A \) and \( B \), with \( A \) to the left of \( B \); let the flea at \( A \) jump to the point \( C \) on the line to the line to the right of \( B \) with \( BC/AB = \lambda \).

In comparing two similar expressions, often they involve a common function. To see which expression is greater, the shape of the graph of the function on an interval is very important. A function \( f \) is said to be convex on an interval \( I \) if for any two points \((x_1, f(x_1))\) and \((x_2, f(x_2))\) on the graph, the segment joining these two points lie on or above the graph of the function over \([x_1, x_2]\). That is,
\[
f((1-t)x_1 + tx_2) \leq (1-t)f(x_1) + tf(x_2)
\]
for every \( t \) in \([0, 1]\). If \( f \) is continuous on \( I \), then it is equivalent to have
\[
f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}
\]
for every \( x_1 \), \( x_2 \) in \( I \). If furthermore \( f \) is differentiable, then it is equivalent to have a nondecreasing derivative. Also, \( f \) is strictly convex on \( I \) if \( f \) is convex on \( I \) and equality holds in the inequalities above only when \( x_1 = x_2 \). We say a function \( g \) is concave on an interval \( I \) if the function \(-g\) is convex on \( I \). Similarly, \( g \) is strictly concave on \( I \) if \(-g\) is strictly convex on \( I \).

The following are examples of strictly convex functions on intervals:
\[
x^p \text{ on } [0, \infty) \text{ for } p > 1,
\]
\[
x^p \text{ on } (0, \infty) \text{ for } p < 0,
\]
\[
a^x \text{ on } (-\infty, \infty) \text{ for } a > 1,
\]
\[
\tan x \text{ on } [0, \frac{\pi}{2}].
\]
The examples of strictly concave functions on intervals:
\[
x^p \text{ on } [0, \infty) \text{ for } 0 < p < 1,
\]
\[
\log_a x \text{ on } (0, \infty) \text{ for } a > 1,
\]
\[
\cos x \text{ on } [-\pi/2, \pi/2],
\]
\[
\sin x \text{ on } [0, \pi].
\]
The most important inequalities of concern in these functions are the following.

**Jensen’s Inequality.** If \( f \) is convex on an interval \( I \) and \( x_1, x_2, \ldots, x_n \) are in \( I \), then
\[
f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \leq \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}.
\]
For strictly convex functions, equality holds if and only if \( x_1 = x_2 = \cdots = x_n \).

**Generalized Jensen’s Inequality.** Let \( f \) be continuous and convex on an interval \( I \). If \( x_1, \ldots, x_n \) are in \( I \) and \( 0 < t_1, t_2, \ldots, t_n < 1 \) with \( t_1 + t_2 + \cdots + t_n = 1 \), then
\[
f(t_1 x_1 + t_2 x_2 + \cdots + t_n x_n) \leq t_1 f(x_1) + t_2 f(x_2) + \cdots + t_n f(x_n)
\]
(with the same equality condition for strictly convex functions).

Jensen’s inequality is proved by doing a forward induction to get the cases \( n = 2^k \), then a backward induction to get case \( n-1 \) from case \( n \) by taking \( x_n \) to be the arithmetic mean of \( x_1, x_2, \ldots, x_{n-1} \). For the generalized Jensen’s inequality, the case all \( t_i \)'s are rational is proved by taking common denominator and the other cases are obtained by using continuity of the function and the density of rational numbers.

There are similar inequalities for concave and strictly concave functions by reversing the inequality signs.

**Example 1.** For a triangle \( ABC \), show that
\[
\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}
\]
determine when equality holds.

**Solution.** Since \( f(x) = \sin x \) is strictly concave on \([0, \frac{\pi}{2}]\), so
\[
\sin A + \sin B + \sin C = f(A) + f(B) + f(C) \leq 3f\left(\frac{A + B + C}{3}\right) = 3 \sin \left(\frac{A + B + C}{3}\right) = \frac{3\sqrt{3}}{2}.
\]
Equality holds if and only if \( A = B = C = \pi/3 \), i.e. \( \triangle ABC \) is equilateral.

**Example 2.** If \( a, b, c > 0 \) and \( a + b + c = 1 \), then find the minimum of
\[
\left( \frac{a + 1}{a} \right)^{10} + \left( \frac{b + 1}{b} \right)^{10} + \left( \frac{c + 1}{c} \right)^{10}.
\]

**Solution.** Note \( 0 < a, b, c < 1 \). Let \( f(x) = \left( \frac{x + 1}{x} \right)^{10} \) on \( I = (0, 1) \), then \( f \) is strictly convex on \( I \) because its second derivative
\[
90\left( x + \frac{1}{x} \right)^{8} + 2 \left( x + \frac{1}{x} \right)^{9} + 2 \left( x + \frac{1}{x} \right)^{2} > 0
\]
is positive on \( I \). By Jensen’s inequality,
\[
\frac{10^{10}}{3^{9}} = 3 \left( \frac{a + b + c}{3} \right)^{10} \leq f(a) + f(b) + f(c) = \left( \frac{a + 1}{a} \right)^{10} + \left( \frac{b + 1}{b} \right)^{10} + \left( \frac{c + 1}{c} \right)^{10}.
\]
So the minimum is \( 10^{10}/3^9 \), attained when \( a = b = c = 1/3 \).

**Example 3.** Prove that AM-GM in-equality, which states that if \( a_1, a_2, \ldots, a_n \geq 0 \), then
\[
a_1 + a_2 + \cdots + a_n \geq \frac{n}{n} \sqrt[n]{a_1 a_2 \cdots a_n}.
\]

**Solution.** If one of the \( a_i \)'s is 0, then the right side is 0 and the inequality is clear. If \( a_1, a_2, \ldots, a_n > 0 \), then since \( f(x) = \log x \) is strictly concave on \( (0, \infty) \), by Jensen’s inequality,
\[
\log \left( \frac{a_1 + a_2 + \cdots + a_n}{n} \right) \geq \frac{1}{n} \log \left( a_1 a_2 \cdots a_n \right).
\]
Exponentiating both sides, we get the AM-GM inequality.

**Remarks.** If we use the generalized Jensen’s inequality instead, we can get the weighted AM-GM inequality. It states that if \( a_1, a_2, \ldots, a_n > 0 \) and \( 0 < t_1, \ldots, t_n < 1 \) satisfying \( t_1 + \cdots + t_n = 1 \), then \( t_1 a_1 + \cdots + t_n a_n \geq a_1^{t_1} \cdots a_n^{t_n} \) with equality if and only if all \( a_i \)'s are equal.

**Example 4.** Prove the power mean inequality, which states that for \( a_1, a_2, \ldots, a_n > 0 \) and \( s < t \), if
\[
S_s = \left( \frac{a_1^s + a_2^s + \cdots + a_n^s}{n} \right)^{1/s}
\]
then \( S_s \leq S_t \). Equality holds if and only if \( a_1 = a_2 = \cdots = a_n \).

**Example 7.** Prove Hölder’s inequality, which states that if \( p, q > 1 \) satisfy \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( a_1, a_2, \ldots, a_n \) are real (or complex) numbers, then
\[
\sum_{i=1}^{n} |a_i| \leq \left( \sum_{i=1}^{n} |a_i|^p \right)^{1/p} \left( \sum_{i=1}^{n} |b_i|^q \right)^{1/q}.
\]
(The case \( p = q = 2 \) is the Cauchy-Schwarz inequality.)

**Solution.** Let
\[
A = |a_1|^p + \cdots + |a_n|^p,
\]
\[
B = |b_1|^p + \cdots + |b_n|^p.
\]
If \( A \) or \( B \) is 0, then either all \( a_i \)'s or all \( b_i \)'s are 0, which will make both sides of the inequality 0.

So we need only consider the case \( A \neq 0 \) and \( B \neq 0 \). Let \( t_1 = 1/p \) and \( t_2 = 1/q \), then \( 0 < t_1, t_2 < 1 \) and \( t_1 + t_2 = 1 \). Let \( x_i = |a_i|^p / A \) and \( y_i = |b_i|^p / B \), then
\[
x_1 + \cdots + x_n = 1, \quad y_1 + \cdots + y_n = 1.
\]
Since \( f(x) = e^x \) is strictly convex on \( (-\infty, \infty) \), by the generalized Jensen’s inequality,
\[
x_1^{1/p} y_1^{1/q} = f(t_1 \ln x_1 + t_2 \ln y_1) \leq t_1 f(\ln x_1) + t_2 f(\ln y_1) = \frac{x_1}{p} + \frac{y_1}{q}.
\]
Adding these for \( i = 1, \ldots, n \), we get
\[
\sum_{i=1}^{n} |a_i|^{1/p} |b_i|^{1/q} \leq \frac{1}{p} \sum_{i=1}^{n} x_i + \frac{1}{q} \sum_{i=1}^{n} y_i = 1.
\]
Therefore,
\[
\sum_{i=1}^{n} |a_i| \leq \left( \sum_{i=1}^{n} |a_i|^p \right)^{1/p} \left( \sum_{i=1}^{n} |b_i|^q \right)^{1/q}.
\]

**Example 1.** Let
\[
x_1 = \sqrt[3]{a/c}, \quad x_2 = \sqrt[5]{b/d}, \quad y_1 = \sqrt[27]{ac}, \quad y_2 = \sqrt[27]{bd}.
\]
By the Cauchy-Schwarz inequality,
\[
\left( \frac{3}{c} + \frac{b}{d} \right)(ac + bd) = x_1^2 + x_2^2 \geq (x_1 y_1 + x_2 y_2)^2
\]
\[
= (a^2 + b^2)^2 \geq ac + bd.
\]
Cancelling \( ac + bd \) on both sides, we get the desired inequality.

**Solution 2.** Let
\[
x = (a^3/c)^{2/3}, \quad y = (b^3/d)^{2/3}.
\]
By the \( p = 3, q = 3/2 \) case of Hölder’s inequality,
\[
a^2 + b^2 = (c^{2/3}) x + (d^{2/3}) y
\]
\[
\leq (c^{2/2} + d^{2/2}) (x^{3/2} + y^{3/2})^{1/3}
\]
Cancelling \( a^2 + b^2 = (a^2 + b^2)^{1/3} \) on both sides, we get \( 1 \leq x^{3/2} + y^{3/2} = (a^3/c) + (b^3/d) \).
Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceeded by the solver’s name, home address and school affiliation. Please send submissions to Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science & Technology, Clear Water Bay, Kowloon. The deadline for submitting solutions is December 10, 2000.

Problem 111. Is it possible to place 100 solid balls in space so that no two of them have a common interior point, and each of them touches at least one-third of the others? (Source: 1997 Czech-Slovak Match)

Problem 112. Find all positive integers (x, n) such that \(x^n + 2^n + 1\) is divisor of \(x^{n+1} + 2^{n+1} + 1\). (Source: 1998 Romanian Math Olympiad)

Problem 113. Let \(a, b, c > 0\) and \(abc \leq 1\). Prove that \(\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq a + b + c\).
(Hint: Consider the case \(abc = 1\) first.)

Problem 114. (Proposed by Mohammad Aassila, Universite Louis Pasteur, Strasbourg, France) An infinite chessboard is given, with \(n\) black squares and the remainder white. Let the collection of black squares be denoted by \(G_0\). At each moment \(t = 1, 2, 3, \ldots\) a simultaneous change of colour takes place throughout the board according to the following rule: every square gets the colour that dominates in the three square configuration consisting of the square itself, the square above and the square to the right. New collections of black squares \(G_1, G_2, G_3, \ldots\) are so formed. Prove that \(G_n\) is empty.

Problem 115. (Proposed by Mohammad Aassila, Universite Louis Pasteur, Strasbourg, France) Find the locus of the points \(P\) in the plane of an equilateral triangle \(ABC\) for which the triangle formed with lengths \(PA, PB\) and \(PC\) has constant area.

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**Solutions**

**Problem 106.** Find all positive integer ordered pairs \((a, b)\) such that \(\gcd(a, b) + \text{lcm}(a, b) = a + b + 6\) where \(\gcd\) stands for greatest common divisor (or highest common factor) and \(\text{lcm}\) stands for least common multiple.

**Solution.** CHAN An Jack and LAW Siu Lun Jack (Mei Kei College, Form 6), CHAN Chin Fei (STFA Leung Kau Kui College), CHAO Khek Lun Harold (St. Paul’s College, Form 6), CHAU Suk Ling (Queen Elizabeth Secondary School, Form 6), CHENG Man Chuen (Tsunen Wan Government Secondary School, Form 7), FUNG Wing Ki Ricky (La Salle College), HUNG Chun Hei (Pui Ching Middle School, Form 5), KO Man Ho (Wah Yan College, Kowloon, Form 7), LAM Shuk Ming Sherman (La Salle College, Form 5), LAW Ka Ho (HKU, Year 1), LEE Kevin (La Salle College), LEUNG Wai Ying (Queen Elizabeth School, Form 6), MAK Ho Kwun Calvin (La Salle College), OR Kin (SKH Bishop Mok Sau Tseng Secondary School), POON Wing Soe Jessica (STFA Leung Kau Kui College, Form 7), TANG Sheung Kon (STFA Leung Kau Kui College, Form 6), TONG Chin Fung (SKH Lam Woo Memorial Secondary School, Form 6), WONG Wing Hong (La Salle College, Form 3) and YEUNG Kai Shing (La Salle College, Form 4).

Let \(m = \gcd(a, b)\), then \(a = mx\) and \(b = my\) with \(\gcd(x, y) = 1\). In that case, \(\text{lcm}(a, b) = mxy\). So the equation becomes \(m + mxy = mx + my + 6\). This is equivalent to \(m(x – 1)(y – 1) = 6\). Taking all possible positive integer factorizations of 6 and requiring \(\gcd(x, y) = 1\), we have \((m, x, y) = (1, 2, 7), (1, 7, 2), (1, 3, 4), (1, 4, 3), (2, 3, 2) and (3, 2, 3)\). Then \((a, b) = (2, 7), (7, 2), (3, 4), (4, 3), (6, 9)\) and \((9, 6)\). Each of these is easily checked to be a solution.

**Other recommended solvers:** CHAN Kin Hang Andy (Bishop Hall Jubilee School, Form 7) and CHENG Kei Tsi Daniel (La Salle College, Form 6).

**Problem 107.** For \(a, b, c > 0\), if \(abc = 1\), then show that \(\frac{b + c}{a} + \frac{c + a}{b} + \frac{a + b}{c} \geq \sqrt{a} + \sqrt{b} + \sqrt{c} + 3\).

**Solution 1.** CHAN Hiu Fai Philip (STFA Leung Kau Kui College, Form 7), LAW Ka Ho (HKU, Year 1) and TSUI Ka Ho Willie (Hoi Ping Chamber of Commerce Secondary School, Form 7).

By the AM-GM inequality and the fact \(abc = 1\), we get \(\sqrt{a} + \sqrt{b} + \sqrt{c} + 3 \geq \frac{b + c}{a} + \frac{c + a}{b} + \frac{a + b}{c} \geq \sqrt{a} + \sqrt{b} + \sqrt{c} + 3\).

**Solution 2.** CHAN Kin Hang Andy (Bishop Hall Jubilee School, Form 7), CHAO Khek Lun Harold (St. Paul’s College, Form 6), CHAU Suk Ling (Queen Elizabeth School, Form 6), CHENG Kei Tsi (La Salle College, Form 6), CHENG Man Chuen (Tsunen Wan Government Secondary School, Form 7), LAW Ka Ho (HKU, Year 1) and LEUNG Wai Ying (Queen Elizabeth School, Form 6).

Without loss of generality, assume \(a \geq b \geq c\). Then \(1/\sqrt{a} \leq 1/\sqrt{b} \leq 1/\sqrt{c}\). By the rearrangement inequality,

\[ \frac{b}{\sqrt{a}} + \frac{c}{\sqrt{b}} + \frac{a}{\sqrt{c}} \geq \frac{a}{\sqrt{b}} + \frac{b}{\sqrt{c}} + \frac{c}{\sqrt{a}} = \sqrt{a} + \sqrt{b} + \sqrt{c}. \]

Also, by the AM-GM inequality,

\[ \frac{a}{\sqrt{b}} + \frac{b}{\sqrt{c}} + \frac{c}{\sqrt{a}} \geq 3. \]

Adding these two inequalities, we get the desired inequality.

**Generalization:** Professor Murray S. Klamkin (University of Alberta, Canada) sent in a solution, which proved a stronger inequality and later generalized it to \(n\) variables. He made the sub-situtions \(x_1 = \sqrt{a}, x_2 = \sqrt{b}, x_3 = \sqrt{c}\) to get rid of square roots and let \(S_m = x_1^m + x_2^m + x_3^m\) so that the inequality became

\[ \frac{x_1^2 + x_2^2 + x_3^2}{x_1} + \frac{x_2^2 + x_3^2 + x_1^2}{x_2} + \frac{x_3^2 + x_1^2 + x_2^2}{x_3} \geq S_m + 3. \]

By the AM-GM inequality, \(S_m \geq 3S_m/(S_1 + S_2)/(S_1 + S_2)^2 \geq S_1/3\) by the power mean inequality, we would get a stronger inequality by replacing \(S_1 + S_2\) by \(S_2\). Rearranging terms, this stronger inequality could be rewritten as \(S_2(S_3 - S_2) \geq S_1 - S_2\). Now the left side is nonnegative, but the right side is nonpositive. So the stronger inequality is true. If we replace 3 by \(n\)
and assume $x_1 \cdots x_n = 1$, then as above, we will get $S_m(S_{1-m} - n) \geq S_1 - S_m$ by the AM-GM and power mean inequalities. Expanding and regrouping terms, we get the stronger inequality in $n$ variables, namely

$$\sum_{i=1}^{n} \frac{S_m}{x_i^{m-1}} \geq (n-1)S_m.$$ 

Other recommended solvers: CHAN Chin Fei (STFA Leung Kau Kui College), LAM Shek Ming Sherman (La Salle College, Form 5), LAW Hiu Fai (Wah Yan College, Kowloon, Form 7), LEE Kevin (La Salle College, Form 5), MAK Hoi Kwan Calvin (La Salle College), OR Kin (SHK Bishop Mok Sau Tseng Secondary School) and YEUNG Kai Shing (La Salle College, Form 4).

Problem 108. Circles $C_1$ and $C_2$ with centers $O_1$ and $O_2$ respectively meet at points $A$, $B$. The radii $O_1B$ and $O_2B$ intersect $C_1$ and $C_2$ at $F$ and $E$. The lines parallel to $EF$ through $B$ meets $C_1$ and $C_2$ at $M$ and $N$, respectively. Prove that $MN = AE + AF$. (Source: 17th Iranian Mathematical Olympiad)

Solution. YEUNG Kai Shing (La Salle College, Form 4).

As the case $F = E = B$ would make the problem nonsensible, the radius $O_1B$ of $C_1$ can only intersect $C_2$, say at $F$. Then the radius $O_2B$ of $C_2$ intersect $C_1$ at $E$. Since $\angle EOF = 180^\circ - 2\angle FBE = \angle EFO$. Thus, $E$, $O_2$, $O_1$, $F$ are concyclic. Then $\angle EFB = (360^\circ - \angle AOB)/2 = 180^\circ - \angle O_2 EF = \angle O_2 EF = \angle EBM$. So arc$\angle ABM = \angle EMA$. Subtracting minor arc$\angle ABM$ from both sides, we get minor arc$\angle ABM = \angle EMA$. So $\angle EMA = \angle EAB$. Similarly, $\angle EAB = \angle EAB$. Then $MN = MB + NB = AE + AF$.

Other recommended solvers: CHAN Kin Hang Andy (Bishop Hall Jubilee School, Form 7), CHAO Khek Lun Harold (St. Paul’s College, Form 6), Chau Suk Ling (Queen Elizabeth School, Form 6), CHENG Kei Tsi (La Salle College, Form 5), CHENG Man Chuen (Tsuen Wan Government Secondary School, Form 7), LAM Shek Ming Sherman (La Salle College, Form 5), LAW Hiu Fai (Wah Yan College, Kowloon, Form 7), LAW Ka Ho (HKU, Year 1) and YEUNG Kai Shing (La Salle College, Form 4).

Let $a_n = n! + 2$. Then for every non-negative integer $k$, if $n \geq k + 2$, then $k + a_n$ is divisible by $k + 2$ and is greater than $k + 2$, hence not prime.

Other commended solvers: CHAN Kin Hang Andy (Bishop Hall Jubilee School, Form 7), KO Man Ho (Wah Yan College, Form 7), LEE Kevin (La Salle College, Form 5) and LEUNG Wai Ying (Queen Elizabeth School, Form 6).

Problem 110. In a park, 1000 trees have been placed in a square lattice. Determine the maximum number of trees that can be cut down so that from any stump, you cannot see any other stump. (Assume the trees have negligible radius compared to the distance between adjacent trees.) (Source: 17th German Mathematical Olympiad)

Solution. CHAN Kin Hang Andy (Bishop Hall Jubilee School, Form 7), CHAO Khek Lun Harold (St. Paul’s College, Form 6), Chau Suk Ling (Queen Elizabeth School, Form 6), CHENG Kei Tsi (La Salle College, Form 5), CHENG Man Chuen (Tsuen Wan Government Secondary School, Form 7), FUNG Wing Kiu Ricky (La Salle College), LAM Shek Ming Sherman (La Salle College, Form 5), LAW Ka Ho (HKU, Year 1), LEE Kevin (La Salle College), LEUNG Wai Ying (Queen Elizabeth School, Form 6), LYN Kwong To and KO Man Ho (Wah Yan College, Kowloon, Form 7), POON Wing Sze Jessica (STFA Leung Kau Kui College, Form 7) and YEUNG Kai Shing (La Salle College, Form 4).

In every $2 \times 2$ subsquare, only one tree can be cut. So a maximum of 2500 trees can be cut down. Now let the trees be at $(x, y)$, where $x, y = 0, 1, 2, \ldots, 99$. If we cut down the 2500 trees at $(x, y)$ with both $x$ and $y$ even, then the condition will be satisfied. To see this, consider the stumps at $(x_1, y_1)$ and $(x_2, y_2)$ with $x_1, y_1, x_2, y_2$ even. The cases $x_1 = x_2$ or $y_1 = y_2$ are clear. Otherwise, write $(y_1 - y_2)(x_1 - x_2) = mn$ in lowest term. Then either $m$ or $n$ is odd and so the tree at $(x_1 + m, y_1 + n)$ will be between $(x_1, y_1)$ and $(x_2, y_2)$.

Other recommended solvers: NG Chok Ming Lewis (STFA Leung Kau Kui College, Form 7).

Olympiad Corner

(continued from page 1)

Problem 3. (cont’d)

Determine all values of $\lambda$ such that, for any point $M$ on the line and any initial position of the $n$ fleas, there is a finite sequence of moves that will take all the fleas to positions to the right of $M$.

Problem 4. A magician has one hundred cards numbered 1 to 100. He puts them into three boxes, a red one, a white one a blue one, so that each contains at least one card.

A member of the audience selects two of the three boxes, chooses one card from each and announces the sum of the numbers on the chosen cards. Given this sum, the magician identifies the box from which no card has been chosen.

How many ways are there to put all the cards into the boxes so that this trick always works? (Two ways are considered different if at least one card is put into a different box.)

Problem 5. Determine whether or not there exists a positive integer $n$ such that $n$ is divisible by exactly 2000 different prime numbers, and $2^n + 1$ is divisible by $n$.

Problem 6. Let $A_1H_1, B_1H_2, C_1H_3$ be the altitudes of an acute-angled triangle ABC. The incircle of the triangle ABC touches the sides BC, CA, AB at $T_1, T_2, T_3$, respectively. Let the lines $\ell_1, \ell_2, \ell_3$ be the reflections of the lines $H_1H_2, H_2H_3, H_3H_1$ in the lines $T_1T_2, T_2T_3, T_3T_1$, respectively.

Prove that $\ell_1, \ell_2, \ell_3$ determine a triangle whose vertices lie on the incircle of the triangle ABC.