1. Let $x = \frac{1}{y}$, then $I = \int_{0}^{+\infty} \frac{\ln x}{1 + x^2} \, dx = \int_{0}^{+\infty} \frac{\ln(1/y)}{1 + (1/y)^2} \left( -\frac{dy}{y^2} \right) = -\int_{0}^{+\infty} \frac{\ln y}{1 + y^2} \, dy = -I$. So $I = 0$.

**Remark.** A number of students used different substitutions, like $x = \tan \theta$ followed by $\phi = \frac{\pi}{2} - \theta$.

2. (a) Let’s refer to the condition $1x_1 \leq 2x_2 \leq 2x_3 \leq \cdots \leq nx_n$ as (*) and so $P_1 = 1$. For $n = 2$, $(x_1, x_2) = (1, 2), (2, 1)$ satisfy (*) and so $P_2 = 2$. For $n = 3$, only $(x_1, x_2, x_3) = (1, 2, 3), (1, 3, 2), (2, 1, 3)$ satisfy (*) and so $P_3 = 3$. For $n = 4$, only $(x_1, x_2, x_3, x_4) = (1, 2, 3, 4), (1, 3, 2, 4), (1, 3, 2, 4), (1, 2, 4, 3), (2, 1, 4, 3)$ satisfy (*) and so $P_4 = 5$.

(b) Observe that in the cases $n = 1$ to 4, $n$ is either $x_n$ or $x_{n-1}$. We claim this is true for $n > 4$. If $x_n = n$, then we can get the $P_{n-1}$ permutations of $1, 2, \ldots, n-1$ from the case $n-1$ and add $x_n = n$ to the end of these permutations for case $n$. Also, if $x_{n-1} = n$, then we can get the $P_{n-2}$ permutations of $1, 2, \ldots, n-2$ from the case $n-2$ and add $x_{n-1} = n, x_n = n-1$ to the end of these permutations for case $n$. So $P_n \geq P_{n-1} + P_{n-2}$.

Next assume $n = x_{n-k}$ for some $k = 2, \ldots, n-1$. Then at least one of the $k$ numbers $x_{n-k+1}, \ldots, x_n$ is less than or equal to $n - k$. Say for some $x_{n-j} \in \{x_{n-k+1}, \ldots, x_n\}$, we have $x_{n-j} \leq n - k$. Then $(n - k)n = (n - k)x_{n-k} \leq (n - j)x_{n-j} \leq (n - j)(n - k)$. This implies $j = 0$, i.e. $x_n = n - k$. Then $ix_i = (n - k)n$ for $i = n - k, \ldots, n$. In particular, $(n - 1)x_{n-1} = (n - k)n$. Since $n$ and $n - 1$ are divisors of $(n - k)n$ and $\gcd(n, n - 1) = 1$. So $n(n - 1)$ is a divisor of $(n - k)n$, which is less than $n(n - 1)$ due to $k \geq 2$. This is a contradiction.

Therefore, $P_n = P_{n-1} + P_{n-2}$. Using $P_3 = 3, P_4 = 5$, we can compute $P_5, P_6, P_7, \ldots$, then finally $P_{20} = 10946$.

3. (Solution 1) Let $L : C^n \to C^n$ be given by $L(x) = Ax$. Let $m$ be the smallest positive integer such that $\ker L^m = \ker L^{m+1}$. From $L^j \neq 0$, we get $m \leq j$. Since

$$0 \subset \ker L \subset \ker L^2 \subset \cdots \subset \ker L^m \subseteq C^n,$$

we have $m \leq n$. Next for every $k \geq m$, if $x \in \ker L^{k+1}$, then $L^{k+1}(x) = 0$ and $L^{k-m}(x) \in \ker L^{m+1} = \ker L^m$. Hence, $L^k(x) = L^m \circ L^{k-m}(x) = 0$. Thus, $\ker L^{k+1} = \ker L^k$ for every $k \geq m$. Since $n, j \geq m$, $\ker L^n = \ker L^j = \ker L^j = \ker L^m$. Since $\ker L^j = \ker 0 = C^n$, so $\ker L^m = C^n$ and $A^m = 0$.

**Remark.** In place of kernels, ZHU Songhao considered the ranges of $A^k$ and came up with essentially the same proof.

(Solution 2) Let $c$ be an eigenvalue of $A$ with nonzero eigenvector $x$. Then $Ax = cx$ implies $0 = A^jx = c^jx$. So $c = 0$. Then the characteristic polynomial of $A$ is $p(t) = t^n$. By the Cayley-Hamilton theorem, we have $A^n = p(A) = 0$.

4. Since $\left( k - \frac{1}{2} \right)^2 = k^2 - k + \frac{1}{4}$ and $\left( k + \frac{1}{2} \right)^2 = k^2 + k + \frac{1}{4}$, it follows that $\langle n \rangle = k$ if and only if $k^2 - k + 1 \leq n \leq k^2 + k$. Hence

$$\lim_{n \to \infty} \sum_{j=1}^{n} \frac{2^j + 2^{-j}}{2^j} = \sum_{j=1}^{n} \sum_{k=1}^{n} 2^{j-k} = \sum_{k=1}^{k^2+k} \frac{2^k + 2^{-k}}{2^n} = \sum_{k=1}^{\infty} (2^k + 2^{-k})(2^{-k^2+k} - 2^{-k^2-k}) = \sum_{k=1}^{\infty} (2^{-k(k-2)} - 2^{-k(k+2)}) = \sum_{k=1}^{\infty} 2^{-k(k-2)} - \sum_{k=3}^{\infty} 2^{-k(k-2)} = 3.$$
5. Since $\ln x$ is strictly increasing, it is the same as proving $x^y \ln y - y^x \ln x > 0$.

**Case 1 ($x^y \geq y^x$)** Then $y > 1$ (otherwise $0 < x < y \leq 1$ implies $x^y < x^x < y^x$, which is a contradiction) and $x^y \ln y > y^x \ln x$.

**Case 2 ($x^y < y^x$)** Either $0 < x < 1$ or $x \geq 1$. If $0 < x < 1$, then $x^y \ln y - y^x \ln x > x^y \ln x - y^x \ln y = (x^y - y^x) \ln x > 0$. If $x \geq 1$, then $y > 1$ and $y \ln x < x \ln y$. Hence

$$x^y \ln y - y^x \ln x > x^{y-1} y \ln x - y^x \ln x = \frac{\ln x}{x} (yx^y - xy^x) \geq 0.$$ 

Next $yx^y - xy^x \geq 0$ iff $\ln(yx^y) \geq \ln(xy^x)$ iff $\ln y + y \ln x - \ln x - x \ln y \geq 0$. Now fix $x$. Let $f(w) = \ln w + w \ln x - \ln x - x \ln w$. Then $f(x) = 0$. If $y > x$, then

$$f'(y) = \frac{1}{y} + \ln x - \frac{x - 1}{y} > \ln x - \frac{x - 1}{x} = \frac{1}{x} \int_1^x \ln t \, dt \geq 0.$$ 

Then $f(y) \geq f(x) = 0$. Therefore, $x^y \ln y - y^x \ln x > 0$.

6. We have $f_{xy}/f_x = f_y/f$. Integrating with respect to $y$, we get $\ln|f_x| = \ln|f| + \alpha(x)$ for some function $\alpha(x)$. Then $\ln|f_x/f| = \alpha(x)$. Let $\beta(x) = f_x/f$. By continuity, either $\beta(x) = e^{\alpha(x)}$ or $\beta(x) = -e^{\alpha(x)}$. Then

$$\ln|f| = \int \beta(x) \, dx + \gamma(y) = \delta(x) + \gamma(y).$$ 

By continuity, either $f(x, y) = e^{\delta(x)+\gamma(y)}$ or $f(x, y) = -e^{\delta(x)+\gamma(y)}$. The result follows.