Directions: This is a three hour test. No calculators are allowed. For every problem, provide complete details of your solution.

Problem 1. Let \( n \) be a positive integer and \( A, B \) be \( n \times n \) matrices over the complex numbers. Prove that \( A \) and \( B \) have a common eigenvalue if and only if \( AX = XB \) for some \( n \times n \) matrix \( X \neq 0 \).

Problem 2. Find all continuous functions \( y : [0, \infty) \rightarrow \mathbb{R} \) such that \( y(0) = 0 \), \( y \) is differentiable on \( (0, \infty) \) satisfying \( y'(x) = \int_{0}^{x} \sin(y(u)) \, du + \cos x \) for all \( x > 0 \).

Problem 3. Let \( a \) and \( b \) be positive integers with \( a > 1 \). If \( a \) and \( b \) are both odd or both even, then prove that \( 2^a - 1 \) does not divide \( 3^b - 1 \).

Problem 4. Let \( f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R} \) be continuous such that for every \( x \in [0, 1] \) and \( y_0 \neq y_1 \) in \( \mathbb{R} \), we have
\[
\frac{1}{2} \leq \frac{f(x, y_0) - f(x, y_1)}{y_0 - y_1} \leq \frac{3}{2}.
\]
Prove that there exists a unique real-valued continuous function \( h \) on \( [0, 1] \) such that for all \( x \in [0, 1] \), \( f(x, h(x)) = 0 \).

Problem 5. Let \( u \cdot v \) denote the usual inner product of \( u, v \in \mathbb{R}^n \). For positive integer \( k < n \), let \( G(k, n) \) be the set of all \( k \)-dimensional linear subspaces in \( \mathbb{R}^n \). For \( v \in \mathbb{R}^n \) and a linear subspace \( S \) in \( \mathbb{R}^n \), let \( d(v, S) \) denote the usual distance from \( v \) to \( S \). For \( V, U \in G(k, n) \), let \( B(V) = \{ v \mid v \in V, v \cdot v = 1 \} \). For \( V, U \in G(k, n) \), let \( d(V, U) = \max \{ d(v, U) \mid v \in B(V) \} \).

(a) Prove that for \( V, W, U \in G(k, n) \), \( d(V, U) \leq d(V, W) + d(W, U) \).

(b) Let \( \{ v_1, v_2, \ldots, v_k \} \), \( \{ w_1, w_2, \ldots, w_k \} \) be orthonormal basis of \( V, W \in G(k, n) \) respectively. Let \( A \) be the \( k \times k \) matrix with \( (i, j) \) entry equal \( v_i \cdot w_j \). Let \( \lambda \) be the smallest eigenvalue of \( A A^T \). Determine the value of \( d(V, W) \) in terms of \( \lambda \).

(c) Prove that \( d(V, W) = d(W, V) \) for all \( V, W \in G(k, n) \).

Problem 6. Let \( S = \{ z \mid z \in \mathbb{C}, \ 0 < |z| < 2 \} \) and \( f : S \rightarrow \mathbb{C} \) be a holomorphic function such that \( \text{Re} \ f(z) \geq 0 \) and \( \text{Im} \ f(z) \geq 0 \) for all \( z \in S \). Prove that \( f \) has a removable singularity at 0.

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