Solutions of 2019 UG Math Competition - Senior Level

Problem 1. Assume the opposite is true. We have
\[
\int_0^\pi |\sin x - \cos x|^2 \, dx = \int_0^\pi (\sin^2 x - 2\sin x \cos x + \cos^2 x) \, dx \\
= \int_0^\pi (1 - \sin 2x) \, dx = \left( x + \frac{\cos 2x}{2} \right)_0^\pi = \pi.
\]
and
\[
\pi = \int_0^\pi |\sin x - \cos x|^2 \, dx \leq \int_0^\pi (|\sin x - f(x)| + |f(x) - \cos x|^2) \, dx \\
\leq 2 \int_0^\pi |f(x) - \sin x|^2 \, dx + 2 \int_0^\pi |f(x) - \cos x|^2 \, dx \\
\leq 2 \left( \frac{3}{4} \right) + 2 \left( \frac{3}{4} \right) = 3,
\]
which is a contradiction.

Problem 2. Let \( S_0 = \emptyset \) and \( S_1, S_2, \ldots, S_{2^n-1} \) be the \( 2^n - 1 \) nonempty distinct subsets of \( \{a_1, a_2, \ldots, a_n\} \), where all \( a_i > 0 \). Let \( F(S_i) \) denote the sum of the elements of \( S_i \) with \( F(S_0) = 0 \). By the pigeonhole principle, there exists distinct \( i, j \) such that \( F(S_i) \equiv F(S_j) \pmod{m} \). Let \( S = (S_i \setminus S_j) \cup \{-x : x \in S_j \setminus S_i\} \). Then all \( a_k \) and \( -a_k \) cannot both be in \( S \) and \( F(S) \equiv 0 \pmod{m} \).

Problem 3. Suppose \( \gcd(k, n) = 1 \). If \( a \in G \) is of order \( m \), then \( m|n \) by Lagrange’s theorem. Then \( kx \equiv 1 \pmod{m} \) has a solution since \( \gcd(k, m) = \gcd(k, n) = 1 \). So \( (ax)^k = 1 \).

Suppose \( \gcd(k, n) > 1 \). Choose a prime \( p \) such that \( p|\gcd(k, n) \). By Cauchy’s theorem, there exists \( b \in G \) with \( b^p = 1 \), then \( b^k = 1 \). For every element in \( G \) to be a \( k \)-th power, it is necessary that the \( k \)-th powers of the \( n \) elements in \( G \) be distinct. Since \( b^k = 1 = 1^k \), this is impossible.

Problem 4. Let \( S_0 = \emptyset \) and \( S_1, S_2, \ldots, S_{2^n-1} \) be the \( 2^n - 1 \) nonempty distinct subsets of \( \{a_1, a_2, \ldots, a_n\} \), where all \( a_i > 0 \). Let \( F(S_i) \) denote the sum of the elements of \( S_i \) with \( F(S_0) = 0 \). By the pigeonhole principle, there exists distinct \( i, j \) such that \( F(S_i) \equiv F(S_j) \pmod{m} \). Let \( S = (S_i \setminus S_j) \cup \{-x : x \in S_j \setminus S_i\} \). Then all \( a_k \) and \( -a_k \) cannot both be in \( S \) and \( F(S) \equiv 0 \pmod{m} \).

Problem 5. Let \( f(z) = z e^{iz}/(1 + z^2)^2 \). For \( R > 0 \), consider the contour going from \(-R\) to \( R \) on the \( x \)-axis followed by the upper semicircle \( C_R \) with the center at 0 and radius \( R \). By the residue theorem,
\[
\int_{-R}^{R} \frac{xe^{ix}}{(1 + x^2)^2} \, dx + \int_{C_R} \frac{ze^{iz}}{(1 + z^2)^2} \, dz = 2\pi \text{Res} \left( \frac{ze^{iz}}{(1 + z^2)^2}, i \right) = 2\pi i \frac{d}{dz} \left( \frac{ze^{iz}}{(1 + z^2)^2} \right) \Big|_{z = i} \frac{\pi i}{2e}.
\]
By Jordan’s inequality, let \( h(z) = \frac{z}{(1 + z^2)^2} \), then
\[
\left| \int_{C_R} h(z) e^{iz} \, dz \right| \leq \int_{C_R} |h(z)e^{iz}| |dz| \leq \frac{R}{(R^2 - 1)^2} \int_0^\pi e^{-R \sin \theta} R \, d\theta \leq \frac{R^2}{(R^2 - 1)^2} \pi R \to 0.
\]
Then we have
\[
\int_{-\infty}^{\infty} \frac{x(\sin x - 2e \cos x)}{(1 + x^2)^2} \, dx = \text{Im} \int_{-\infty}^{\infty} \frac{xe^{ix}}{(1 + x^2)^2} \, dx - 2e \text{Re} \int_{-\infty}^{\infty} \frac{xe^{ix}}{(1 + x^2)^2} \, dx = \frac{\pi}{2e}.
\]

Problem 6. If \( x^2 + ry^2 = p \), then \( x^2 \equiv -ry^2 \pmod{p} \). So \(-r \) is a quadratic residue of \( p \). Hence, \( \left( \frac{-r}{p} \right) = 1 \) for \( 1 \leq r \leq 10 \). It is sufficient to have \(-1, 2, 3, 5 \) and 7 are quadratic residues of \( p \). This follows from having \( p \equiv 1 \pmod{2^3} \), \( p \equiv 1 \pmod{3} \), \( p \equiv 1 \) or \(-1 \pmod{5} \) and \( p \equiv 1, 2 \) or 4 \pmod{7} \). Then \( p \) must satisfy \( p \equiv 1^2, 11^2, 13^2, 17^2, 19^2 \) or 23 \pmod{840} \). The smallest such \( p \) is 1009. We have
\[
1009 = 15^2 + 28^2 = 19^2 + 2 \cdot 18^2 = 31^2 + 3 \cdot 4^2 = 15^2 + 4 \cdot 14^2 = 17^2 + 5 \cdot 12^2 \\
= 25^2 + 6 \cdot 8^2 = 1^2 + 7 \cdot 12^2 = 19^2 + 8 \cdot 9^2 = 28^2 + 9 \cdot 5^2 = 3^2 + 10 \cdot 10^2.
\]