GEOMETRIC PROGRESSION

Examples  The following are called geometric progressions:

1. 3, 6, 12, 24, ⋯
2. 1, −1/3, 1/9, −1/27, ⋯
3. a, ar, ar^2, ar^3 ⋯.

We note that the ratio between any two consecutive terms of each of the above sequences is always the same.

1. 6/3 = 12/6 = 24/12 = ⋯ = 2.
2. −1/3 = 1/9 = −1/27 = ⋯ = −1/3.
3. ar/a = ar^2/ar = ar^3/ar^2 = ⋯ = r.

The ratios that appear in the above examples are called the common ratio of the geometric progression. It is usually denoted by r. The first term (e.g. 3, 1, a in the above examples) is called the initial term, which is usually denoted by the letter a.

Example  Consider the geometric progression

\[ a, ar, ar^2, ar^3, \cdots. \]

The third, sixth and twentieth terms of the progression are given by \(ar^2\), \(ar^5\) and \(ar^{19}\) respectively. The \(p\)th term is given by \(ar^{p-1}\).

GEOMETRIC SERIES

The sum of the geometric progression

\[ a, ar, ar^2, ar^3, \cdots, \]

denoted by

\[ S_n = a + ar + ar^2 + \cdots + ar^{n-1}, \]

is called the geometric series.

Multiplying the above geometric series by \(r\), and then subtracting the resulting series by the geometric series gives

\[ S_n - rS_n = a - ar^n. \]

So we obtain

\[ S_n = a \left( \frac{1 - r^n}{1 - r} \right). \]
Example  Find the sum of the first seven terms of the sequence

\[ \frac{2}{3}, -1, \frac{3}{2}, \ldots. \]

The common ratio is \(-\frac{3}{2}\). Thus

\[
S_7 = \frac{\frac{2}{3} - 1}{1 - (-\frac{3}{2})} + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^3 + \left(\frac{3}{2}\right)^4 + \left(\frac{3}{2}\right)^5
\]

\[
= \frac{2/3(1 - (-3/2))^7}{1 - (-3/2)}
\]

\[
= \frac{2 \cdot \frac{2}{5} \cdot \frac{1}{3} + 2178}{128}
\]

\[
= \frac{1153}{240}.
\]

Example  Find the sum of the first \(n\) terms of the sequence

\[ 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots. \]

We note that the first term \(a = 1\) and the common ratio is clearly \(1/2\). So by the geometric series formula

\[
S_n = \frac{1 - (1/2)^n}{1 - 1/2} = 2 \left(1 - \frac{1}{2^n}\right).
\]

Remark  Notice that the term \(1/2^n\) can be made arbitrary small by choosing the \(n\) sufficiently large.

Exercises  Find the sum of the first

1. seven terms of \(\frac{1}{2}, \frac{1}{3}, \frac{2}{5}, \ldots.\) \((\frac{2059}{1458})\)
2. six terms of \(-2, 2\frac{1}{2}, -3\frac{1}{5}, \ldots.\) \((\frac{1281}{512})\)
3. eight terms of \(\frac{3}{4}, 1\frac{3}{4}, 3, \ldots.\) \((191\frac{1}{4})\).
4. ten terms of \(2, -4, 8, \ldots.\) \((-682)\)
5. seven terms of \(16.2, 5.4, 18, \ldots.\) \((\frac{1093}{35})\)
6. \(p\) terms of \(1, 5, 25, \ldots.\) \((\frac{5^p - 1}{4})\)
7. \(2n\) terms of \(3, -4, \frac{16}{7}, \ldots.\) \((\frac{2}{7}(1 - (\frac{4}{3})^{2n}))\)
8. twelve terms of \(1, \sqrt{3}, 3, \ldots.\) \((364(\sqrt{3} + 1))\)