**Note:** As with the midterm review, the following list of problems is longer than what will appear on the actual final. Some problems may also turn out to be more difficult than the problems you'll see on the exam. On average, however, these problems should be fairly similar in difficulty to the exam problems, and they cover most of the material that you should review.

These exercises focus more on the second half of the course.

For problems related to the first half, see the midterm review materials.

1. Give the definitions of (a) *vector space*, (b) *subspace* of a vector space, and (c) *linear transformation* between vector spaces

**Solution.** See the textbook and the lecture notes.

2. Let $V$ be the set of polynomials $f(x)$ in one variable of degree at most 3.

   This means that $x^3 + x \in V$ and $x^2 - 4 \in V$ but $x^4 \notin V$.

   Let $D$ be the subset of polynomials $f(x) \in V$ with $f(0) = 0$.

   Let $E$ be the subset of polynomials $f(x) \in V$ with $f(1) = 0$.

   (a) Explain why $V$ is a vector space.

   (b) Give a basis for $V$. What is $\dim V$?

   (c) Explain why $D$ and $E$ are subspaces of $V$.

   (d) Give a basis for $D$. What is $\dim D$?

   (e) Find an invertible linear function $T : D \to E$.

   (f) Use the previous two parts to find a basis for $E$. What is $\dim E$?

**Solution.**

(a) Addition and scalar multiplication satisfy the axioms of a vector space.

(b) A basis for $V$ is $1, x, x^2, x^3$, and $\dim V = 4$.

(c) If $f(c) = g(c) = 0$ then $(f + g)(c) = 0$ and $\lambda f(c) = 0$ for all $\lambda \in \mathbb{R}$.

Taking $c = 0$ and $c = 1$ shows that $D$ and $E$ are subspaces.

(d) A basis for $D$ is $x, x^2, x^3$, and $\dim D = 3$.

(e) The function $T : D \to E$ given by $T(f(x)) = f(x - 1)$ is linear and invertible.

(f) A basis for $E$ is $x - 1 = T(x)$, $(x - 1)^2 = T(x^2)$, $(x - 1)^3 = T(x^3)$, and $\dim E = 3$.

3. Give the definitions of (a) *eigenvector*, (b) *eigenvalue*, and (c) *diagonalisable*.

**Solution.** See the textbook and the lecture notes.

4. Consider the matrix
$$A = \begin{bmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix}.$$

   (a) Find the eigenvalues of $A$. Do this without using a calculator.

   (b) Find a basis for each eigenspace of $A$.

   (c) Is $A$ diagonalisable? If it is, find an invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$. Then find an exact formula for $A^n$ for any $n$.

**Solution.** (a) The eigenvalues of $A$ are 6, $-5$, and 3.

(b) Each eigenspace is 1-dimensional (since eigenvectors from distinct eigenvalues are linearly independent and we are working in $\mathbb{R}^3$).

A basis for the 6-eigenspace is $\begin{bmatrix} 1 \\ 6 \\ 16 \end{bmatrix}$.

A basis for the $-5$-eigenspace is $\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$.

A basis for the 3-eigenspace is $\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$.

(c) $A$ is diagonalisable since it is $3 \times 3$ with 3 distinct eigenvalues.

For $P$ and $D$ we can take
$$P = \begin{bmatrix} 1 & -2 & -2 \\ 6 & -1 & 3 \\ 16 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

5. Consider the matrix
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

   (a) Is $A$ invertible? Explain why or why not.

   (b) Is $A$ diagonalisable? Explain why or why not.

   (c) Find an exact formula for $A^n$ for any positive integer $n$.

**Solution.** (a) $A$ is invertible since $\det A = 1 \neq 0$.

(b) $A$ is not diagonalisable since its only eigenvalue is 1 but its 1-eigenspace is 1-dimensional. Also, we proved in class that the only upper triangular matrix with all ones on the diagonal which is diagonalisable is the identity matrix.

(c) Although $A$ is not diagonalisable, we can still guess a formula for $A^n$:

$$A^n = \begin{bmatrix} 1 & n & n(n-1)/2 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & n & 1+2+3+\cdots+n-1 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}.$$

Multiplying this matrix by $A$ gives the same formula with $n$ replaced by $n+1$.

6. Find examples of the following:

(a) A matrix which is not invertible or diagonalisable.

(b) A matrix which is symmetric but not invertible.

(c) A matrix which is not diagonal or invertible, but is diagonalisable.

(d) A $3 \times 3$ matrix which is diagonalisable but not diagonal, with only two eigenvalues.

(e) A $3 \times 3$ matrix with all real entries and two complex eigenvalues which are not in $\mathbb{R}$.

**Solution.** (a) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(c) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(d) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

(e) $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

7. Find an invertible matrix $P$ and a matrix $C$ of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that

$$\begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix} = PCP^{-1}.$$

**Solution.** The matrix $A = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$ has characteristic polynomial $(5-x)(1-x)+5 = x^2 - 6x + 10$ which, by the quadratic formula has roots $3+i$ and $3-i$. An eigenvector with eigenvalue $3-i$ is found by row reducing

$$A - (3-i)I = \begin{bmatrix} 2+i & -5 \\ 1 & -2+i \end{bmatrix} \sim \begin{bmatrix} 1 & -2+i \\ 2+i & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2+i \\ 0 & 0 \end{bmatrix}.$$

This indicates that $v = \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $3-i$.

By the last theorem in Lecture 18, it then holds that $A = PCP^{-1}$ for the matrices

$$P = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}.$$

8. Consider the sequence $a_n$ with $a_0 = 1$, $a_1 = 3$, which satisfies $a_{n+2} = a_n + a_{n+1}$ for $n \geq 0$.

   (a) Find a matrix $A$ such that $A \begin{bmatrix} a_{2n} \\ a_{2n+1} \end{bmatrix} = \begin{bmatrix} a_{2n+2} \\ a_{2n+3} \end{bmatrix}$ for all $n \geq 0$.

   (b) Find an exact formula for the $n$th term $a_n$ of the sequence.

   (c) More generally, suppose $p, q \in \mathbb{R}$ are any real numbers.

   Find an exact formula for the $n$th term for the sequence $b_n$ which begins as $b_0 = p$, $b_1 = q$, and satisfies $b_{n+2} = b_n + b_{n+1}$ for $n \geq 0$.

**Solution.** This challenging problem is more difficult than would be reasonable for an exam question. However, in principle you could find a solution by a method similar to how we computed an exact formula for the Fibonacci numbers in class. The sequence $a_n$ gives the *Tribonacci numbers*. An exact formula is discussed at `http://mathworld.wolfram.com/TribonacciNumber.html`, where $T_n$ is defined as $a_{n-1}$.

9. Give definitions of $u \bullet v$ and $\|v\|$ for vectors $u, v \in \mathbb{R}^n$. What is a *unit vector*? Define what it means for a set of vectors to be *orthogonal* and *orthonormal*.

**Solution.** See the textbook and the lecture notes.

10. Find an orthonormal basis for the column space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 3 & 5 & 8 \\ 1 & -1 & -3 & -2 \end{bmatrix}.$$

Then find an orthonormal basis for $(\operatorname{Col} A)^\perp$.

**Solution.** Find, we find a basis for $\operatorname{Col} A$ by row reducing:

$$A \sim \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 3 & 5 & 8 \\ 0 & -3 & -7 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 3 & 5 & 8 \\ 0 & 0 & -2 & -1 \end{bmatrix}.$$

This reveal columns 1, 2, and 3 of $A$ to be pivot columns. Hence

$$u = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad v = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \quad w = \begin{bmatrix} 4 \\ 5 \\ -3 \end{bmatrix}$$

is a basis for Col $A$. We can replace $w$ by $\frac{1}{2}(w-v)$. This vector $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is already orthogonal to $u$. To convert $v$ to a vector orthogonal to $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, we compute

$$\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{6}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2-1/2-2 \\ 3-0-2 \\ -1-1/2+2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \\ 1/2 \end{bmatrix}.$$

Thus the vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1 \\ 1/2 \end{bmatrix}$ forms an orthogonal basis for Col $A$. To make this basis orthonormal, we normalise to get:

$$\boxed{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.}$$

Since in this case Col $A = \mathbb{R}^3$, we have $(\text{Col}\,A)^\perp = \text{Nul}(A^T) = \{0\}$, so the empty set is an orthonormal basis for $(\text{Col}\,A)^\perp$.

11. Give a formula for the orthogonal projection of a vector $y \in \mathbb{R}^3$ onto the plane

$$H = \{v \in \mathbb{R}^3 : v_1 + 2v_2 + 3v_3 = 0\}.$$

**Solution.** A basis for $H$ is given by the vectors $u = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$. These vectors are not orthogonal, but if

$$w = v - \frac{u \bullet v}{u \bullet u} u = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} - \frac{-3}{5}\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6/5 \\ 12/5 \\ -2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 6 \\ 12 \\ -10 \end{bmatrix}$$

then $u$ and $w$ are an orthogonal basis for $H$. Let $u_1 = u$ and $u_2 = 5w$. Then $u_1$ and $u_2$ are an even simpler orthogonal basis for $H$. The formula for $\text{proj}_H(y)$ is then

$$\text{proj}_H(y) = \frac{y \bullet u_1}{5}\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + \frac{y \bullet u_2}{280}\begin{bmatrix} 6 \\ 12 \\ -10 \end{bmatrix}.$$

12. Find the best approximation to $z$ by vectors of the form $c_1v_1 + c_2v_2$ when

$$z = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -1 \end{bmatrix}, \qquad v_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix}, \qquad \text{and} \qquad v_2 = \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix}.$$

**Solution.** The best approximation is found by finding a least-squares solution to $Ac = z$ where

$$A = \begin{bmatrix} 2 & 5 \\ 0 & -2 \\ -1 & 4 \\ -3 & 2 \end{bmatrix} \qquad \text{and} \qquad c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Our least-squares solution is found as the exact solution to $A^T Ac = A^T z$. We have

$$A^T A = \begin{bmatrix} 14 & 0 \\ 0 & 49 \end{bmatrix} \quad \text{and} \quad A^T z = \begin{bmatrix} 7 \\ 0 \end{bmatrix}.$$

The system $A^T Ac = A^T z$ can be solved by row reducing:

$$\begin{bmatrix} 14 & 0 & 7 \\ 0 & 49 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$$

which indicates that there is a unique least-squares solution $c = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$. Therefore

$$\frac{1}{2}v_1 + 0v_2 = \begin{bmatrix} 1 \\ 0 \\ -1/2 \\ -3/2 \end{bmatrix}$$

is the best approximation to $z$ of the desired form.

13. Suppose a function $f : \mathbb{R} \to \mathbb{R}$ has the following values:

| $x$ | $f(x)$ |
|---|---|
| 0 | 0 |
| 1 | 6 |
| 2 | 5 |
| 3 | 10 |
| 4 | 7 |

(a) Find the equation of the line $y = ax + b$ that best approximates $f(x)$ is the sense of least-squares.

(b) Find the equation of the parabola $y = ax^2 + bx + c$ that best approximates $f(x)$ is the sense of least-squares.

(c) How would you find a function of the form $g(x) = 2^{ax+b}$ that is a good approximation for $f(x)$?

**Solution.** (a) If $y = ax + b$ is a line of best fit then $\begin{bmatrix} a \\ b \end{bmatrix}$ is a least-squares solution to $Ax = B$ for

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 6 \\ 5 \\ 10 \\ 7 \end{bmatrix}.$$

The least-squares solutions to $Ax = B$ are the exact solutions to $A^T Ax = A^T B$, which can be written as

$$\begin{bmatrix} 30 & 10 \\ 10 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 74 \\ 28 \end{bmatrix}.$$

Row reducing the augmented matrix of this system gives

$$\begin{bmatrix} 30 & 10 & 74 \\ 10 & 5 & 28 \end{bmatrix} \sim \begin{bmatrix} 10 & 5 & 28 \\ 30 & 10 & 74 \end{bmatrix} \sim \begin{bmatrix} 10 & 5 & 28 \\ 0 & -5 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & .5 & 2.8 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1.8 \\ 0 & 1 & 2 \end{bmatrix}.$$

Thus the line of best fix is $\boxed{y = 1.8x + 2}$.

(b) To find $a, b, c$ we compute the least-squares solution to $Ax = B$ for

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 6 \\ 5 \\ 10 \\ 7 \end{bmatrix}.$$

To find a least-squares solution, we find an exact solution to $A^T Ax = A^T B$, whose augmented matrix is

$$\begin{bmatrix} 354 & 100 & 39 & 228 \\ 100 & 30 & 10 & 74 \\ 30 & 10 & 5 & 28 \end{bmatrix}.$$

By the usual methods of row reduction, you can compute that the unique exact solution to $A^T Ax = A^T b$ is

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -6/7 \\ 183/35 \\ 2/7 \end{bmatrix}$$

so the best fix parabola is $\boxed{y = (-6/7)x^2 + (183/35)x + 2/7.}$

(c) To find a function of the desired form, you could solve the least-squares problem

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \log_2 6 \\ \log_2 5 \\ \log_2 10 \\ \log_2 7 \end{bmatrix}.$$

In this computation, we ignore the first data point $(x, f(x)) = (0, 0)$ since $\log_2 x$ is undefined when $x = 0$.

14. Consider the symmetric matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}.$$

Find an orthogonal matrix $U$ (that is, an invertible matrix with $U^T = U^{-1}$) and a diagonal matrix $D$ such that

$$A = UDU^T.$$

Repeat this exercise with 1, 2, 3, 4, 5, 6 replaced by a random list of six numbers.

**Solution.** Note that for an arbitrarily chosen symmetric matrix, it may be difficult to compute an orthogonal diagonalization without a calculator. In such cases, problems asking you to compute this wouldn't appear on exams when calculators are prohibited.

The important thing is to review the general produce for constructing an orthogonal diagonalization:

1. First find the characteristic polynomial of $A$.
2. Factor this polynomial to compute the eigenvalues $\lambda$ of the matrix.
3. For each eigenvalue $\lambda$, find a basis for $\mathrm{Nul}(A - \lambda I)$.
4. Apply the Gram-Schmidt process to turn this basis into an orthogonal basis.
5. Distinct eigenspaces of a symmetric matrix are automatically orthogonal, so you just need to put the bases found in the last step together and then normalise so that your have an orthonormal basis $v_1, v_2, \dots, v_n$ for $\mathbb{R}^n$.
6. The matrices $U$ and $D$ are then

$$U = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

15. Find a singular value decomposition for the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 3 & 0 \end{bmatrix}$$

**Solution.** A matrix $A$ may have many different singular value decompositions $A = U\Sigma V^T$. In each SVD, the middle term $\Sigma$ is uniquely determined by $A$, but $U$ and $V$ may vary. For the current problem, one possible singular value decomposition is

$$A = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{=U} \underbrace{\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{=\Sigma} \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}}_{=V^T}.$$

The diagonal entries of $\Sigma$ are the singular values of $A$ which are the squares roots of the eigenvalues of $A^TA$.

The columns $v_1, v_2, \dots, v_5$ of $V$ (the transposes of the rows of $V^T$) are an orthonormal basis of eigenvectors for $A^TA$.

The columns of $U$ are the normalised vectors obtained from $Av_1, Av_2, \dots, Av_r$, where $r = 4$ is the rank of $A$.

For example, since $v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $Av_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$, the second column of $U$ is $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

16. Do the first problem in each section of supplementary exercises for Chapters 1-7.

**Solution.** Answers are in the back of the textbook.