Summary

Quick summary of today's notes. Lecture starts on next page.

Matrix-vector products:

- We can multiply an $m \times n$ matrix A by a vector $v \in \mathbb{R}^n$. The result, written Av, belongs to $\mathbb{R}^m$.

  If $a_1, a_2, \dots, a_n \in \mathbb{R}^m$ are the columns of A and $v_1, v_2, \dots, v_n \in \mathbb{R}$ are the entries of v then

$$Av = \left[\begin{array}{cccc} a_1 & a_2 & \dots & a_n \end{array}\right] \left[\begin{array}{c} v_1 \\ v_2 \\ \vdots \\ v_n \end{array}\right] = v_1 a_1 + v_2 a_2 + \dots + v_n a_n.$$

  Here is a concrete example:

$$\left[\begin{array}{rrrr} 1 & 2 & 3 & 4 \\ -1 & 0 & -2 & -3 \end{array}\right] \left[\begin{array}{r} 1 \\ 10 \\ 1000 \\ 10000 \end{array}\right] = \left[\begin{array}{r} 1 \\ -1 \end{array}\right] + 10 \left[\begin{array}{r} 2 \\ 0 \end{array}\right] + 1000 \left[\begin{array}{r} 3 \\ -2 \end{array}\right] + 10000 \left[\begin{array}{r} 4 \\ -3 \end{array}\right] = \left[\begin{array}{r} 4321 \\ -3201 \end{array}\right].$$

- If A is $m \times n$, $u, v \in \mathbb{R}^n$, and $c \in \mathbb{R}$ then $A(u + v) = Au + Av \in \mathbb{R}^m$ and $A(cv) = c(Av) \in \mathbb{R}^m$. We say that $v \mapsto Av$ (the function whose output, given input $v \in \mathbb{R}^n$, is $Av \in \mathbb{R}^m$) is *linear*.

Matrix equations:

- If A is an $m \times n$ matrix, $b \in \mathbb{R}^m$, and

$$x = \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array}\right]$$

  is a vector of n variables, then $Ax = b$ is a *matrix equation*. This equation has the same solutions as the linear system with augmented matrix $\left[\begin{array}{cc} A & b \end{array}\right]$.

- The matrix equation $Ax = b$ has a solution for all b if and only if A has a pivot in every row.

Linear independence:

- Vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ are *linearly independent* if the only way to express $0 \in \mathbb{R}^n$ as a linear combination $c_1 v_1 + c_2 v_2 + \dots + c_p v_p$ for $c_1, c_2, \dots, c_p \in \mathbb{R}$ is by taking $c_1 = c_2 = \dots = c_p = 0$.

  Vectors that are not linearly independent are *linearly dependent*. Two or more vectors in $\mathbb{R}^n$ are linearly dependent precisely when one of the vectors is in the span of all of the others.

- A single nonzero vector is linearly independent. A list of vectors containing $0 \in \mathbb{R}^n$ is linearly dependent. Two vectors are linearly dependent if and only if one is a scalar multiple of the other.

- Any sufficiently large set of vectors in $\mathbb{R}^n$ is linearly dependent. Specifically, if $p > n$ then any vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ are linearly dependent.

- To determine if a general list of vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ is linearly dependent (when $p \leq n$) form the $n \times p$ matrix $A = \left[\begin{array}{cccc} v_1 & v_2 & \dots & v_p \end{array}\right]$. The matrix equation $Ax = 0$ always has at least one solution $x = 0$. The vectors are linearly dependent if and only if this equation has a second solution (and therefore, infinitely many solutions). This happens if and only if at least one column of A is not a pivot column, so to find the answer you just need to compute $\mathsf{RREF}(A)$.

1 Last time: Vectors

A *(column) vector* of size n is an $n \times 1$ matrix:

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

A vector has the same data as a list of real numbers.

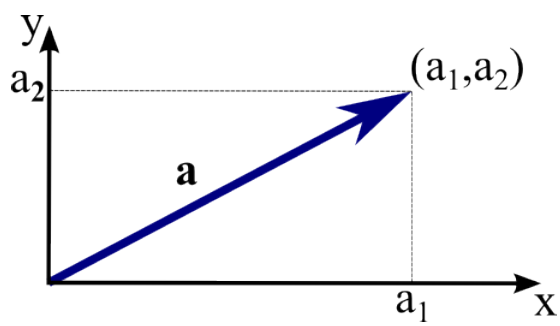
Let $\mathbb{R}^n$ be the set of all vectors with exactly n rows.

We can add two vectors of the same size: $\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$.

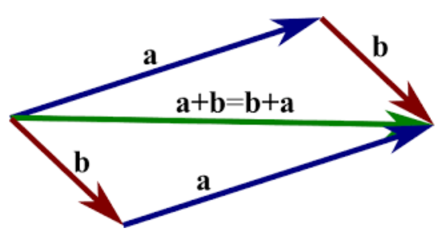
We can multiply a vector by a *scalar*: $cv = c \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}$ for $c \in \mathbb{R}$ and $v \in \mathbb{R}^n$

The word "scalar" is a synonym for number.

We visualize vectors $a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2$ as arrows in the Cartesian plane from the origin to $(x, y) = (a_1, a_2)$:



Relative to this picture, the sum $a + b$ of two vectors $a, b \in \mathbb{R}^2$ is the vector represented by the arrow from the origin to the point which is the opposite vertex of the parallelogram with sides a and b:



The *zero vector* $0 \in \mathbb{R}^n$ is the vector

$$0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

whose entries are all zero. We have $0 + v = v + 0 = v$ for any vector v.

A *linear combination* of vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ is any vector of the form

$$y = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p \in \mathbb{R}^n$$

where $c_1, c_2, \dots, c_p \in \mathbb{R}$.

The *span* of some vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ is the set of all of their linear combinations. Denote this by

$$\mathbb{R}\text{-span}\{v_1, v_2, \dots, v_p\} \qquad \text{or} \qquad \text{span}\{v_1, v_2, \dots, v_p\}.$$

Proposition. If $v_1, v_2, \dots, v_p \in \mathbb{R}^n$, then a vector $y \in \mathbb{R}^n$ belongs to $\mathbb{R}$-span$\{v_1, v_2, \dots, v_p\}$ if and only if the matrix $\begin{bmatrix} v_1 & v_2 & \dots & v_p & y \end{bmatrix}$ is the augmented matrix of a consistent linear system.

In terms of geometry, the span of a set of vectors in $\mathbb{R}^2$ is either a point (at the origin), a line (through the origin), or the whole plane $\mathbb{R}^2$. The span of a set of vectors in $\mathbb{R}^3$ is either a point (at the origin), a line (through the origin), a plane (containing the origin), or all of $\mathbb{R}^3$.

2 Multiplying matrices and vectors

So far we have been using matrices as a compact notation for representing linear systems.

Today we introduce a second, perhaps more fundamental way of viewing a matrix: namely, as an operator that transforms one vector to another.

Definition. If A is a matrix with columns $a_1, a_2, \dots, a_n \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$, so that

$$A = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \qquad \text{and} \qquad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

then the *matrix-vector product* Av is the vector in $\mathbb{R}^m$ given by:

$$Av = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1 a_1 + v_2 a_2 + \cdots + v_n a_n \in \mathbb{R}^m.$$

Thus Av is the linear combination of the columns of A with coefficients given by the entries of v.

Example. If $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix}$ and $v = \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$ then $a_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $a_2 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, and $a_3 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ so

$$\boxed{Av = 4a_1 + 3a_2 + 7a_3 = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -15 \end{bmatrix} + \begin{bmatrix} -7 \\ 21 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.}$$

Example. If $A = \begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix}$ and $v = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ then $a_1 = \begin{bmatrix} 2 \\ 8 \\ -5 \end{bmatrix}$ and $a_2 = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$ so we have

$$\boxed{Av = 4a_1 + 7a_2 = 4\begin{bmatrix} 2 \\ 8 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 32 \\ -20 \end{bmatrix} + \begin{bmatrix} -21 \\ 0 \\ 14 \end{bmatrix} = \begin{bmatrix} -13 \\ 32 \\ -6 \end{bmatrix}.}$$

If A is $m \times n$ then Av is only defined for $v \in \mathbb{R}^n$ (when v is an $n \times 1$ matrix), and in this case $Av \in \mathbb{R}^m$. Thus A transforms vectors in $\mathbb{R}^n$ to vectors in $\mathbb{R}^m$.

This transformation is *linear*:

1. If A is an $m \times n$ matrix and $u, v \in \mathbb{R}^n$ then $A(u + v) = Au + Av$.
2. If A is an $m \times n$ matrix and $v \in \mathbb{R}^n$ and $c \in \mathbb{R}$ then $A(cv) = c(Av)$.

Let A and v be the general $m \times n$ matrix and n-row vector given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

Quick way to compute Av: match up entries in the ith column of A with the entry in the ith row of v.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \dots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \dots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \dots + a_{mn}v_n \end{bmatrix}.$$

For example, $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = 1 \cdot 5 + 2 \cdot 6 + 3 \cdot 7 + 4 \cdot 8 = 5 + 12 + 21 + 32 = 70.$

3 Matrix equations

If A is an $m \times n$ matrix with columns $a_1, a_2, \dots, a_n \in \mathbb{R}^m$ and

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^m$$

where each x_i is a variable, then we call $Ax = b$ a *matrix equation*.

Proposition. The matrix equation $Ax = b$ has the same solutions as both the vector equation $x_1a_1 + x_2a_2 + \dots + x_na_n = b$ and the linear system whose augmented matrix is $\begin{bmatrix} a_1 & a_2 & \dots & a_n & b \end{bmatrix}$.

Proposition. The matrix equation $Ax = b$ has a solution if and only if b is a linear combination of the columns of A, that is, $b \in \mathbb{R}\text{-span}\{a_1, a_2, \dots, a_n\}$.

Example. Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Does $Ax = b$ have a solution for all choices of $b_1, b_2, b_3 \in \mathbb{R}$?

The system $Ax = b$ has a solution if and only if

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix}$$

is the augmented matrix of a consistent linear system. We can determine if this system is consistent by row reducing the matrix to echelon form:

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ 0 & 7 & 5 & 3b_1 + b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ 0 & 0 & 0 & b_1 - \frac{1}{2}b_2 + b_3 \end{bmatrix}.$$

The last matrix is in echelon form, so its leading entries are the pivot positions of our first matrix. The corresponding linear system is consistent if and only if the last column does not contain a pivot position. This occurs precisely when $b_1 - \frac{1}{2}b_2 + b_3 = 0$.

But we can choose numbers such that $b_1 - \frac{1}{2}b_2 + b_3 \neq 0$: take $b_1 = 1$ and $b_2 = b_3 = 0$. Therefore our original matrix equation $Ax = b$ does not always have a solution.

We can generalize this example:

Theorem. Let A be an $m \times n$ matrix. The following properties are equivalent, meaning that if one of them holds, then they all hold, but if one of them fails to hold, then they all fail:

1. For each vector $b \in \mathbb{R}^m$, the matrix equation $Ax = b$ has a solution.
2. Each vector $b \in \mathbb{R}^m$ is a linear combination of the columns of A.
3. The span of the columns of A is the set $\mathbb{R}^m$ (say this as: "the columns of A span $\mathbb{R}^m$").
4. A has a pivot position in every row.

Proof. (1)-(3) are different ways of saying the same thing.

We must check that (1)-(3) are equivalent to (4), which is less obvious.

If A has a pivot position in every row, then the augmented matrix $\begin{bmatrix} A & b \end{bmatrix}$ cannot have a pivot position in the last column; saying that A has a pivot position in every row means that $\begin{bmatrix} A & b \end{bmatrix}$ has to be row equivalent to something like

$$\begin{bmatrix} 0 & 1 & * & * & * & c_1 \\ 0 & 0 & 0 & 4 & * & c_2 \\ 0 & 0 & 0 & 0 & 3 & c_3 \end{bmatrix}$$

where c_1, c_2, c_3 are some numbers that depend on b_1, b_2, b_3. Regardless of what c_1, c_2, c_3 are, the given matrix has pivot columns 2, 4 and 5 but not 6.

We saw last time that not having a pivot position in the last column means that $\begin{bmatrix} A & b \end{bmatrix}$ is the augmented matrix of a consistent linear system. On the other hand, if A doesn't have a pivot position in some row, then it is always possible to choose b such that $\begin{bmatrix} A & b \end{bmatrix}$ has a pivot position in the last column, in which case the corresponding linear system has no solution. (Think about why this is true!) $\square$

4 Linear independence

Let $v_1, v_2, \dots, v_p$ be vectors in $\mathbb{R}^n$. These vectors are *linearly independent* if the only solution to the vector equation $x_1v_1 + x_2v_2 + \cdots + x_pv_p = 0$ is given by $x_1 = x_2 = \cdots = x_p = 0$.

The vectors $v_1, v_2, \dots, v_p$ are *linearly dependent* otherwise, that is, if there are numbers $c_1, c_2, \dots, c_p \in \mathbb{R}$, at least one of which is nonzero, such that $c_1v_1 + c_2v_2 + \cdots + c_pv_p = 0$.

Example. If $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ then $v_1 + v_3 = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ and $v_2 + v_3 = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}$, so

$$2(v_1 + v_3) - (v_2 + v_3) = 2v_1 - v_2 + v_3 = 0.$$

Hence v_1, v_2, v_3 are linearly dependent.

It is usually not so easy to guess whether a given list of vectors is linearly independent or not. In general, to do this we have to determine whether a certain linear system has a nonzero solution, which involves reducing its matrix to echelon form.

The columns of a matrix A are linearly independent if and only if $Ax = 0$ has no solution except $x = 0$.

Example. Some useful observations:

1. A list of just one vector v is linearly independent if and only if $v \neq 0$.
2. Two vectors $u, v \in \mathbb{R}^n$ are linearly dependent if and only if we can write $au + bv = 0$ for numbers $a, b \in \mathbb{R}$ with $a \neq 0$ or $b \neq 0$. If $a \neq 0$ then we have $u = (-b/a)v$. If $b \neq 0$ then $v = (-a/b)u$. (Both of these cases could occur.) Thus:

   Two vectors are linearly independent if and only if neither is a scalar multiple of the other.
3. If some $v_i = 0$ then $v_1, v_2, \dots, v_p$ are linearly dependent, since then
   $$0v_1 + \cdots + 0v_{i-1} + 5v_i + 0v_{i+1} + \cdots + 0v_p = 0.$$
   (The scalar 5 here can be replaced by any number.)

Proposition (Characterization of linear dependence)**.** The vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ are linearly dependent if and only if some vector v_i is a linear combination of the other vectors $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_p$.

Proof. We first show that if the vectors are linearly dependent then some vector is a linear combination of the others. Suppose $c_1v_1 + \cdots + c_pv_p = 0$ where $c_i \neq 0$. Then
$$v_i = (-c_1/c_i)v_1 + (-c_2/c_i)v_2 + \cdots + (-c_{i-1}/c_i)v_{i-1} + (-c_{i+1}/c_i)v_{i+1} + \cdots + (-c_p/c_i)v_p$$
so v_i is a linear combination of $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_p$.

Conversely, if $v_i = c_1v_1 + \cdots + c_{i-1}v_{i-1} + c_{i+1}v_{i+1} + \cdots + c_pv_p$ for some coefficients in $\mathbb{R}$, so that v_i is a linear combination of the remaining vectors, then $c_1v_1 + \cdots + c_{i-1}v_{i-1} - v_i + c_{i+1}v_{i+1} + \cdots + c_pv_p = 0$ which means that the vectors are linearly dependent, since the coefficient of at least v_i is nonzero. □

We conclude this lecture with a useful, non-obvious fact:

Theorem. Suppose $v_1, v_2, \dots, v_p \in \mathbb{R}^n$. If $p > n$ then these vectors are linearly dependent.

Proof. Saying these vectors are linearly dependent is the same thing as saying that the $n \times (p+1)$ matrix
$$A = \begin{bmatrix} v_1 & v_2 & \dots & v_p & 0 \end{bmatrix}$$
is the augmented matrix of a linear system with at least one free variable. A variable x_i for $1 \leq i \leq p$ is free for this system precisely when i is not a pivot column of A. There can only be 1 pivot position in each row, so there can be at most n pivot columns in A. If $p > n$, it follows that there will be at least $p - n > 0$ free variables, so our vectors must be linearly dependent. □

Example. Suppose $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $w = \begin{bmatrix} 5 \\ 60 \end{bmatrix}$. Then
$$A = \begin{bmatrix} 1 & 1 & 5 & 0 \\ 2 & 3 & 60 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 5 & 0 \\ 0 & 1 & 50 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -45 & 0 \\ 0 & 1 & 50 & 0 \end{bmatrix} = \mathsf{RREF}(A)$$
so the pivot columns of A are 1 and 2, while x_3 is a free variable. Therefore u, v, w are linearly dependent.

In fact we have $x_1u + x_2v + x_3w = 0$ if and only if $x_1 - 45x_3 = x_2 + 50x_3 = 0$.

If $x_3 = 1$ then this equation holds for $x_1 = 45$ and $x_2 = -50$, so $45u - 50v + w = 0$.

Similarly, if $x_3 = 2$ then the equation holds for $x_1 = 90$ and $x_2 = -100$, so also $90u - 100v + w = 0$.

5 Vocabulary

Keywords from today's lecture:

1. The **product** of a matrix A and a vector v.

   This is only defined if A is $m \times n$ and $v \in \mathbb{R}^n$.

   In this case, if

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

   then their product is

$$Av = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \dots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \dots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \dots + a_{mn}v_n \end{bmatrix} \in \mathbb{R}^m.$$

   Example: $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 5 + 12 + 21 + 32 \\ 6 + 8 \end{bmatrix} = \begin{bmatrix} 70 \\ 14 \end{bmatrix}.$

2. A **matrix equation**.

   An equation of the form $Ax = b$ where A is an $m \times n$ matrix with columns $a_1, a_2, \dots, a_n \in \mathbb{R}^m$ and

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

   is a vector where each x_i is a variable and $b \in \mathbb{R}^m$.

   This equation has the same solutions as the linear system with augmented matrix $\begin{bmatrix} A & b \end{bmatrix}$.

   There are several equivalent ways of characterizing whether this system has a solution.

   Example: $\begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$

3. **Linearly independent** vectors.

   The vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ are linearly independent when $x_1v_1 + \dots + x_pv_p = 0$ if and only if $x_1 = x_2 = \dots = x_p = 0$; equivalently, when the matrix equation

$$\begin{bmatrix} v_1 & v_2 & \dots & v_p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = 0$$

   has no solutions other than $x = 0$.

Vectors that are not linearly independent are **linearly dependent**.

Example: The three vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$ are linearly independent.

The four vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$ are linearly dependent.