

## Summary

Quick summary of today's notes. Lecture starts on next page.

- A *vector space* is a nonempty set with a “zero vector” and two operations that can be thought of a “vector addition” and “scalar multiplication.” The operations must obey several conditions.
- There are notions of *subspaces*, *linear functions*, *linear combinations*, *spans*, *linear independence*, and *bases* for vector spaces. The definitions are essentially the same as for  $\mathbb{R}^n$ , with one minor caveat when we are considering linear combinations and independence of infinite sets of vectors.
- Every vector space has a basis, and every basis for a given vector space has the same number of elements, which could be infinite. This number of elements is the *dimension* of the vector space.
- If  $X$  and  $Y$  are sets, then let  $\text{Fun}(X, Y)$  be the set of functions  $f : X \rightarrow Y$ .

The set  $\text{Fun}(X, \mathbb{R})$  is naturally a vector space. If  $X$  is finite then  $\boxed{\dim \text{Fun}(X, \mathbb{R}) = |X|}$ .

- If  $U$  and  $V$  are vector spaces, then let  $\text{Lin}(U, V)$  be the set of linear functions  $f : U \rightarrow V$ .

The set  $\text{Lin}(U, V)$  is naturally a vector space. If  $\dim U < \infty$  then  $\boxed{\dim \text{Lin}(U, \mathbb{R}) = \dim U}$ .

Moreover, if  $W$  is another vector space and  $f \in \text{Lin}(V, W)$  and  $g \in \text{Lin}(U, V)$ , then  $f \circ g \in \text{Lin}(U, W)$ .

- Suppose  $f : U \rightarrow V$  is a linear function between vector spaces.

Define  $\text{range}(f) = \{f(u) : u \in U\} \subseteq V$  and  $\text{kernel}(f) = \{u \in U : f(u) = 0\} \subseteq U$ .

These sets are subspaces. If  $\dim U < \infty$  then  $\boxed{\dim \text{range}(f) + \dim \text{kernel}(f) = \dim U}$ .

- Let  $A$  be an  $n \times n$  matrix. Let  $\lambda$  be a number and suppose  $0 \neq v \in \mathbb{R}^n$ .

If  $Av = \lambda v$  then we say that  $v$  is an *eigenvector* for  $A$  and that  $\lambda$  is an *eigenvalue* for  $A$ .

More specifically,  $v$  is an *eigenvector with eigenvalue  $\lambda$*  for  $A$ .

This happens if and only if  $0 \neq v \in \text{Nul}(A - \lambda I_n)$ .

For example,  $v = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$  is an eigenvector with eigenvalue  $\lambda = -4$  for  $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$  since

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix}.$$

The zero vector is not allowed to be an eigenvector, but 0 can occur as an eigenvalue.

- The eigenvalues  $\lambda$  for  $A$  are the numbers such that  $\det(A - \lambda I_n) = 0$ .
- The eigenvectors with eigenvalue  $\lambda$  for  $A$  are the nonzero elements of  $\text{Nul}(A - \lambda I_n)$ .
- If  $A$  is a triangular matrix, then its eigenvalues are its diagonal entries.

For example, the eigenvalues of  $\begin{bmatrix} 1 & 6 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$  are 0 and 1.

# 1 Last time: vector spaces

A (*real*) *vector space*  $V$  is a set containing a *zero vector*, denoted  $0$ , with *vector addition* and *scalar multiplication* operations that let us produce new vectors  $u + v \in V$  and  $cv \in V$  from given elements  $u, v \in V$  and  $c \in \mathbb{R}$ . Several conditions must be satisfied so that these operations behave exactly like vector addition and scalar multiplication for  $\mathbb{R}^n$ . Most importantly, we require that

1.  $u + v = v + u$  and  $(u + v) + w = u + (v + w)$ .
2.  $v - v = 0$  where we define  $u - v = u + (-1)v$ .
3.  $v + 0 = v$
4.  $cv = v$  if  $c = 1$ .

There are a few other more conditions to give the full definition (see the notes from last time).

By convention, we refer to elements of vector spaces as *vectors*.

**Example.** All subspace of  $\mathbb{R}^n$  are vector spaces, with the usual zero vector and vector operations.

The set of  $m \times n$  matrices is a vector space, with the usual addition and scalar multiplication operations. The zero vector in this vector space is the  $m \times n$  zero matrix.

Most vector spaces that we encounter are either subspaces of  $\mathbb{R}^n$  or subspaces of the following construction.

**Proposition.** Let  $X$  be a set and let  $V$  be a vector space.

Then the set  $\text{Fun}(X, V)$  of all functions  $f : X \rightarrow V$  is a vector space once we define

$$\begin{aligned} f + g &= (\text{ the function that maps } x \mapsto f(x) + g(x) \text{ for } x \in X ), \\ cf &= (\text{ the function that maps } x \mapsto c \cdot f(x) \text{ for } x \in X ), \\ 0 &= (\text{ the function that maps } x \mapsto 0 \in V \text{ for } x \in X ), \end{aligned}$$

for  $f, g \in \text{Fun}(X, V)$  and  $c \in \mathbb{R}$ .

**Definition.** The definitions of a *subspace* of a vector space and of *linear transformations* between vector spaces are identical to the ones we have already seen for subspaces of  $\mathbb{R}^n$ :

- A subset  $H \subseteq V$  is a *subspace* if  $0 \in H$  and if  $u + v \in H$  and  $cv \in H$  for all  $u, v \in H$  and  $c \in \mathbb{R}$ .
- A function  $f : U \rightarrow V$  is *linear* if  $f(u + v) = f(u) + f(v)$  and  $f(cv) = cf(v)$  for all  $u, v \in U$  and  $c \in \mathbb{R}$ .

**Proposition.** If  $U, V, W$  are vector spaces and  $f : V \rightarrow W$  and  $g : U \rightarrow V$  are linear functions then  $f \circ g : U \rightarrow W$  is also linear, where we define  $f \circ g(x) = f(g(x))$  for  $x \in U$ .

**Example.** If  $U$  and  $V$  are vector spaces then let  $\text{Lin}(U, V)$  be the set of linear functions  $f : U \rightarrow V$ .

Then  $\text{Lin}(U, V)$  is a subspace of  $\text{Fun}(U, V)$ .

Can you make sense of this statement? “ $\text{Lin}(\mathbb{R}^n, \mathbb{R}^m)$  is the vector space of  $m \times n$  matrices.”

**Example.** A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a *polynomial* if it has the formula

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

for some nonnegative integer  $n$  and some coefficients  $a_0, a_1, \dots, a_n \in \mathbb{R}$ .

The set of polynomial functions  $\mathbb{R} \rightarrow \mathbb{R}$  is a subspace of  $\text{Fun}(\mathbb{R}, \mathbb{R})$ .

**Example.** Suppose  $V$  is a vector space. Choose  $v \in V$ . Given a linear function  $f : V \rightarrow \mathbb{R}$ , define

$$v^*(f) = f(v).$$

Then  $v^*$  is a linear function  $\text{Lin}(V, \mathbb{R}) \rightarrow \mathbb{R}$ .

Let's go deeper: the function with the formula  $v \mapsto v^*$  is a linear function  $V \rightarrow \text{Lin}(\text{Lin}(V, \mathbb{R}), \mathbb{R})$ .

If  $V = \mathbb{R}^n$  then this function  $V \rightarrow \text{Lin}(\text{Lin}(V, \mathbb{R}), \mathbb{R})$  is invertible.

Let  $V$  be a vector space. The definitions of *linear combinations* and *linear independence* for vectors in  $V$  are mostly the same as for vectors in  $\mathbb{R}^n$ , with one caveat.

**Definition.** A *linear combination* of a **finite** list of vectors  $v_1, v_2, \dots, v_k \in V$  is a vector of the form

$$c_1 v_1 + c_2 v_2 + \dots + c_k v_k$$

for some scalars  $c_1, c_2, \dots, c_k \in \mathbb{R}$ .

We must be a little careful when defining linear combinations for infinite sets. Specifically: a *linear combination* of an infinite set of vectors is a linear combination of some **finite** subset of the vectors.

**Definition.** The *span* of a set of vectors is the set of all linear combinations that can be formed from the vectors. The span of a set of vectors in  $V$  is a subspace of  $V$ .

**Example.** The subspace of polynomials in  $\text{Fun}(\mathbb{R}, \mathbb{R})$  is the span of the set of functions  $1, x, x^2, x^3, \dots$

The infinite sum  $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots + \frac{1}{n!}x^n + \dots$  does not belong to this subspace.

**Definition.** A finite list of vectors  $v_1, v_2, \dots, v_k \in V$  is *linearly independent* if it is impossible to express  $0 = c_1 v_1 + c_2 v_2 + \dots + c_k v_k$  except when  $c_1 = c_2 = \dots = c_k = 0$ .

An infinite list of vectors is defined to be *linearly independent* if every finite subset of the vectors is linearly independent.

**Definition.** A *basis* of a vector space  $V$  is a subset of linearly independent vectors whose span is  $V$ . Saying  $b_1, b_2, b_3, \dots$  is a basis for  $V$  is the same as saying that for each  $v \in V$ , there are unique coefficients  $x_1, x_2, x_3, \dots \in \mathbb{R}$ , **all but finitely many of which are zero**, such that  $v = x_1 b_1 + x_2 b_2 + x_3 b_3 + \dots$

**Theorem.** Let  $V$  be a vector space.

1.  $V$  has at least one basis.
2. Every basis of  $V$  has the same number of elements (but this could be infinite).
3. If  $A$  is a subset of linearly independent vectors in  $V$  then  $V$  has a basis  $B$  with  $A \subseteq B$ .
4. If  $C$  is a subset of vectors in  $V$  whose span is  $V$  then  $V$  has a basis  $B$  with  $B \subseteq C$ .

**Definition.** The *dimension* of a vector space  $V$  is the number  $\dim V$  of elements in any of its bases.

**Example.** If  $X$  is a finite set then  $\dim \text{Fun}(X, \mathbb{R}) = |X|$  where  $|X|$  is the size of  $X$ .

## 2 More on dimension

If  $V$  is a finite-dimensional vector space then I claim that  $\dim \text{Lin}(V, \mathbb{R}) = \dim V$ .

Suppose  $b_1, b_2, \dots, b_n$  is a basis for  $V$ .

Then a basis for  $\text{Lin}(V, \mathbb{R})$  is given by the linear functions  $\phi_1, \phi_2, \dots, \phi_n : V \rightarrow \mathbb{R}$  with the formulas

$$\phi_i(x_1 b_1 + x_2 b_2 + \dots + x_n b_n) = x_i \quad \text{for } x_1, x_2, \dots, x_n \in \mathbb{R}.$$

The unique way to express a linear function  $f : V \rightarrow \mathbb{R}$  as a linear combination of these functions is

$$f = f(b_1)\phi_1 + f(b_2)\phi_2 + \dots + f(b_n)\phi_n.$$

Assume  $V = \mathbb{R}^n$ . Then we can think of  $\text{Lin}(\mathbb{R}^n, \mathbb{R})$  as the vector space of  $1 \times n$  matrices.

If  $b_1 = e_1, b_2 = e_2, \dots, b_n = e_n$  is the standard basis, then  $\phi_1 = e_1^T, \phi_2 = e_2^T, \dots, \phi_n = e_n^T$ .

**Definition.** Suppose  $U$  and  $V$  are vector spaces and  $f : U \rightarrow V$  is a linear function.

Define  $\text{range}(f) = \{f(x) : x \in U\} \subseteq V$  and  $\text{kernel}(f) = \{x \in U : f(x) = 0\} \subseteq U$ .

These sets are subspaces which generalize the column space and null space of a matrix.

We have a version of the rank-nullity theorem for arbitrary vector spaces:

**Theorem** (Rank-Nullity Theorem). If  $\dim U < \infty$  then  $\dim \text{range}(f) + \dim \text{kernel}(f) = \dim U$ .

This specializes to our earlier statement about matrices when  $U = \mathbb{R}^n$  and  $V = \mathbb{R}^m$ .

We can prove the theorem in a self-contained, completely abstract way, but it's a little involved.

*Proof.* If  $b_1, b_2, \dots, b_n$  is a basis for  $U$  then the span of  $f(b_1), f(b_2), \dots, f(b_n)$  must be equal to  $\text{range}(f)$ .

Therefore  $\dim \text{range}(f) \leq \dim U < \infty$ . Since  $\text{kernel}(f) \subseteq U$ , we also have  $\dim \text{kernel}(f) < \infty$ .

Let  $k = \dim \text{range}(f)$  and  $l = \dim \text{kernel}(f)$ .

Choose  $u_1, u_2, \dots, u_k \in U$  such that  $f(u_1), f(u_2), \dots, f(u_k)$  is a basis for  $\text{range}(f)$ .

Choose a basis  $v_1, v_2, \dots, v_l$  for  $\text{kernel}(f)$ . We will check that  $u_1, u_2, \dots, u_k, v_1, v_2, \dots, v_l$  is a basis for  $U$ .

To show linear independence, suppose  $a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_l \in \mathbb{R}$  are such that

$$a_1 u_1 + a_2 u_2 + \dots + a_k u_k + b_1 v_1 + b_2 v_2 + \dots + b_l v_l = 0.$$

Applying  $f$  to both sides gives  $a_1 f(u_1) + a_2 f(u_2) + \dots + a_k f(u_k) = 0$ , so  $a_1 = a_2 = \dots = a_k = 0$ .

But this implies  $b_1 v_1 + b_2 v_2 + \dots + b_l v_l = 0$ , so we also have  $b_1 = b_2 = \dots = b_l = 0$ .

Our vectors  $u_1, u_2, \dots, u_k, v_1, v_2, \dots, v_l$  are therefore linearly independent in  $U$ .

Now let  $x \in U$ . By assumption  $f(x) = c_1 f(u_1) + c_2 f(u_2) + \dots + c_k f(u_k)$  for some  $c_1, c_2, \dots, c_k \in \mathbb{R}$ .

The vector  $x - c_1 u_1 - c_2 u_2 - \dots - c_k u_k$  is then in the span of  $v_1, v_2, \dots, v_l$  since it belongs to  $\text{kernel}(f)$ .

We conclude that  $x$  is a linear combination of  $u_1, u_2, \dots, u_k, v_1, v_2, \dots, v_l$ , so this is a basis for  $U$ .  $\square$

### 3 Eigenvectors and eigenvalues

We return to the concrete setting of  $\mathbb{R}^n$  and its subspaces. Let  $A$  be a square  $n \times n$  matrix.

**Definition.** An *eigenvector* of  $A$  is a **nonzero** vector  $v \in \mathbb{R}^n$  such that

$$Av = \lambda v$$

for a number  $\lambda \in \mathbb{R}$ . ( $\lambda$  is the Greek letter “lambda.”)

The number  $\lambda$  is called the *eigenvalue* of  $A$  for the eigenvector  $v$ .

We require eigenvectors to be nonzero because if  $v = 0$  then  $Av = \lambda v = 0$  for all numbers  $\lambda \in \mathbb{R}$ .

The number 0 is allowed to be an eigenvalue of  $A$ , however.

**Example.** If we are given  $A$  and  $v$ , it is easy to check whether  $v$  is an eigenvector: just compute  $Av$ .

For example, if  $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$  and  $v = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$  then  $Av = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4v$ .

Therefore  $v$  is an eigenvector of  $A$  with eigenvalue  $-4$ .

**Example.** What are the eigenvectors of the matrix  $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ?

If  $v \in \mathbb{R}^4$  were an eigenvector with eigenvalue  $\lambda$  then

$$Av = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} v_2 \\ v_3 \\ v_4 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}.$$

The last equation implies that  $0 = \lambda v_4$  and  $v_4 = \lambda v_3$  and  $v_3 = \lambda v_2$  and  $v_2 = \lambda v_1$ . In other words,

$$0 = \lambda v_4 = \lambda^2 v_3 = \lambda^3 v_2 = \lambda^4 v_1.$$

If  $\lambda \neq 0$  then this would mean that  $v_1 = v_2 = v_3 = v_4 = 0$ , but remember that  $v$  should be nonzero. Therefore the only possible eigenvalue of  $A$  is  $\lambda = 0$ . The eigenvectors of  $A$  with eigenvalue 0 are

$$v = \begin{bmatrix} v_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{where } v_1 \neq 0.$$

To say that “ $\lambda$  is an eigenvalue of  $A$ ” means that there exists a nonzero vector  $v \in \mathbb{R}^n$  such that  $Av = \lambda v$ .

Recall that  $I_n$  denotes the  $n \times n$  identity matrix. We abbreviate by setting  $I = I_n$ .

**Proposition.** A number  $\lambda \in \mathbb{R}$  is an eigenvalue of  $A$  if and only if  $A - \lambda I$  is not invertible.

*Proof.* The equation  $Ax = \lambda x$  has a nonzero solution  $x \in \mathbb{R}^n$  if and only if  $(A - \lambda I)x = 0$  has a nonzero solution, which occurs if and only if  $\text{Nul}(A - \lambda I) \neq \{0\}$ , or equivalently when  $A - \lambda I$  is not invertible.  $\square$

**Example.** If  $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$  then

$$A - 7I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \text{RREF}(A - 7I).$$

Since  $\text{RREF}(A - 7I) \neq I$ , the matrix  $A - 7I$  is not invertible so 7 is an eigenvalue of  $A$ .

**Corollary.** A number  $\lambda \in \mathbb{R}$  is an eigenvalue of  $A$  if and only if  $\det(A - \lambda I) = 0$ .

*Proof.* Remember that  $A - \lambda I$  is not invertible if and only if  $\det(A - \lambda I) = 0$ . □

Another way of defining an eigenvector: the eigenvectors of  $A$  with eigenvalue  $\lambda$  are precisely the nonzero elements of the null space  $\text{Nul}(A - \lambda I)$ . Since we know how to construct a basis for the null space of any matrix, we also know how to find all eigenvectors of a matrix for any given eigenvalue.

**Example.** In the previous example,  $\text{RREF}(A - 7I) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$  so  $Ax = 7x$  if and only if  $(A - 7I)x = 0$  if and only if  $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$  where  $x_1 - x_2 = 0$ . In this linear system,  $x_2$  is a free variable, and we can rewrite  $x$  as  $x = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . This means  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  is a basis for  $\text{Nul}(A - 7I)$ .

Therefore every eigenvector of  $A$  with eigenvalue 7 has the form  $\begin{bmatrix} a \\ a \end{bmatrix}$  for some  $a \in \mathbb{R}$ .

One calls the set of all  $v \in \mathbb{R}^n$  with  $Av = \lambda v$  the *eigenspace* of  $A$  for  $\lambda$ . We also call this the  *$\lambda$ -eigenspace* of  $A$ . Note that this is just the null space of  $A - \lambda I$ . A number is an eigenvalue of  $A$  if and only if the corresponding eigenspace is nonzero (that is, contains a nonzero vector).

**Example.** Suppose we were told that  $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$  has 2 as an eigenvalue.

To find a basis for the 2-eigenspace of  $A$ , we row reduce

$$A - 2I = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{RREF}(A - 2I).$$

Thus  $Ax = 2x$  if and only if  $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$  where  $x_1 - \frac{1}{2}x_2 + 3x_3 = 0$ , that is, if and only if

$$x = \begin{bmatrix} \frac{1}{2}x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

The vectors  $\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$  are then a basis for the 2-eigenspace of  $A$ .

Recall that a matrix is *triangular* if its nonzero entries all appear on or above the main diagonal, or all appear on or below the main diagonal.

**Theorem.** The eigenvalues of a triangular square matrix  $A$  are its diagonal entries.

*Proof.* If  $A$  has diagonal entries  $d_1, d_2, \dots, d_n$  then  $A - \lambda I$  is triangular with diagonal entries  $d_1 - \lambda, d_2 - \lambda, \dots, d_n - \lambda$ , so  $\det(A - \lambda I) = (d_1 - \lambda)(d_2 - \lambda) \cdots (d_n - \lambda)$  which is zero if and only if  $\lambda \in \{d_1, d_2, \dots, d_n\}$ . □

**Example.** The eigenvalues of the matrix  $\begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$  are 3, 0, and 2.

## 4 Vocabulary

Keywords from today's lecture:

1. **Subspace** of a vector space.

A nonempty subset closed under linear combinations.

2. **Linearly combination** and **span** of elements in a vector space.

A linear combination of a finite set of vectors  $v_1, v_2, \dots, v_p \in V$  is a vector of the form

$$c_1v_1 + c_2v_2 + \dots + c_pv_p$$

where  $c_1, c_2, \dots, c_p \in \mathbb{R}$ . A linear combination of an infinite set of vectors is a linear combination of some finite subset. The set of all linear combinations of a set of vectors is the span of the vectors.

3. **Linearly independent** elements in a vector space.

A list of elements in a vector space is **linearly dependent** if one vector can be expressed as a linear combination of a finite subset of the other vectors. If this is impossible, then the vectors are linearly independent.

Example:  $\cos(x)$  and  $\sin(x)$  are linearly independently in  $\text{Fun}(\mathbb{R}, \mathbb{R})$ .

Example: the infinite list of functions  $1, x, x^2, x^3, x^4, \dots$  are linearly independent in  $\text{Fun}(\mathbb{R}, \mathbb{R})$ .

4. **Basis** and **dimension** of a vector space.

A set of linearly independent elements whose span is the entire vector space.

Every basis in a vector space has the same number of elements. This number is defined to be the **dimension** of the vector space.

5. **Linear functions**.

If  $U$  and  $V$  are vector spaces, then a function  $f : U \rightarrow V$  is linear when

$$f(u + v) = f(u) + f(v) \quad \text{and} \quad f(cv) = cf(v)$$

for all  $u, v \in U$  and  $c \in \mathbb{R}$ .

6. **Eigenvector** for an  $n \times n$  matrix  $A$ .

A *nonzero* vector  $v \in \mathbb{R}^n$  such that  $Av = \lambda v$  for some real number  $\lambda \in \mathbb{R}$ .

The number  $\lambda$  is the **eigenvalue** of  $A$  for  $v$ .

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ is an eigenvector for } \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ with eigenvalue } 2 \text{ as } \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

7.  **$\lambda$ -eigenspace** for an  $n \times n$  matrix  $A$ , where  $\lambda \in \mathbb{R}$ .

The subspace  $\text{Nul}(A - \lambda I) \subseteq \mathbb{R}^n$  where  $I$  is the  $n \times n$  identity matrix.

If  $\lambda$  is not an eigenvalue of  $A$ , then this subspace is  $\{0\}$ .

But if  $\lambda$  is an eigenvalue of  $A$ , then the subspace is nonzero.