FINAL EXAMINATION SOLUTIONS - MATH 2121, FALL 2018.

Name:

Student ID:

Email:

Tutorial: T1A T1B T2A T2B T3A T3B

Problem #	Max points possible	Actual score
1	20	
2	15	
3	10	
4	15	
5	20	
6	10	
7	15	
8	15	
Total	120	

You have **180 minutes** to complete this exam.

No books, notes, or electronic devices can be used on the test.

Clearly label your answers by putting them in a box.

Partial credit can be given on some problems if you show your work. Good luck!

Problem 1. (4 + 4 + 4 + 4 + 4 = 20 points)

(a) State the definition of a *linear transformation* $T : \mathbb{R}^n \to \mathbb{R}^m$.

A *linear transformation* $T : \mathbb{R}^n \to \mathbb{R}^m$ is a function such that

$$T(u + v) = T(u) + T(v) \quad \text{and} \quad T(cv) = cT(v)$$

for all vectors $u, v \in \mathbb{R}^n$ and scalars $c \in \mathbb{R}$.

(b) Suppose $v_1 = \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix}$, $v_2 = \begin{bmatrix} -3 \\ 8 \\ -7 \end{bmatrix}$, $v_3 = \begin{bmatrix} -4 \\ 6 \\ -7 \end{bmatrix}$, and $w = \begin{bmatrix} 6 \\ -10 \\ 11 \end{bmatrix}$.

Determine if w is in the span of v_1, v_2, and v_3.

Justify your answer to receive full credit.

The reduced echelon form of $\begin{bmatrix} v_1 & v_2 & v_3 & w \end{bmatrix}$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2/9 \\ 0 & 0 & 1 & -9/7 \end{bmatrix}.$$

Since there is no pivot in the last column, we conclude that w is in the span of v_1, v_2, v_3, and more specifically that

$$w = (-2/9)v_2 + (-9/7)v_3.$$

(c) State the definition of the *dimension* a subspace V of $\mathbb{R}^n$.

The dimension of a subspace V is the size of any basis for V.

(d) Let $v_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 3 \\ 3 \\ 0 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ -2 \end{bmatrix}$, $v_4 = \begin{bmatrix} 9 \\ 8 \\ 4 \\ 2 \end{bmatrix}$, and $v_5 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$.

Determine if the vectors v_1, v_2, v_3, v_4, v_5 are linearly independent.

Justify your answer to receive full credit.

The vectors are not linearly independent, since they consist of 5 vectors in $\mathbb{R}^4$, but n vectors in $\mathbb{R}^m$ are never linearly independent if $n > m$.

(e) Consider the matrix

$$A = \begin{bmatrix} 1 & 7 & 0 \\ -2 & -3 & -3 \end{bmatrix}.$$

Find a 3×2 matrix B such that

$$Au \bullet v = u \bullet Bv$$

for all $u \in \mathbb{R}^3$ and $v \in \mathbb{R}^2$, where $\bullet$ denotes the vector inner product.

Since $Au \bullet v = (Au)^T v = u^T A^T v$ and $u \bullet Bv = u^T Bv$, the desired matrix is

$$\boxed{B = A^T = \begin{bmatrix} 1 & -2 \\ 7 & -3 \\ 0 & -3 \end{bmatrix}.}$$

Problem 2. (15 points) In the following statements, A, B, C, etc., are matrices (with all real entries), and b, u, v, w, x, etc., are vectors, unless otherwise noted.

Indicate which of the following is TRUE or FALSE.

One point will be given for each correct answer (no penalty for incorrect answers).

(1) Every linear system with fewer equations than variables has a solution.

FALSE Consider the system $\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 + x_2 + x_3 = 1 \end{cases}$

(2) If w is a linear combination of u and v in $\mathbb{R}^n$, then u is a linear combination of v and w.

FALSE Suppose $w = 0$ but u and v are linearly independent.

(3) If A is an $n \times n$ matrix and I is the $n \times n$ identity matrix and $A^m = 0$ for some positive integer m, then $I - A$ is invertible.

TRUE Check that $(I - A)^{-1} = I + A + A^2 + \cdots + A^{m-1}$.

(4) If A is a 2×2 matrix and $\det A = 0$, then one row of A is a scalar multiple of the other row.

TRUE

Since $\det A^T = \det A = 0$, the columns of A^T are linearly dependent.

(5) If A and B are row equivalent $m \times n$ matrices and the columns of A span $\mathbb{R}^m$, then so do the columns of B.

TRUE

The columns of A span $\mathbb{R}^m$ if A has a pivot in every row. If A has this property and B is row equivalent to A, then B also has a pivot in every row so its columns span $\mathbb{R}^m$.

(6) If A is an $m \times n$ matrix and the linear system $Ax = b$ has more free variables than basic variables, then $\operatorname{rank} A < \frac{n}{2}$.

TRUE

The rank of A is the number of basic variables, and the sum of the number of free and basic variables is n.

(7) If A is an $m \times n$ matrix then the function $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(x) = Ax$ for $x \in \mathbb{R}^n$ is one-to-one only if $\mathrm{Nul}A = \{0\}$.

TRUE

Assume $\mathrm{Nul}A = \{0\}$. If $x, y \in \mathbb{R}^n$ and $x \neq y$, then $x - y \neq 0$, so

$$0 \neq A(x - y) = Ax - Ay = T(x) - T(y)$$

and $T(x) \neq T(y)$. Therefore T is one-to-one.

(8) If A is a 3×3 matrix and at least 6 entries in A are zero, then A is not invertible.

FALSE Consider $A = I$ the 3×3 identity matrix

(9) Each eigenvector of an invertible square matrix A is an eigenvector of A^{-1}.

TRUE Since if $Av = \lambda v$ then $A^{-1}v = \lambda^{-1}v$.

(10) If A is an $n \times n$ matrix with fewer than n distinct eigenvalues, then A is not diagonalizable.

FALSE

The identity matrix $A = I$ has only one distinct eigenvalue but is diagonal and there diagonalizable.

(11) An $n \times n$ matrix can have n distinct eigenvalues and exactly $n-1$ real eigenvalues.

FALSE

In the problem instructions, we are told that A has all real entries. Therefore if λ is a complex eigenvalue that is not real then $\overline{\lambda}$ is a second complex eigenvalue that is not real.

(12) If the columns of A are orthonormal then A has at least as many rows as columns.

TRUE

If the columns are orthonormal then they are linearly independent.

(13) If W is a subspace of $\mathbb{R}^n$ then $(W^\perp)^\perp = W$.

TRUE

We have $W \subset (W^\perp)^\perp$ since $u \bullet v = v \bullet u$. The dimension of $W^\perp$ is $n - \dim W$ and the dimension of $(W^\perp)^\perp = n - \dim W^\perp = \dim W$. Since W is a subspace of $(W^\perp)^\perp$ with the same dimension, $(W^\perp)^\perp = W$.

(14) If A is symmetric then A has all real eigenvalues.

TRUE This was a theorem presented in class.

(15) Matrices with only real entries that have non-real eigenvalues can have singular value decompositions involving matrices with only real entries.

TRUE

Every matrix has an SVD and every SVD only involves real matrices

Problem 3. (5 + 5 = 10 points)

(a) Compute the inverse of the matrix

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}.$$

One checks that

$$\mathsf{RREF}\left(\begin{bmatrix} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -2 & 3 & -2 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{bmatrix}$$

so the inverse matrix is

$$\boxed{A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 3 & -2 \\ 1 & -1 & 1 \end{bmatrix}.}$$

(b) Compute the determinant of the matrix

$$A = \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & 4 & 4 \\ -1 & 0 & 1 & 0 \\ 2 & 0 & 4 & 1 \end{bmatrix}.$$

Performing several row operations gives

$$\begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & 4 & 4 \\ -1 & 0 & 1 & 0 \\ 2 & 0 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & 4 & 4 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 6 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & 4 & 4 \\ 0 & 4 & 3 & 3 \\ 0 & 0 & 6 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & 4 & 4 \\ 0 & 0 & -13 & -13 \\ 0 & 0 & 6 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & 4 & 4 \\ 0 & 0 & -13 & -13 \\ 0 & 0 & 0 & -5 \end{bmatrix}.$$

None of these row operations swapped or rescaled rows, so the determinant of the last matrix is equal to $\det A$. The last matrix is triangular so its determinant is $1 \cdot 1 \cdot -13 \cdot -5 = 65$. Therefore

$$\boxed{\det A = 65.}$$

Problem 4. (5 + 5 + 5 = 15 points) Let

$$\mathcal{V} = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} : a, b, c, d, e, f, g, h, i \in \mathbb{R} \right\}$$

be the vector space of 3×3 matrices.

Define $L : \mathcal{V} \to \mathcal{V}$ to be the linear transformation

$$L\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \right) = \begin{bmatrix} a+e+i & b+f & c \\ d+h & a+e+i & b+f \\ g & d+h & a+e+i \end{bmatrix}.$$

(a) Find a basis for the subspace $\mathcal{R} = \{L(A) : A \in \mathcal{V}\}$. What is $\dim \mathcal{R}$?

The matrices

$$\boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}$$

form a basis for $\mathcal{R}$, so $\dim \mathcal{R} = 5$.

(b) Find a basis for the subspace $\mathcal{N} = \{A \in \mathcal{V} : L(A) = 0\}$. What is $\dim \mathcal{N}$?

We have $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \in \mathcal{N}$ if and only if

$$c = b + f = a + e + i = d + h = g = 0$$

in which case we can write

$$A = \begin{bmatrix} a & b & 0 \\ d & e & -b \\ 0 & -d & -a-e \end{bmatrix}.$$

The matrices

$$\boxed{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}$$

therefore form a basis for $\mathcal{N}$, and $\dim \mathcal{N} = 4$.

Note that $\dim \mathcal{R} + \dim \mathcal{N} = 9 = \dim \mathcal{V}$ which is consistent with the rank-nullity theorem.

(c) A number $\lambda \in \mathbb{R}$ is an *eigenvalue* for L if there exists a nonzero matrix $A \in \mathcal{V}$ with $L(A) = \lambda A$, in which case say that A is an *eigenvector* for L.

Find the distinct eigenvalues λ for L. For each eigenvalue λ, provide a nonzero matrix $A \in \mathcal{V}$ with $L(A) = \lambda A$.

Part (b) shows that $\lambda = 0$ is an eigenvalue, with eigenvector $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

If $\lambda \neq 0$ is an eigenvector and $L(A) = \lambda A$, then A must have the form

$$A = \begin{bmatrix} a & b & c \\ d & a & b \\ e & d & a \end{bmatrix}$$

for some a, b, c, d, e. But then

$$L(A) = \begin{bmatrix} 3a & 2b & c \\ 2d & 3a & 2b \\ e & 2d & 3a \end{bmatrix} = \lambda A$$

so $\lambda \in \{1, 2, 3\}$.

An eigenvector for $\lambda = 1$ is $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

An eigenvector for $\lambda = 2$ is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

An eigenvector for $\lambda = 3$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Problem 5. (5 + 10 + 5 = 20 points)

(a) Compute the distinct eigenvalues of the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & -1 & -2 \\ 3 & 3 & 4 \end{bmatrix}$.

We have

$$\begin{aligned} \det(A - xI) &= \det \begin{bmatrix} 2-x & 0 & 0 \\ -3 & -1-x & -2 \\ 3 & 3 & 4-x \end{bmatrix} \\ &= (2-x)((-1-x)(4-x) - (-2)(3)) \\ &= (2-x)((x+1)(x-4) + 6) \\ &= (2-x)(x^2 - 4x + x - 4 + 6) \\ &= (2-x)(x^2 - 3x + 2) \\ &= (2-x)(2-x)(1-x). \end{aligned}$$

Therefore the distinct eigenvalues of A are 1 and 2.

(b) Again let $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & -1 & -2 \\ 3 & 3 & 4 \end{bmatrix}$.

For each eigenvalue λ of A, find a basis for the eigenspace $\mathrm{Nul}(A - \lambda I)$.

First suppose $\lambda = 2$. To find a basis of $\mathrm{Nul}(A - 2I)$ we row reduce

$$A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ -3 & -3 & -2 \\ 3 & 3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathsf{RREF}(A - 2I).$$

This tells us that $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathrm{Nul}(A - 2I)$ if and only if

$$x = \begin{bmatrix} -x_2 - 2/3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2/3 \\ 0 \\ 1 \end{bmatrix},$$

so a basis for $\mathrm{Nul}(A - 2I)$ is given by the two vectors

$$\boxed{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2/3 \\ 0 \\ 1 \end{bmatrix}.}$$

Next suppose $\lambda = 1$. To find a basis of $\mathrm{Nul}(A - I)$ we row reduce

$$A - I = \begin{bmatrix} 1 & 0 & 0 \\ -3 & -2 & -2 \\ 3 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \mathsf{RREF}(A - I).$$

This tells us that $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathrm{Nul}(A - I)$ if and only if

$$x = \begin{bmatrix} 0 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix},$$

so a basis for $\mathrm{Nul}(A - I)$ is given by the since vectors

$$\boxed{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.}$$

(c) Continue to let $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & -1 & -2 \\ 3 & 3 & 4 \end{bmatrix}$.

Determine if A is diagonalizable. If A is diagonalizable, then give an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}.$$

If A is not diagonalizable, give an explanation why.

The dimension the 2-eigenspace of A is 2 and the dimension of the 1-eigensapce of A is 1. Since these dimensions sum to 3 which is the number of columns of A, it follows that A is diagonalizable with $A = PDP^{-1}$ for

$$\boxed{P = \begin{bmatrix} -1 & -2/3 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.}$$

Problem 6. (5 + 5 = 10 points) Consider the symmetric matrix

$$A = \begin{bmatrix} 9/2 & 11/2 \\ 11/2 & 9/2 \end{bmatrix}.$$

(a) Find a 2×2 orthogonal matrix U and a 2×2 diagonal matrix D such that

$$A = UDU^T = UDU^{-1}.$$

The sum of the eigenvalues of A is $\mathrm{tr}A = 9$ while the product of the eigenvalues is $\det A = 81/4 - 121/4 = -10$. From these constraints, we can guess that the eigenvalues must be 10 and -1. (Alternatively, one can find the eigenvalues by factoring $\det(A - xI)$.)

To find eigenvalues corresponding to these eigenvalues, we row reduce

$$A + I = \begin{bmatrix} 11/2 & 11/2 \\ 11/2 & 11/2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \mathsf{RREF}(A + I)$$

which implies that $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is eigenvector with eigenvalue 1, while

$$A - 10I = \begin{bmatrix} -11/2 & 11/2 \\ 11/2 & -11/2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \mathsf{RREF}(A + 10I)$$

while implies that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 10. These eigenvalues are already orthogonal, so converting them to unit vectors gives an orthonormal basis of $\mathbb{R}^2$ consisting of the eigenvectors

$$\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We conclude that $A = UDU^T$ for the matrices

$$\boxed{U = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 10 \end{bmatrix}.}$$

(b) Continue to let

$$A = \begin{bmatrix} 9/2 & 11/2 \\ 11/2 & 9/2 \end{bmatrix}.$$

Find exact formulas for the functions $a(n)$, $b(n)$, $c(n)$, $d(n)$ such that

$$A^n = \begin{bmatrix} a(n) & b(n) \\ c(n) & d(n) \end{bmatrix}$$

for all positive integers n.

We have $A^n = UD^nU^T$ where U and D as in the previous part. Therefore

$$\begin{aligned} A^n &= \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (-1)^n & 0 \\ 0 & 10^n \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (-1)^n & 0 \\ 0 & 10^n \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} -(-1)^n & 10^n \\ (-1)^n & 10^n \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} (-1)^n + 10^n & -(-1)^n + 10^n \\ -(-1)^n + 10^n & (-1)^n + 10^n \end{bmatrix}. \end{aligned}$$

Thus we have

$$\boxed{A^n = \begin{bmatrix} a(n) & b(n) \\ c(n) & d(n) \end{bmatrix} = \begin{bmatrix} (10^n + (-1)^n)/2 & (10^n - (-1)^n)/2 \\ (10^n - (-1)^n)/2 & (10^n + (-1)^n)/2 \end{bmatrix}.}$$

Problem 7. (5 + 5 + 5 = 15 points) Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \\ -1 & 1 & -1 \end{bmatrix}.$$

(a) Find an orthogonal basis for the column space of A.

By row reducing A to echelon form, one finds that the three columns of A are linearly independent and therefore give a basis for the column space. The first two columns are already orthogonal, while

$$\frac{\begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}} = \frac{2+1}{1+1+1} = 1$$

and

$$\frac{\begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}} = \frac{2+2-1}{1+1+1} = 1$$

The vector

$$\begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

is therefore orthogonal to the first two columns and the three vectors

$$\boxed{\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}}$$

are an orthogonal basis for the column space of A.

(b) Find a least-squares solution to the linear system $Ax = b$ where

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \\ -1 & 1 & -1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}.$$

A least-squares solution is an exact solution to $A^T Ax = A^T b$, and we have

$$A^T A = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 3 \\ 3 & 3 & 9 \end{bmatrix} \quad \text{and} \quad A^T b = \begin{bmatrix} 1 \\ 14 \\ 10 \end{bmatrix}.$$

Row reducing gives

$$\begin{bmatrix} A^T A & A^T b \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 0 & 3 & 3 & 14 \\ 3 & 3 & 9 & 10 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & 3 & 1 \\ 0 & 3 & 3 & 14 \\ 0 & 3 & 6 & 9 \end{bmatrix}$$
$$\to \begin{bmatrix} 3 & 0 & 3 & 1 \\ 0 & 3 & 3 & 14 \\ 0 & 0 & 3 & -5 \end{bmatrix}$$
$$\to \begin{bmatrix} 3 & 0 & 0 & 6 \\ 0 & 3 & 0 & 19 \\ 0 & 0 & 3 & -5 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 19/3 \\ 0 & 0 & 1 & -5/3 \end{bmatrix}$$

so $\boxed{x = \begin{bmatrix} 2 \\ 19/3 \\ -5/3 \end{bmatrix}}$ is a least-squares solution.

(c) Again let $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \\ -1 & 1 & -1 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}$.

Compute the orthogonal projection of b onto the **orthogonal complement** of the column space of A.

If $x = \begin{bmatrix} 2 \\ 19/3 \\ -5/3 \end{bmatrix}$ is the least-squares solution in the previous part then

$$Ax = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}$$

is the orthogonal projection of b onto the column space of A.

The orthogonal projection of b onto the orthogonal complement of the column space of A is therefore

$$b - Ax = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix} - \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix} = \boxed{\begin{bmatrix} -3 \\ 3 \\ 3 \\ 0 \end{bmatrix}}$$

Problem 8. (5 + 10 = 15 points)

(a) Compute the singular values $\sigma_1 \geq \sigma_2$ of the matrix

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 4 & 0 \\ 0 & 1 \end{bmatrix}.$$

The singular values of A are the square roots of the eigenvalues of

$$A^T A = \begin{bmatrix} 25 & 0 \\ 0 & 2 \end{bmatrix}.$$

Since this matrix is diagonal, its eigenvalues are 25 and 2, so the singular values of A are

$$\boxed{\sigma_1 = 5 \quad \text{and} \quad \sigma_2 = \sqrt{2}}$$

(b) Find a singular value decomposition for

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 4 & 0 \\ 0 & 1 \end{bmatrix}.$$

In other words, find a 4×4 invertible matrix U and a 2×2 invertible matrix V with $U^{-1} = U^T$ and $V^{-1} = V^T$ such that

$$A = U\Sigma V^T \quad \text{where} \quad \Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Let $\sigma_1 = 5$ and $\sigma_2 = \sqrt{2}$ be the singular values of A.

Since $A^T A$ is diagonal, the vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

are an orthonormal basis of eigenvectors.

Next define $u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{5} \begin{bmatrix} 3 \\ 0 \\ 4 \\ 0 \end{bmatrix}$ and $u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

Adding the vectors

$$u_3 = \frac{1}{5} \begin{bmatrix} -4 \\ 0 \\ 3 \\ 0 \end{bmatrix} \quad \text{and} \quad u_4 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

gives an orthonormal basis u_1, u_2, u_3, u_4 of $\mathbb{R}^4$.

The desired matrices U, Σ, and V are then

$$U = \begin{bmatrix} 3/5 & 0 & 3/5 & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 4/5 & 0 & -4/5 & 0 \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 5 & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$