This document is intended as an **exact transcript** of the lecture, with extra summary and vocabulary sections for your convenience. By design, the material covered in lecture is exactly the same as what is in these notes. Due to time constraints, the notes may sometimes only contain limited illustrations, proofs, and examples; for a more thorough discussion of the course content, **consult the textbook**.

Summary

Quick summary of today's notes. Lecture starts on next page.

- A matrix is in *echelon form* if looks something like this:

$$\begin{bmatrix} 0 & 0 & 5 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 6 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & -7 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

  Here each $*$ can be any number. A matrix is in *reduced echelon form* if it looks like this:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & * & * & 0 & * & * & 0 \\ 0 & 0 & 0 & 1 & * & * & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1}$$

- Each matrix A is row equivalent to exactly one matrix in reduced echelon form, denoted $\mathsf{RREF}(A)$.
- There is an algorithm to compute $\mathsf{RREF}(A)$, called *row reduction*. The algorithm gives a sequence of row operations that we perform on A to obtain $\mathsf{RREF}(A)$. The algorithm is hard to summarize in one line, but it's not too complicated; you will want to become very familiar with the definition.
- The matrices A and $\mathsf{RREF}(A)$ are row equivalent. If A is the augmented matrix of a linear system, then $\mathsf{RREF}(A)$ is the augmented matrix of another system with the same solutions. Key advantage: it is easy to read off the form of all solutions to the linear system corresponding to $\mathsf{RREF}(A)$.
- The *pivot positions* in a matrix A are the locations of the leading 1s in $\mathsf{RREF}(A)$. To find these, you must first compute $\mathsf{RREF}(A)$. If $\mathsf{RREF}(A)$ is (1) then the pivot positions are $(1, 3)$, $(2, 4)$, $(3, 7)$, $(4, 10)$. The *pivot columns* are the columns with pivots; in the example, these are 3, 4, 7, 10.

  Here's (1) again with the pivot positions in boxes:

$$\begin{bmatrix} 0 & 0 & \boxed{1} & 0 & * & * & 0 & * & * & 0 \\ 0 & 0 & 0 & \boxed{1} & * & * & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Consider a linear system in n variables with augmented matrix A.
  - The variable x_i is a *basic variable* if i is a pivot column of A.
  - All non-basic variables are *free variables*.
  - The system has 0 solutions if the last column of A has a pivot (e.g., if $\mathsf{RREF}(A)$ is (1)).
  - If this doesn't occur, then the system has only one solution if all variables are basic.
  - If there is at least one free variable and the system has a solution, then it has infinitely many.

1 Last time: linear systems and row operations

Here's what we did last time: a *system of linear equations* or *linear system* is a list of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where $x_1, x_2, \dots, x_n$ are variables and each a_{ij} and b_i for $1 \le i \le m$ and $1 \le j \le n$ is a number.

The *coefficient matrix* and *augmented matrix* of such a system are respectively

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{bmatrix}.$$

The coefficient matrix is $m \times n$: it has m rows and n columns.

The augmented matrix has one extra column, so has size $m \times (n+1)$.

A *solution* to a linear system is a list of numbers $(s_1, s_2, \dots, s_n)$ such that setting $x_1 = s_1$, $x_2 = s_2$, $\dots$, $x_n = s_n$, all at the same time, makes each equation in the system a true statement.

Two linear systems are *equivalent* if they have the same solutions.

Important fact: Any linear system has either 0, 1, or infinitely many solutions.

We solve a linear system by performing *row operations* on its augmented matrix.

The following are row operations:

(1) Replace one row by the sum of itself and a multiple of another row.

(2) Multiply all entries in one row by a fixed nonzero number.

(3) Interchange two rows.

Let's do an example to see these rules in action.

Example. Consider the linear system

$$\begin{aligned} x_1 + 2x_2 + 5x_3 &= 1 \\ x_1 + x_3 &= 0 \\ x_2 + x_3 &= 7 \end{aligned} \quad \text{which has augmented matrix} \quad \begin{bmatrix} 1 & 2 & 5 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 7 \end{bmatrix}$$

Adding -1 times the second row to the first is an example of row operation (1):

$$\begin{bmatrix} 1 & 2 & 5 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 7 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} \mathbf{0} & \mathbf{2} & \mathbf{4} & \mathbf{1} \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 7 \end{bmatrix}.$$

Next let's add -2 times the last row to the first row:

$$\begin{bmatrix} 0 & 2 & 4 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 7 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{2} & \mathbf{-13} \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 7 \end{bmatrix}.$$

Now lets use rule (3) to swap some rows:

$$\begin{bmatrix} 0 & 0 & 2 & -13 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 7 \end{bmatrix} \xrightarrow{(3)} \begin{bmatrix} 0 & 1 & 1 & 7 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -13 \end{bmatrix} \xrightarrow{(3)} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 2 & -13 \end{bmatrix}.$$

Now lets scale the third row by 1/2:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 2 & -13 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 7 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-13/2} \end{bmatrix}.$$

Finally, lets use (1) twice to cancel the entries in rows 1 and 2 in column 3:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 1 & -13/2 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{13/2} \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 1 & -13/2 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 & 0 & 13/2 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{27/2} \\ 0 & 0 & 1 & -13/2 \end{bmatrix}.$$

Two linear systems are *row equivalent* if their augmented matrices can be transformed to each other by a sequence of zero or more row operations.

Theorem. Row equivalent linear systems are equivalent, which means they have the same solutions.

Therefore the original system in our example has the same solutions as the system corresponding to the last matrix, which consists of the three equations

$$\begin{aligned} x_1 &= 13/2 \\ x_2 &= 27/2 \\ x_3 &= -13/2. \end{aligned}$$

This system has only one solution $(13/2, 27/2, -13/2)$, so the original system also has only one solution.

2 Row reduction to echelon form

The goal today is to give an algorithm to determine whether a linear system has 0, 1, or infinitely many solutions, and to find out what these solutions are when they exist.

The algorithm will be called *row reduction to echelon form* and will formalize the way we solved the linear system in the last example. Sometimes, this algorithm is also called *Gaussian elimination.*

Key ideas:

- For some linear systems it is easy to determine all solutions.
- We want to describe the special form of augmented matrices that correspond to such systems.
- Then we want to find a way to transform any matrix to this special form using row operations.
- Row operations don't change the solution set, so the solutions to an "easy" linear system are the same as any other linear system with a row equivalent augmented matrix.

To motivate what will be called the *(reduced) echelon form* of a matrix, let's consider what kinds of linear systems are easy to solve. Think of each equation in the system as giving a way to express **the first variable** that appears with nonzero coefficient in terms of the others. For example,

$$3x_2 + 2x_4 + 5x_5 = 1 \quad \text{can be rewritten as expressing} \quad x_2 = \tfrac{1}{3}(1 - 2x_4 - 5x_5).$$

With this in mind, here are some properties that would make it easier to solve a linear system:

(E1) The equations in our system should be ordered so any trivial equations $0 = 0$ are listed at the end.

(E2) If the first variable appearing in an equation is x_i then the first variable appearing in the next equation should be a later variable x_j with $i < j$. (Since the given equation "determines" x_i we don't need other equations to start with x_i.)

If we have a linear system in this form then we can solve it by choosing arbitrary values for any variables that are never the first variable in an equation (call these *free variables*), and then expressing each variable that does appear as the first variable in some equation (call these *basic variables*) in terms of these.

However, when expressing the basic variables in terms of the free variables, we need to go through the equations in the system **in reverse order** because the equation that determines a given basic variable x_i may involve both free variables and basic variables x_j with $i < j$. Consider this example:

Example. Here is a linear system in variables x_1, x_2, x_3, x_4 with properties (E1) and (E2):

$$\begin{cases} 2x_1 + x_2 + x_3 & = 1 \\ 3x_3 + 3x_4 & = 5 \\ 0 & = 0. \end{cases}$$

The free variables are x_2 and x_4. The basic variables are x_1 and x_3. If we set $x_2 = a$ and $x_4 = b$ then

$$3x_3 + 3x_4 = 5 \quad \Rightarrow \quad x_3 = \tfrac{1}{3}(5 - 3x_4) = \tfrac{1}{3}(5 - 3b) = \tfrac{5}{3} - b$$

and now we can substitute this into the first equation to figure out x_1 in terms of a and b:

$$2x_1 + x_2 + x_3 = 1 \quad \Rightarrow \quad x_1 = \tfrac{1}{2}(1 - x_2 - x_3) = \tfrac{1}{2}(1 - a - (\tfrac{5}{3} - b)) = -\tfrac{1}{3} - \tfrac{1}{2}a + \tfrac{1}{2}b$$

so the general solution is $(x_1, x_2, x_3, x_4) = (-\frac{1}{3} - \frac{1}{2}a + \frac{1}{2}b, a, \frac{5}{3} - b, b)$ where $a, b \in \mathbb{R}$ are arbitrary.

This linear system is *consistent* with infinitely many solutions.

A few other properties that would make this solution process even easier to carry out:

(R1) The coefficient of the first variable x_i appearing in an equation should be 1.

(R2) The first variable x_i appearing in an equation should not appear in any earlier equations.

If these properties also hold, then we can solve our linear system in the same way, but we won't have to do any arithmetic besides moving free variables to the right side of each equation. Also, when expressing the basic variables in terms of the free variables, we can go through the equations **in any order**.

Example. Here is a linear system in variables x_1, x_2, x_3, x_4 with properties (E1)-(E2) and (R1)-(R2):

$$\begin{cases} x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_4 & = -\frac{1}{3} \\ x_3 + x_4 & = \frac{5}{3} \\ 0 & = 0. \end{cases}$$

The free variables are x_2 and x_4. The basic variables are x_1 and x_3. If we set $x_2 = a$ and $x_4 = b$ then

$$x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_4 = -\tfrac{1}{3} \quad \Rightarrow \quad x_1 = -\tfrac{1}{3} - \tfrac{1}{2}a + \tfrac{1}{2}b \qquad \text{while} \qquad x_3 + x_4 = \tfrac{5}{3} \quad \Rightarrow \quad x_3 = \tfrac{5}{3} - b$$

so the general solution is $(x_1, x_2, x_3, x_4) = (-\frac{1}{3} - \frac{1}{2}a + \frac{1}{2}b, a, \frac{5}{3} - b, b)$ where $a, b \in \mathbb{R}$ are arbitrary.

This set of solutions is the same as before (in fact, the augmented matrices in both examples are row equivalent), but notice how much easier it is here to express x_1 and x_3 in terms of the other variables.

We now discuss what properties (E1)-(E2) and (R1)-(R2) **mean in terms of the augmented matrix**.

A row in a matrix $\begin{bmatrix} a & b & \dots & z \end{bmatrix}$ is *nonzero* if not every entry in the row is zero.

A *nonzero column* in a matrix is defined similarly.

The *leading entry* in a row of a matrix is the first nonzero entry from left going right.

For example, $\begin{bmatrix} 0 & 0 & 7 & 0 & 5 \end{bmatrix}$ has leading entry 7. The leading entry occurs in column 3.

Definition. A matrix with m rows and n columns is in *echelon form* if it has the following properties:

(E1) If a row is nonzero, then every row above it is also nonzero.

(E2) The leading entry in a nonzero row is strictly to the right of the leading entry of any earlier row.

The second property implies this additional property of a matrix in echelon form:

(E3) If a row is nonzero, then every entry below its leading entry in the same column is zero.

Some examples are helpful to understand this definition. The following is in echelon form:

$$\begin{bmatrix} 0 & 0 & 5 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 6 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}$$

Here each * can be replaced by an arbitrary number.

The matrix $\begin{bmatrix} 0 & 0 & 5 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 6 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is also in echelon form.

The matrix $\begin{bmatrix} 0 & 0 & 0 & 6 & * & * & * & * & * & * \\ 0 & 0 & 5 & 0 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}$ is **not** in echelon form: condition (E2) fails.

The matrix $\begin{bmatrix} 0 & 0 & 6 & * & * & * & * & * & * & * \\ 0 & 0 & 5 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}$ is **not** in echelon form: conditions (E2) and (E3) fail.

A sort of degenerate case: every one-row matrix is in echelon form. (Why?)

The only one-column matrices in echelon form are ones like $\begin{bmatrix} * \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ where $*$ can be any number.

There is a more restrictive version of echelon form that will be useful.

Definition. A matrix in echelon form is *reduced* if

(R1) Each nonzero row has leading entry 1.

(R2) The leading 1 in each nonzero row is the only nonzero number in its column.

A matrix in echelon form that is reduced is said to be in *reduced echelon form*.

The following matrix is in echelon form but is not reduced:

$$\begin{bmatrix} 0 & 0 & 5 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 6 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}$$

But we can apply row operations to turn it into reduced echelon form:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & * & * & 0 & * & * & 0 \\ 0 & 0 & 0 & 1 & * & * & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The fundamental theorem of today is the following:

Theorem. Each matrix A is row equivalent to exactly one matrix $\mathsf{RREF}(A)$ in reduced echelon form.

The proof of this result is included in an appendix of the textbook. By the end of today, the result should at least seem plausible, once we understand how to construct the matrix $\mathsf{RREF}(A)$ from A.

We call $\mathsf{RREF}(A)$ the *(row) reduced echelon form* of A.

A *pivot position* in a matrix A is the location containing a leading 1 in the reduced echelon form for A.

A *pivot column* in a matrix A is a column containing a pivot position.

If a matrix E is in echelon form and is row equivalent to A, then we say that E is *an echelon form* of A.

Proposition. In any echelon form E of a matrix A, the locations of the leading entries are the same.

This means we can compute the pivot positions of A from any echelon form E, and we do not necessarily have to compute the unique reduced echelon form $\mathsf{RREF}(A)$ which can take more work.

Example. Suppose

$$A = \begin{bmatrix} 0 & 3 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 3 & 1 & 1 \end{bmatrix}.$$

Let's find $\mathsf{RREF}(A)$. Add -1 times first row to third row, then $-2/3$ times first row to second row:

$$A = \begin{bmatrix} 0 & 3 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 3 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 3 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & \boxed{3} & 1 & 0 \\ 0 & 0 & \boxed{-2/3} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}.$$

The last matrix is in echelon form, but is not reduced. Pivot positions are boxed. To get to the reduced echelon form, rescale rows 1 and 2 by $1/3$ and $-3/2$, then add a multiple of the second row to the first:

$$\begin{bmatrix} 0 & 3 & 1 & 0 \\ 0 & 0 & -2/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1/3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathsf{RREF}(A).$$

Columns 2, 3, and 4 are the pivot columns.

We sometimes refer to an entry in a pivot position of a matrix as a *pivot*.

We are now ready to describe the row reduction algorithm. Let's first go over what this algorithm does to another specific matrix. Then we'll say what the procedure is for a generic matrix.

Example (Row reduction to echelon form, for a specific matrix)**.**

Input: for the general algorithm, the input is an $m \times n$ matrix A. Suppose this matrix is

$$A = \begin{bmatrix} 0 & 3 & -6 & 6 \\ 3 & -7 & 8 & -5 \\ 3 & -9 & 12 & -9 \end{bmatrix}.$$

Procedure:

1. Begin with the leftmost nonzero column.

   This is a pivot column. The pivot position is the top position of the column.

   For our matrix, the leftmost nonzero column is the first column; the pivot position is boxed:

$$\begin{bmatrix} \boxed{\mathbf{0}} & 3 & -6 & 6 \\ \mathbf{3} & -7 & 8 & -5 \\ \mathbf{3} & -9 & 12 & -9 \end{bmatrix}.$$

2. Select a nonzero entry in the current pivot column.

   If needed, perform a row operation to swap the row with this entry and the top row.

   For example, we can select the 3 in the second row of the first column and then swap rows 1 and 2:

$$\begin{bmatrix} \boxed{\mathbf{0}} & 3 & -6 & 6 \\ \mathbf{3} & -7 & 8 & -5 \\ \mathbf{3} & -9 & 12 & -9 \end{bmatrix} \to \begin{bmatrix} \boxed{3} & -7 & 8 & -5 \\ 0 & 3 & -6 & 6 \\ 3 & -9 & 12 & -9 \end{bmatrix}.$$

3. Use row operations to create zeros below the boxed pivot position:

$$\begin{bmatrix} \boxed{3} & -7 & 8 & -5 \\ 0 & 3 & -6 & 6 \\ 3 & -9 & 12 & -9 \end{bmatrix} \to \begin{bmatrix} \boxed{3} & -7 & 8 & -5 \\ 0 & 3 & -6 & 6 \\ 0 & -2 & 4 & -4 \end{bmatrix}$$

4. Repeat steps 1-3 on the bottom right submatrix:

$$\left[\begin{array}{c|ccc} 3 & -7 & 8 & -5 \\ \hline 0 & \boxed{3} & -6 & 6 \\ 0 & -2 & 4 & -4 \end{array}\right] \to \left[\begin{array}{c|ccc} 3 & -7 & 8 & -5 \\ \hline 0 & \boxed{3} & -6 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{c|c|cc} 3 & -7 & 8 & -5 \\ \hline 0 & 3 & -6 & 6 \\ \hline 0 & 0 & 0 & 0 \end{array}\right].$$

5. We now have a matrix in echelon form: $\begin{bmatrix} \boxed{3} & -7 & 8 & -5 \\ 0 & \boxed{3} & -6 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

   Start with the row containing the rightmost pivot position in our matrix, now in echelon form.

   Rescale rightmost pivot, then cancel entries above rightmost pivot position in same column:

$$\begin{bmatrix} 3 & -7 & 8 & -5 \\ 0 & \boxed{3} & -6 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & -7 & 8 & -5 \\ 0 & \boxed{1} & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & -6 & 9 \\ 0 & \boxed{1} & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

   Repeat with the next pivot position, going right to left:

$$\begin{bmatrix} \boxed{3} & 0 & -6 & 9 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 0 & -2 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

   The result is the reduced echelon form $\mathsf{RREF}(A) = \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

With this example in mind, we should now be able to follow the steps in the general algorithm:

Algorithm (Row reduction to echelon form, for a generic matrix)**.**

Input: an $m \times n$ matrix A.

Procedure:

1. Begin with the leftmost nonzero column.

   This is a pivot column. The pivot position is the top position of the column.

2. Select a nonzero entry in the current pivot column.

   If needed, perform a row operation to swap the row with this entry and the top row.

3. Use row operations to create zeros in the entries below the pivot position.

4. Cover the row containing the current pivot position, and then apply the previous steps to the $(m-1)\times n$ submatrix that remains. Repeat until the entire matrix is in echelon form.

5. Start with the row containing the rightmost pivot position in our matrix, now in echelon form.

   Use row operations to rescale this row to have leading entry 1.

   Then use row operations to create zeros in the entries in the same column above each leading entry.

   Repeat this for each successive pivot position going left, until the matrix is in reduced echelon form.

Output: $\mathsf{RREF}(A)$.

3 Solutions of linear systems

We now return to the problem of solving linear systems.

We talked about basic variables and free variables earlier.

The formal definition is: for a linear system in variables $x_1, x_2, \dots, x_n$ with augmented matrix matrix A, the variable x_i is *basic* if i is a pivot column of A and is *free* if i is not a pivot column of A.

Example. If a linear system has augmented matrix A with

$$\mathsf{RREF}(A) = \begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{then the system is equivalent to} \quad \begin{cases} x_1 - 5x_3 = 1 \\ x_2 + x_3 = 4 \\ 0 = 0. \end{cases}$$

The pivot columns of A are 1 and 2, so the basic variables are x_1 and x_2. The only free variable is x_3.

To find all solutions to the system, **choose any values for the free variables** and then **solve for the basic variables**. In the above system, we have $x_1 = 5x_3 + 1$ and $x_2 = 4 - x_3$.

Hence all solutions to this system have the form $(s_1, s_2, s_3) = (5a + 1, 4 - a, a)$ for $a \in \mathbb{R}$.

Theorem. Consider a linear system whose augmented matrix is A.

- The system has 0 solutions (and is *inconsistent*) if the last column of A contains a pivot.

  In this case $\mathsf{RREF}(A)$ has a row of the form $\begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix}$ so our system is equivalent to a linear system containing the false equation $0 = 1$.

- The system has only 1 solution if there are no free variables and the last column is not a pivot.
- Otherwise, the system has infinitely many solutions.

Once we have computed $\mathsf{RREF}(A)$ and identified the free and basic variables, we can write down all solutions to the system (if there are solutions) exactly as in our earlier examples: by letting each free variable be arbitrary, and then solving for the basic variables in terms of the free variables.

4 Vocabulary

Keywords from today's lecture:

1. **Leading entry** in a row of a matrix.

   The first nonzero entry in a given row, going left to right.

   Example: The leading entry in the second row of $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 8 \\ 4 & 5 & 6 \end{bmatrix}$ is 8.

2. **Echelon form**.

   A matrix is in echelon form if it has these properties:

   (a) If a row is nonzero, then every row above it is also nonzero.

   (b) The leading entry in one row is in a column to the right of the leading entry in each row above.

   (c) If a row is nonzero, then every entry below its leading entry in the same column is zero.

   Example: $\begin{bmatrix} 0 & 0 & 5 & 1 & 1 & 1 & 1 & 2 & 3 & 0.1 \\ 0 & 0 & 0 & 6 & 6 & 7 & 6 & 4 & 5 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

3. **Reduced echelon form**

   A matrix that is in echelon form, has 1 as the leading entry in each nonzero row, and has no other nonzero entries in the same column as a leading entry in a row.

   For each matrix A, there is a unique matrix $\mathsf{RREF}(A)$ in reduced echelon form that is row equivalent to A. This is the **row reduced echelon form** of A.

   Example: $\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 6 & 7 & 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is in reduced echelon form.

4. **Pivot position** and **pivot column** of a matrix.

   The location (respectively, column) containing a leading 1 in the reduced echelon form for A.

   Example: If $A = \begin{bmatrix} 0 & 3 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 3 & 1 & 1 \end{bmatrix}$ then $\mathsf{RREF}(A) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

   The pivot positions of A are $(1, 2)$ and $(2, 3)$ and $(3, 4)$. The pivot columns of A are 2 and 3 and 4.

5. **Basic variables** and **free variables** of a linear system.

   If A is the augmented matrix of a linear system in $x_1, x_2, \dots, x_n$ and $i \in \{1, 2, \dots, n\}$ is a pivot column in A then the variable x_i is basic; otherwise x_i is free.

   Example: $A = \begin{bmatrix} 0 & 3 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 3 & 1 & 1 \end{bmatrix}$ is the augmented matrix of $\begin{cases} 3x_2 + x_3 = 0 \\ 2x_2 = 0 \\ 3x_2 + x_3 = 1. \end{cases}$

   In previous example, saw that pivot columns of A are 2, 3, and 4. So x_2 and x_3 are basic variables while x_1 is free. Since the last column of A is a pivot column, the linear system has 0 solutions.