This document is intended as an **exact transcript** of the lecture, with extra summary and vocabulary sections for your convenience. By design, the material covered in lecture is exactly the same as what is in these notes. Due to time constraints, the notes may sometimes only contain limited illustrations, proofs, and examples; for a more thorough discussion of the course content, **consult the textbook**.

Summary

Quick summary of today's notes. Lecture starts on next page.

Linear independence:

- Vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ are *linearly independent* if the only way to express
$$0 = c_1v_1 + c_2v_2 + \dots + c_pv_p$$
for $c_1, c_2, \dots, c_p \in \mathbb{R}$ is by taking $c_1 = c_2 = \dots = c_p = 0$. This happens if and only if
$$\{0\} \neq \mathbb{R}\text{-span}\{v_1\} \neq \mathbb{R}\text{-span}\{v_1, v_2\} \neq \mathbb{R}\text{-span}\{v_1, v_2, v_3\} \neq \dots \neq \mathbb{R}\text{-span}\{v_1, v_2, \dots, v_p\}.$$
- If the vectors are not linearly independent, then they are *linearly dependent*. This happens when
$$\mathbb{R}\text{-span}\{v_1, v_2, \dots, v_{i-1}\} = \mathbb{R}\text{-span}\{v_1, v_2, \dots, v_i\}$$
for at least one $i \in \{1, 2, \dots, p\}$. Here we interpret "$\mathbb{R}\text{-span}\{v_1, v_2, \dots, v_{i-1}\}$" to be $\{0\}$ if $i = 1$.
- Two or more vectors are linearly dependent if one of the vectors is in the span of all of the others.
- If $p > n$ then any vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ are linearly dependent.
- A list of vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ is linearly dependent if the $n \times p$ matrix
$$A = \left[\begin{array}{cccc} v_1 & v_2 & \dots & v_p \end{array}\right]$$
has at least one column that is not a pivot column.

Functions and linearity:

- Writing $f : X \to Y$ means that f is a function that transforms inputs $x \in X$ to outputs $f(x) \in Y$.

  The set X is called the *domain* while Y is called the *codomain* of f.

  The *range* of f is the subset $\mathsf{range}(f) = \{f(x) : x \in X\}$ of Y.
- Let m, n be positive integers. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a function then the following mean the same thing:
  - For any $u, v \in \mathbb{R}^n$ and $c \in \mathbb{R}$ it holds that $f(u + v) = f(u) + f(v)$ and $f(c \cdot v) = c \cdot f(v)$.
  - There exists an $m \times n$ matrix A such that $f(v) = Av$ for all $v \in \mathbb{R}^n$.

  Such functions f are said to be *linear*. The matrix A is called the *standard matrix* of f.
- Every linear function $f : \mathbb{R}^n \to \mathbb{R}^m$ has exactly one standard matrix.
- If $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear then its standard matrix is $A = \left[\begin{array}{cccc} f(e_1) & f(e_2) & \dots & f(e_n) \end{array}\right]$ where
$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n, \quad \dots \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^n.$$

1 Last time: multiplying vectors and matrices

Given a matrix $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$ and a vector $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ we define

$$\boxed{Av = v_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + v_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + v_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \in \mathbb{R}^m.}$$

We refer to Av as the product of A and v, or the vector given by multiplying v by A.

Example. We have $\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = - \begin{bmatrix} 1 \\ 5 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 6 \end{bmatrix} + \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} -1+0+3 \\ -5+0+7 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

If A is an $m \times n$ matrix and $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $b \in \mathbb{R}^m$, then we call $\boxed{Ax = b}$ a *matrix equation*.

A matrix equation $Ax = b$ has the same solutions as the linear system with augmented matrix $\begin{bmatrix} A & b \end{bmatrix}$.

Theorem. Let A be an $m \times n$ matrix. The following are equivalent:

1. $Ax = b$ has a solution for any $b \in \mathbb{R}^m$.
2. The span of the columns of A is all of $\mathbb{R}^m$.
3. A has a pivot position in every row.

Example. The matrix equation

$$\begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

may fail to have a solution since

$$\mathsf{RREF}\left(\begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$$

has pivot positions only in rows 1 and 2.

2 Linear independence

We briefly introduced the notion of linear independence last time.

Suppose we have some vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$. Recall that the *span* of a set of vectors is the set of all possible linear combinations that can be formed using the vectors. If you have a smaller set of vectors inside a bigger set, then the span of the smaller set is always contained in the span of the bigger set.

Moreover, if $y = c_1v_1 + c_2v_2 + \cdots + c_pv_p$ for $c_i \in \mathbb{R}$ is any linear combination of our vectors then $\mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_p\} = \mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_p, y\}$, since if $a_1, \ldots, a_p, b \in \mathbb{R}$ then

$$a_1v_1 + \cdots + a_pv_p + by = (a_1 + bc_1)v_1 + (a_2 + bc_2)v_2 + \cdots + (a_p + bc_p)v_p \in \mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_p\}.$$

If S and T are sets then we write $S \subseteq T$ to mean that every element of S is also an element of T.

Definition. Consider the p sets given by

$$\{0\} \subseteq \mathbb{R}\text{-span}\{v_1\} \subseteq \mathbb{R}\text{-span}\{v_1, v_2\} \subseteq \mathbb{R}\text{-span}\{v_1, v_2, v_3\} \subseteq \cdots \subseteq \mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_p\}.$$

The vectors $v_1, v_2, \ldots, v_p$ are *linearly independent* if these sets are all distinct. That is, if $\mathbb{R}\text{-span}\{v_1\}$ is strictly bigger than the set $\{0\}$ consisting of just the zero vector, and $\mathbb{R}\text{-span}\{v_1, v_2\}$ is strictly bigger than $\mathbb{R}\text{-span}\{v_1\}$, and $\mathbb{R}\text{-span}\{v_1, v_2, v_3\}$ is strictly bigger than $\mathbb{R}\text{-span}\{v_1, v_2\}$, and so on.

Example. If $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ then v_1, v_2, v_3 are linearly independent, since

$$\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\} \subsetneq \mathbb{R}\text{-span}\{v_1\} = \left\{\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} : a \in \mathbb{R}\right\} \subsetneq \mathbb{R}\text{-span}\{v_1, v_2\} = \left\{\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} : a, b \in \mathbb{R}\right\} \subsetneq \mathbb{R}\text{-span}\{v_1, v_2, v_3\} = \left\{\begin{bmatrix} a \\ b \\ c \end{bmatrix} : a, b, c \in \mathbb{R}\right\}.$$

Here we write $S \subsetneq T$ to mean that both $S \subseteq T$ and $S \neq T$.

Example. If $v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ then v_1, v_2, v_3 are not linearly independent as

$$\mathbb{R}\text{-span}\{v_1, v_2\} = \mathbb{R}\text{-span}\{v_1, v_2, -v_1 - v_2\} = \mathbb{R}\text{-span}\{v_1, v_2, v_3\}.$$

When vectors are not linearly independent, we say they are *linearly dependent*.

A *linear dependence* among $v_1, v_2, \ldots, v_p$ is a way of writing the zero vector as a linear combination $0 = c_1v_1 + c_2v_2 + \cdots + c_pv_p$ for some scalar coefficients $c_1, c_2, \ldots, c_p \in \mathbb{R}$ that are *not all zero*.

If $0 = c_1v_1 + c_2v_2 + \cdots + c_pv_p$ is a linear dependence then the matrix equation

$$\begin{bmatrix} v_1 & v_2 & \ldots & v_p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = 0$$

has two solutions given by $(0, 0, \ldots, 0)$ and $(c_1, c_2, \ldots, c_p)$.

Proposition (Another characterization of linear independence)**.** The vectors $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$ are linearly independent if and only if no linear dependence exists among them.

Proof. If i is minimal such that there exists a linear dependence $c_1v_1 + c_2v_2 + \cdots + c_iv_i = 0$ then we must have $c_i \neq 0$ (since if $c_i = 0$ then $c_1v_1 + c_2v_2 + \cdots + c_{i-1}v_{i-1} = 0$ would be a shorter dependence). Then

$$v_i = -\tfrac{c_1}{c_i}v_1 - \tfrac{c_2}{c_i}v_2 - \cdots - \tfrac{c_{i-1}}{c_i}v_{i-1}$$

so $\mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_{i-1}\} = \mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_i\}$.

Conversely, if $\mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_{i-1}\} = \mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_i\}$ then $v_i \in \mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_{i-1}\}$, which means $v_i = a_1v_1 + a_2v_2 + \ldots a_{i-1}v_{i-1}$ for some coefficients $a_1, a_2, \ldots, a_{i-1} \in \mathbb{R}$. But then we get a linear dependence $c_1v_1 + c_2v_2 + \cdots + c_iv_i = 0$ by taking $c_1 = a_1$, $c_2 = a_2$, $\ldots$, $c_{i-1} = a_{i-1}$ and $c_i = -1$. $\square$

How to determine if $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ are linearly independent.

- Form the $n \times p$ matrix $A = \begin{bmatrix} v_1 & v_2 & \dots & v_p \end{bmatrix}$.
- Reduce A to echelon form to find its pivot columns.
- If every column of A is a pivot column, then the vectors are linearly independent.

  If some column of A is not a pivot column, then the vectors are linearly dependent.

Example. The vectors $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$, and $\begin{bmatrix} 5 \\ 9 \\ 16 \end{bmatrix}$ are linearly dependent since

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 9 \\ -1 & 5 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 9 \\ 0 & 7 & 21 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \mathsf{RREF}(A)$$

where $\sim$ denotes row equivalence. The last matrix has no pivot position in column 3. In fact, we have

$$-\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 5 \\ 9 \\ 16 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0.$$

The vectors $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$, and $\begin{bmatrix} 5 \\ 9 \\ 15 \end{bmatrix}$ are linearly independent, since

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 9 \\ -1 & 5 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 9 \\ 0 & 7 & 20 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathsf{RREF}(A).$$

Every column of A contains a pivot position, so the linear system with coefficient matrix A has no free variables, so $Ax = 0$ have no nontrivial solutions, meaning the columns of A are linearly independent.

Facts about linear independence.

1. A single vector v is linearly independent if and only if $v \neq 0$.
2. A list of vectors in $\mathbb{R}^n$ is linearly dependent if it includes the zero vector.
3. Vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ are linearly dependent if and only if some vector v_i is a linear combination of the other vectors $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_p$.

   We saw this in the previous example: $\begin{bmatrix} 5 \\ 9 \\ 16 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.
4. If $p > n$ then any list of p vectors in $\mathbb{R}^n$ is linearly dependent.

Example. The vectors $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 5 \\ 60 \end{bmatrix}$ are linearly dependent since $3 > 2$.

3 Linear transformations

A function f takes an input x from some set X and produces an output $f(x)$ in another set Y.

We write $f : X \to Y$ to mean that f is a function that takes inputs from X and gives outputs in Y.

The set X is called the *domain* of the function f. The set Y is called the *codomain* of f.

A function has **three components**: a choice of **domain**, a choice of **codomain**, and a **formula/rule/algorithm** to assign an output in the codomain to each input in the domain. Two functions are the same only when all three of these components are equal.

For example, the formula $f(x) = \sqrt{x}$ defines a function $X \to Y$ with $X = \{x \in \mathbb{R} : x \geq 0\}$ and $Y = \mathbb{R}$.

The formula $f(x) = \sqrt{x}$ also defines a function $X \to X$ with $X = \{x \in \mathbb{R} : x \geq 0\}$. We consider this to be a different function from the previous example, because it has a different codomain.

The formula $f(x) = |x|$ defines a function $\mathbb{R} \to \mathbb{R}$.

For every x in the domain X of f, we get an output $f(x)$.

It is possible that some values y in the codomain Y may never occur as outputs of f.

The *image* of an input x in X under f is the output $f(x)$. The *range* of the function f (sometimes called the *image* of f) is the set $\mathsf{range}(f) = \{f(x) : x \in X\}$ of images of all inputs in the domain. This is the subset of the codomain Y which gives all actual outputs of f.

Definition. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function whose domain and codomain are sets of vectors. The function f is a *linear transformation* (also called a *linear function*) if both of these properties hold:

(1) $f(u + v) = f(u) + f(v)$ for all vectors $u, v \in \mathbb{R}^n$.

(2) $f(cv) = cf(v)$ for all vectors $v \in \mathbb{R}^n$ and scalars $c \in \mathbb{R}$.

Example. If A is an $m \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^m$ is the function with the formula $T(v) = Av$ for $v \in \mathbb{R}^n$ then T is a linear function.

Linear transformations have some additional properties worth noting:

Proposition. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then

(3) $f(0) = 0$.

(4) $f(u - v) = f(u) - f(v)$ for $u, v \in \mathbb{R}^n$.

(5) $f(au + bv) = af(u) + bf(v)$ for all $a, b \in \mathbb{R}$ and $u, v \in \mathbb{R}^n$.

Proof. We have $2f(0) = f(0 + 0) = f(0)$ so $f(0) = 0$.

We have $f(u - v) = f(u) + f(-v) = f(u) + (-1)f(v) = f(u) - f(v)$.

Finally, we have $f(au + bv) = f(au) + f(bv) = af(u) + bf(v)$. □

Example. Suppose $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$ and $T : \mathbb{R}^2 \to \mathbb{R}^3$ is the function defined by $T(v) = Av$.

(a) The image of a vector $v \in \mathbb{R}^2$ under T is by definition $T(v) = Av$.

The image of $v = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ under T is $T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$.

(b) Is the range of T all of $\mathbb{R}^3$? If it was, then (from results last time) A would have a pivot position in every row. This is impossible since each column can only contain one pivot position, but A has three rows and only two columns. Therefore $\mathsf{range}(T) \neq \mathbb{R}^3$.

The fundamental theorem relating matrices and linear transformations:

Theorem. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Then there is a unique $m \times n$ matrix A such that $T(v) = Av$ for all $v \in \mathbb{R}^n$.

Moral: **matrices uniquely represent linear transformations** $\mathbb{R}^n \to \mathbb{R}^m$**.**

Proof. Define $e_1, e_2, \dots, e_n \in \mathbb{R}^n$ as the vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad e_{n-1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

so that e_i has a 1 in the ith row and 0 in all other rows.

Define $a_i = T(e_i) \in \mathbb{R}^m$ and $A = \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_n \end{bmatrix}$. If w is any vector $w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{R}^n$ then

$$T(w) = T(w_1e_1 + \cdots + w_ne_n) = w_1T(e_1) + \cdots + w_nT(e_n) = w_1a_1 + \cdots + w_na_n = Aw.$$

Thus A is one matrix such that $T(v) = Av$ for all vectors $v \in \mathbb{R}^n$.

To show that A is the only such matrix, suppose B is a $m \times n$ matrix with $T(v) = Bv$ for all $v \in \mathbb{R}^n$.

Then $T(e_i) = Ae_i = Be_i$ for all $i = 1, 2, \dots, n$.

But Ae_i and Be_i are the ith columns of A and B. For example,

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} e_3 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}.$$

Therefore A and B have the same columns, so they are the same matrix: $A = B$. □

We call the matrix A in this theorem the *standard matrix* of the linear transformation T.

Example. Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is the function $T(v) = 3v$.

This is a linear transformation. What is the standard matrix A of T?

As we saw in the proof of the theorem, the standard matrix of $T : \mathbb{R}^n \to \mathbb{R}^n$ is

$$A = \begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_n) \end{bmatrix} = \begin{bmatrix} 3e_1 & 3e_2 & \dots & 3e_n \end{bmatrix} = \begin{bmatrix} 3 & 0 & \dots & 0 \\ 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 3 \end{bmatrix}.$$

In words, A is the matrix with 3 in each position $(1,1), (2,2), \dots, (n,n)$ and 0 in all other positions.

One calls such a matrix *diagonal*.

Example. Suppose $T : \mathbb{R}^n \to \mathbb{R}$ is the function

$$T\left(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right) = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1^2 + v_2^2 + \cdots + v_n^2.$$

This function is *not* linear: we have $T(2v) = 4T(v) \neq 2T(v)$ for any nonzero vector $v \in \mathbb{R}^n$.

Example. Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is the function

$$T\left(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right) = \begin{bmatrix} v_n \\ \vdots \\ v_2 \\ v_1 \end{bmatrix}.$$

This function is a linear transformation. (Why?) Its standard matrix is

$$A = \begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_{n-1}) & T(e_n) \end{bmatrix} = \begin{bmatrix} e_n & e_{n-1} & \dots & e_2 & e_1 \end{bmatrix} = \begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix}.$$

In the matrix on the right, we adopt the convention of only writing the nonzero entries: all positions in the matrix which are blank contain zero entries.

4 Vocabulary

Keywords from today's lecture:

1. **Linearly independent** vectors.

   Vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ are **linearly independent** if $x_1v_1 + \dots + x_pv_p = 0$ holds only if $x_1 = x_2 = \dots = x_p = 0$; or when $\begin{bmatrix} v_1 & v_2 & \dots & v_p \end{bmatrix}$ has a pivot position in every column.

   Vectors that are not linearly independent are **linearly dependent**.

   Example: The three vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$ are linearly independent.

   The four vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$ are linearly dependent.

2. **Domain** and **codomain** of a function $f : X \to Y$.

   The **domain** X is the set of inputs for the function.

   The **codomain** Y is a set that contains the output of the function. This set can also contain elements that are not outputs of the function.

   Example: If A is an $m \times n$ matrix then the function $T(v) = Av$ has domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$.

3. **Range** of a function $f : X \to Y$.

   The set $\mathsf{range}(f) = \{f(x) : x \in X\} \subset Y$ of all possible outputs of the function f.

   Example: If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$ has $T(v) = Av$ then $\mathsf{range}(T) = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : x, y \in \mathbb{R} \right\}$.

4. **Linear function** $f : \mathbb{R}^n \to \mathbb{R}^m$.

   A function with $f(cv) = cf(v)$ and $f(u + v) = f(u) + f(v)$ for $c \in \mathbb{R}$ and $u, v \in \mathbb{R}^n$.

   Example: Every such function has the form $f(v) = Av$ for a unique $m \times n$ matrix A.

   The matrix A is called the **standard matrix** of f if $f(v) = Av$ for all $v \in \mathbb{R}^n$.

5. **Diagonal** matrix

   A matrix which has 0 in position (i, j) if $i \neq j$.

   Example: $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$.