This document is intended as an **exact transcript** of the lecture, with extra summary and vocabulary sections for your convenience. By design, the material covered in lecture is exactly the same as what is in these notes. Due to time constraints, the notes may sometimes only contain limited illustrations, proofs, and examples; for a more thorough discussion of the course content, **consult the textbook**.

Summary

Quick summary of today's notes. Lecture starts on next page.

- A *vector space* is a nonempty set with a "zero vector" and two operations that can be thought of a "vector addition" and "scalar multiplication." The operations must obey several conditions.
- There are notions of *subspaces*, *linear functions*, *linear combinations*, *spans*, *linear independence*, and *bases* for vector spaces. The definitions are essentially the same as for $\mathbb{R}^n$, with one minor caveat when we are considering linear combinations and independence of infinite sets of vectors.
- Every vector space has a basis, and every basis for a given vector space has the same number of elements, which could be infinite. This number of elements is the *dimension* of the vector space.
- If X and Y are sets, then let $\mathsf{Fun}(X, Y)$ be the set of functions $f : X \to Y$.

  The set $\mathsf{Fun}(X, \mathbb{R})$ is naturally a vector space. If X is finite then $\boxed{\dim \mathsf{Fun}(X, \mathbb{R}) = |X|}$.
- If U and V are vector spaces, then let $\mathsf{Lin}(U, V)$ be the set of linear functions $f : U \to V$.

  The set $\mathsf{Lin}(U, V)$ is naturally a vector space. If $\dim U < \infty$ then $\boxed{\dim \mathsf{Lin}(U, \mathbb{R}) = \dim U}$.

  Moreover, if W is another vector space and $f \in \mathsf{Lin}(V, W)$ and $g \in \mathsf{Lin}(U, V)$, then $f \circ g \in \mathsf{Lin}(U, W)$.
- Suppose $f : U \to V$ is a linear function between vector spaces.

  Define $\mathsf{range}(f) = \{f(u) : u \in U\} \subseteq V$ and $\mathsf{kernel}(f) = \{u \in U : f(u) = 0\} \subseteq U$.

  These sets are subspaces. If $\dim U < \infty$ then $\boxed{\dim \mathsf{range}(f) + \dim \mathsf{kernel}(f) = \dim U}$.
- Let A be an $n \times n$ matrix. Let λ be a number and suppose $0 \neq v \in \mathbb{R}^n$.

  If $Av = \lambda v$ then we say that v is an *eigenvector* for A and that λ is an *eigenvalue* for A.

  More specifically, v is an *eigenvector with eigenvalue* λ for A.

  This happens if and only if $0 \neq v \in \mathrm{Nul}(A - \lambda I_n)$.

  For example, $v = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda = -4$ for $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ since

  $$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4\begin{bmatrix} 6 \\ -5 \end{bmatrix}.$$

  The zero vector is not allowed to be an eigenvector, but 0 can occur as an eigenvalue.
- The eigenvalues λ for A are the numbers such that $\det(A - \lambda I_n) = 0$.
- The eigenvectors with eigenvalue λ for A are the nonzero elements of $\mathrm{Nul}(A - \lambda I_n)$.
- If A is a triangular matrix, then its eigenvalues are its diagonal entries.

  For example, the eigenvalues of $\begin{bmatrix} 1 & 6 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ are 0 and 1.

1 Last time: vector spaces

A *(real) vector space* V is a set containing a *zero vector*, denoted 0, with *vector addition* and *scalar multiplication* operations that let us produce new vectors $u + v \in V$ and $cv \in V$ from given elements $u, v \in V$ and $c \in \mathbb{R}$. Several conditions must be satisfied so that these operations behave exactly like vector addition and scalar multiplication for $\mathbb{R}^n$. Most importantly, we require that

1. $u + v = v + u$ and $(u + v) + w = u + (v + w)$.
2. $v - v = 0$ where we define $u - v = u + (-1)v$.
3. $v + 0 = v$
4. $cv = v$ if $c = 1$.

There are a few other more conditions to give the full definition (see the notes from last time).

By convention, we refer to elements of vector spaces as *vectors*.

Example. All subspace of $\mathbb{R}^n$ are vector spaces, with the usual zero vector and vector operations.

The set of $m \times n$ matrices is a vector space, with the usual addition and scalar multiplication operations. The zero vector in this vector space is the $m \times n$ zero matrix.

Most vector spaces that we encounter are either subspaces of $\mathbb{R}^n$ or subspaces of the following construction.

Proposition. Let X be a set and let V be a vector space.

Then the set $\mathsf{Fun}(X, V)$ of all functions $f : X \to V$ is a vector space once we define

$$\begin{aligned} f + g &= (\text{ the function that maps } x \mapsto f(x) + g(x) \text{ for } x \in X\ ),\\ cf &= (\text{ the function that maps } x \mapsto c \cdot f(x) \text{ for } x \in X\ ),\\ 0 &= (\text{ the function that maps } x \mapsto 0 \in V \text{ for } x \in X\ ), \end{aligned}$$

for $f, g \in \mathsf{Fun}(X, V)$ and $c \in \mathbb{R}$.

Definition. The definitions of a *subspace* of a vector space and of *linear transformations* between vector spaces are identical to the ones we have already seen for subspaces of $\mathbb{R}^n$:

- A subset $H \subseteq V$ is a *subspace* if $0 \in H$ and if $u + v \in H$ and $cv \in H$ for all $u, v \in H$ and $c \in \mathbb{R}$.
- A function $f : U \to V$ is *linear* if $f(u + v) = f(u) + f(v)$ and $f(cv) = cf(v)$ for all $u, v \in U$ and $c \in \mathbb{R}$.

Proposition. If U, V, W are vector spaces and $f : V \to W$ and $g : U \to V$ are linear functions then $f \circ g : U \to W$ is also linear, where we define $f \circ g(x) = f(g(x))$ for $x \in U$.

Example. If U and V are vector spaces then let $\mathsf{Lin}(U, V)$ be the set of linear functions $f : U \to V$.

Then $\mathsf{Lin}(U, V)$ is a subspace of $\mathsf{Fun}(U, V)$.

Can you make sense of this statement? "$\mathsf{Lin}(\mathbb{R}^n, \mathbb{R}^m)$ is the vector space of $m \times n$ matrices."

Example. A function $f : \mathbb{R} \to \mathbb{R}$ is a *polynomial* if it has the formula

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

for some nonnegative integer n and some coefficients $a_0, a_1, \ldots, a_n \in \mathbb{R}$.

The set of polynomial functions $\mathbb{R} \to \mathbb{R}$ is a subspace of $\mathsf{Fun}(\mathbb{R}, \mathbb{R})$.

Example. Suppose V is a vector space. Choose $v \in V$. Given a linear function $f : V \to \mathbb{R}$, define

$$v^*(f) = f(v).$$

Then v^* is a linear function $\mathsf{Lin}(V, \mathbb{R}) \to \mathbb{R}$.

Let's go deeper: the function with the formula $v \mapsto v^*$ is a linear function $V \to \mathsf{Lin}(\mathsf{Lin}(V, \mathbb{R}), \mathbb{R})$.

If $V = \mathbb{R}^n$ then this function $V \to \mathsf{Lin}(\mathsf{Lin}(V, \mathbb{R}), \mathbb{R})$ is invertible.

Let V be a vector space. The definitions of *linear combinations* and *linear independence* for vectors in V are mostly the same as for vectors in $\mathbb{R}^n$, with one caveat.

Definition. A *linear combination* of a **finite** list of vectors $v_1, v_2, \dots, v_k \in V$ is a vector of the form

$$c_1v_1 + c_2v_2 + \dots + c_kv_k$$

for some scalars $c_1, c_2, \dots, c_k \in \mathbb{R}$.

We must be a little careful when defining linear combinations for infinite sets. Specifically: a *linear combination* of an infinite set of vectors is a linear combination of some **finite** subset of the vectors.

Definition. The *span* of a set of vectors is the set of all linear combinations that can be formed from the vectors. The span of a set of vectors in V is a subspace of V.

Example. The subspace of polynomials in $\mathsf{Fun}(\mathbb{R}, \mathbb{R})$ is the span of the set of functions $1, x, x^2, x^3, \dots$.

The infinite sum $e^x = 1 + x + \frac{1}{2}x + \frac{1}{6}x^2 + \frac{1}{24}x^3 + \dots + \frac{1}{n!}x^n + \dots$ does not belong to this subspace.

Definition. A finite list of vectors $v_1, v_2, \dots, v_k \in V$ is *linearly independent* if it is impossible to express $0 = c_1v_1 + c_2v_2 + \dots + c_kv_k$ except when $c_1 = c_2 = \dots = c_k = 0$.

An infinite list of vectors is defined to be *linearly independent* if every finite subset of the vectors is linearly independent.

Definition. A *basis* of a vector space V is a subset of linearly independent vectors whose span is V. Saying $b_1, b_2, b_3, \dots$ is a basis for V is the same as saying that for each $v \in V$, there a unique coefficients $x_1, x_2, x_3, \dots \in \mathbb{R}$, **all but finitely many of which are zero**, such that $v = x_1b_1 + x_2b_2 + x_3b_3 + \dots$.

Theorem. Let V be a vector space.

1. V has at least one basis.
2. Every basis of V has the same number of elements (but this could be infinite).
3. If A is a subset of linearly independent vectors in V then V has a basis B with $A \subseteq B$.
4. If C is a subset of vectors in V whose span is V then V has a basis B with $B \subseteq C$.

Definition. The *dimension* of a vector space V is the number $\dim V$ of elements in any of its bases.

Example. If X is a finite set then $\dim \mathsf{Fun}(X, \mathbb{R}) = |X|$ where $|X|$ is the size of X.

2 More on dimension

If V is a finite-dimensional vector space then I claim that $\dim \mathsf{Lin}(V, \mathbb{R}) = \dim V$.

Suppose $b_1, b_2, \ldots, b_n$ is a basis for V.

Then a basis for $\mathsf{Lin}(V, \mathbb{R})$ is given by the linear functions $\phi_1, \phi_2, \ldots, \phi_n : V \to \mathbb{R}$ with the formulas

$$\phi_i(x_1 b_1 + x_2 b_2 + \ldots x_n b_n) = x_i \qquad \text{for } x_1, x_2, \ldots, x_n \in \mathbb{R}.$$

The unique way to express a linear function $f : V \to \mathbb{R}$ as a linear combination of these functions is

$$f = f(b_1)\phi_1 + f(b_2)\phi_2 + \cdots + f(b_n)\phi_n.$$

Assume $V = \mathbb{R}^n$. Then we can think of $\mathsf{Lin}(\mathbb{R}^n, \mathbb{R})$ as the vector space of $1 \times n$ matrices.

If $b_1 = e_1$, $b_2 = e_2$, $\ldots$, $b_n = e_n$ is the standard basis, then $\phi_1 = e_1^T$, $\phi_2 = e_2^T$, $\ldots$, $\phi_n = e_n^T$.

Definition. Suppose U and V are vector spaces and $f : U \to V$ is a linear function.

Define $\mathsf{range}(f) = \{f(x) : x \in U\} \subseteq V$ and $\mathsf{kernel}(f) = \{x \in U : f(x) = 0\} \subseteq U$.

These sets are subspaces which generalize the column space and null space of a matrix.

We have a version of the rank-nullity theorem for arbitrary vector spaces:

Theorem (Rank-Nullity Theorem)**.** If $\dim U < \infty$ then $\dim \mathsf{range}(f) + \dim \mathsf{kernel}(f) = \dim U$.

This specializes to our earlier statement about matrices when $U = \mathbb{R}^n$ and $V = \mathbb{R}^m$.

We can prove the theorem in a self-contained, completely abstract way, but it's a little involved.

Proof. If $b_1, b_2, \ldots, b_n$ is a basis for U then the span of $f(b_1), f(b_2), \ldots, f(b_n)$ must be equal to $\mathsf{range}(f)$.

Therefore $\dim \mathsf{range}(f) \leq \dim U < \infty$. Since $\mathsf{kernel}(f) \subseteq U$, we also have $\dim \mathsf{kernel}(f) < \infty$.

Let $k = \dim \mathsf{range}(f)$ and $l = \dim \mathsf{kernel}(f)$.

Choose $u_1, u_2, \ldots, u_k \in U$ such that $f(u_1), f(u_2), \ldots, f(u_k)$ is a basis for $\mathsf{range}(f)$.

Choose a basis $v_1, v_2, \ldots, v_l$ for $\mathsf{kernel}(f)$. We will check that $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_l$ is a basis for U.

To show linear independence, suppose $a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_l \in \mathbb{R}$ are such that

$$a_1 u_1 + a_2 u_2 + \cdots + a_k u_k + b_1 v_1 + b_2 v_2 + \cdots + b_l v_l = 0.$$

Applying f to both sides gives $a_1 f(u_1) + a_2 f(u_2) + \cdots + a_k f(u_k) = 0$, so $a_1 = a_2 = \cdots = a_k = 0$.

But this implies $b_1 v_1 + b_2 v_2 + \cdots + b_l v_l = 0$, so we also have $b_1 = b_2 = \cdots = b_l = 0$.

Our vectors $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_l$ are therefore linearly independent in U.

Now let $x \in U$. By assumption $f(x) = c_1 f(u_1) + c_2 f(u_2) + \ldots c_k f(u_k)$ for some $c_1, c_2, \ldots, c_k \in \mathbb{R}$.

The vector $x - c_1 u_1 - c_2 u_2 - \cdots - c_k u_k$ is then in the span of $v_1, v_2, \ldots, v_l$ since it belongs to $\mathsf{kernel}(f)$.

We conclude that x is a linear combination of $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_l$, so this is a basis for U. □

3 Eigenvectors and eigenvalues

We return to the concrete setting of $\mathbb{R}^n$ and its subspaces. Let A be a square $n \times n$ matrix.

Definition. An *eigenvector* of A is a **nonzero** vector $v \in \mathbb{R}^n$ such that

$$Av = \lambda v$$

for a number $\lambda \in \mathbb{R}$. (λ is the Greek letter "lambda.")

The number λ is called the *eigenvalue* of A for the eigenvector v.

We require eigenvectors to be nonzero because if $v = 0$ then $Av = \lambda v = 0$ for all numbers $\lambda \in \mathbb{R}$.

The number 0 is allowed to be an eigenvalue of A, however.

Example. If we are given A and v, it is easy to check whether v is an eigenvector: just compute Av.

For example, if $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ and $v = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ then $Av = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4v$.

Therefore v is an eigenvector of A with eigenvalue -4.

Example. What are the eigenvectors of the matrix $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$?

If $v \in \mathbb{R}^4$ were an eigenvector with eigenvalue λ then

$$Av = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} v_2 \\ v_3 \\ v_4 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}.$$

The last equation implies that $0 = \lambda v_4$ and $v_4 = \lambda v_3$ and $v_3 = \lambda v_2$ and $v_2 = \lambda v_1$. In other words,

$$0 = \lambda v_4 = \lambda^2 v_3 = \lambda^3 v_2 = \lambda^4 v_1.$$

If $\lambda \neq 0$ then this would mean that $v_1 = v_2 = v_3 = v_4 = 0$, but remember that v should be nonzero. Therefore the only possible eigenvalue of A is $\lambda = 0$. The eigenvectors of A with eigenvalue 0 are

$$v = \begin{bmatrix} v_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{where } v_1 \neq 0.$$

To say that "λ is an eigenvalue of A" means that there exists a nonzero vector $v \in \mathbb{R}^n$ such that $Av = \lambda v$.

Recall that I_n denotes the $n \times n$ identity matrix. We abbreviate by setting $I = I_n$.

Proposition. A number $\lambda \in \mathbb{R}$ is an eigenvalue of A if and only if $A - \lambda I$ is not invertible.

Proof. The equation $Ax = \lambda x$ has a nonzero solution $x \in \mathbb{R}^n$ if and only if $(A - \lambda I)x = 0$ has a nonzero solution, which occurs if and only if $\mathrm{Nul}(A - \lambda I) \neq \{0\}$, or equivalently when $A - \lambda I$ is not invertible. □

Example. If $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ then

$$A - 7I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \mathsf{RREF}(A - 7I).$$

Since $\mathsf{RREF}(A - 7I) \neq I$, the matrix $A - 7I$ is not invertible so 7 is an eigenvalue of A.

Corollary. A number $\lambda \in \mathbb{R}$ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$.

Proof. Remember that $A - \lambda I$ is not invertible if and only if $\det(A - \lambda I) = 0$. □

Another way of defining an eigenvector: the eigenvectors of A with eigenvalue λ are precisely the nonzero elements of the null space $\mathrm{Nul}(A - \lambda I)$. Since we know how to construct a basis for the null space of any matrix, we also know how to find all eigenvectors of a matrix for any given eigenvalue.

Example. In the previous example, $\mathsf{RREF}(A-7I) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so $Ax = 7x$ if and only if $(A-7I)x = 0$ if and only if $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 - x_2 = 0$. In this linear system, x_2 is a free variable, and we can rewrite x as $x = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. This means $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a basis for $\mathrm{Nul}(A - 7I)$.

Therefore every eigenvector of A with eigenvalue 7 has the form $\begin{bmatrix} a \\ a \end{bmatrix}$ for some $a \in \mathbb{R}$.

One calls the set of all $v \in \mathbb{R}^n$ with $Av = \lambda v$ the *eigenspace* of A for λ. We also call this the *λ-eigenspace* of A. Note that this is just the null space of $A - \lambda I$. A number is an eigenvalue of A if and only if the corresponding eigenspace is nonzero (that is, contains a nonzero vector).

Example. Suppose we were told that $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ has 2 as an eigenvalue.

To find a basis for the 2-eigenspace of A, we row reduce

$$A - 2I = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathsf{RREF}(A - 2I).$$

Thus $Ax = 2x$ if and only if $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 - \frac{1}{2}x_2 + 3x_3 = 0$, that is, if and only if

$$x = \begin{bmatrix} \frac{1}{2}x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

The vectors $\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ are then a basis for the 2-eigenspance of A.

Recall that a matrix is *triangular* if its nonzero entries all appear on or above the main diagonal, or all appear on or below the main diagonal.

Theorem. The eigenvalues of a triangular square matrix A are its diagonal entries.

Proof. If A has diagonal entries $d_1, d_2, \ldots, d_n$ then $A - \lambda I$ is triangular with diagonal entries $d_1 - \lambda$, $d_2 - \lambda$, $\ldots$, $d_n - \lambda$, so $\det(A - \lambda I) = (d_1 - \lambda)(d_2 - \lambda) \cdots (d_n - \lambda)$ which is zero if and only if $\lambda \in \{d_1, d_2, \ldots, d_n\}$. □

Example. The eigenvalues of the matrix $\begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ are 3, 0, and 2.

4 Vocabulary

Keywords from today's lecture:

1. **Subspace** of a vector space.

   A nonempty subset closed under linear combinations.

2. **Linearly combination** and **span** of elements in a vector space.

   A linear combination of a finite set of vectors $v_1, v_2, \dots v_p \in V$ is a vector of the form

   $$c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$$

   where $c_1, c_2, \dots, c_p \in \mathbb{R}$. A linear combination of an infinite set of vectors is a linear combination of some finite subset. The set of all linear combinations of a set of vectors is the span of the vectors.

3. **Linearly independent** elements in a vector space.

   A list of elements in a vector space is **linearly dependent** if one vector can be expressed as a linear combination of a finite subset of the other vectors. If this is impossible, then the vectors are linearly independent.

   Example: $\cos(x)$ and $\sin(x)$ are linearly independently in $\mathsf{Fun}(\mathbb{R}, \mathbb{R})$.

   Example: the infinite list of functions $1, x, x^2, x^3, x^4, \dots$ are linearly independent in $\mathsf{Fun}(\mathbb{R}, \mathbb{R})$.

4. **Basis** and **dimension** of a vector space.

   A set of linearly independent elements whose span is the entire vector space.

   Every basis in a vector space has the same number of elements. This number is defined to be the **dimension** of the vector space.

5. **Linear functions**.

   If U and V are vector spaces, then a function $f : U \to V$ is linear when

   $$f(u + v) = f(u) + f(v) \qquad \text{and} \qquad f(cv) = cf(v)$$

   for all $u, v \in U$ and $c \in \mathbb{R}$.

6. **Eigenvector** for an $n \times n$ matrix A.

   A *nonzero* vector $v \in \mathbb{R}^n$ such that $Av = \lambda v$ for some real number $\lambda \in \mathbb{R}$.

   The number λ is the **eigenvalue** of A for v.

   $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ with eigenvalue 2 as $\begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$.

7. **λ-eigenspace** for an $n \times n$ matrix A, where $\lambda \in \mathbb{R}$.

   The subspace $\mathrm{Nul}(A - \lambda I) \subseteq \mathbb{R}^n$ where I is the $n \times n$ identity matrix.

   If λ is not an eigenvalue of A, then this subspace is $\{0\}$.

   But if λ is an eigenvalue of A, then the subspace is nonzero.