This document is intended as an **exact transcript** of the lecture, with extra summary and vocabulary sections for your convenience. By design, the material covered in lecture is exactly the same as what is in these notes. Due to time constraints, the notes may sometimes only contain limited illustrations, proofs, and examples; for a more thorough discussion of the course content, **consult the textbook**.

Summary

Quick summary of today's notes. Lecture starts on next page.

- Given real numbers $a, b \in \mathbb{R}$, define $a + bi = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

  In this notation, we think of 1 as the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and i as the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

  The set of *complex numbers* is $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\} = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : a, b \in \mathbb{R} \right\}$.

  We view $\mathbb{R}$ as a subset of $\mathbb{C}$ by setting $a = a + 0i = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$.

- We can add, subtract, multiply, and invert complex numbers, since they are 2×2 matrices.

  The identity "$i^2 = -1$" holds in the sense that $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

- Once we get used to these operations, another useful way to view the elements of $\mathbb{C}$ is as formal expressions $a + bi$ where $a, b \in \mathbb{R}$ and i is a symbol that satisfies $i^2 = -1$.

  Addition, subtraction, and multiplication work just like polynomials, but substituting -1 for i^2.

- Suppose $p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots a_1 x + a_0$ is a polynomial with coefficients $a_0, a_1, \dots, a_n \in \mathbb{C}$.

  Assume $a_n \neq 0$ so that $p(x)$ has *degree* n.

  Then there are are n (not necessarily distinct) complex numbers $r_1, r_2, \dots, r_n \in \mathbb{C}$ such that

  $$p(x) = a_n(x - r_1)(x - r_2)\cdots(x - r_n).$$

  The numbers $r_1, r_2, \dots, r_n$ are the *roots* of $p(x)$.

- The characteristic equation of an $n \times n$ matrix A is a degree n polynomial with real coefficients.

  Counting multiplicities, $\det(A - xI)$ has exactly n roots but some roots may be complex numbers.

- Define $\mathbb{C}^n$ to be the set of vectors $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ with n rows and entries $v_1, v_2, \dots, v_n \in \mathbb{C}$.

  We have $\mathbb{R}^n \subset \mathbb{C}^n$ since $\mathbb{R} = \{a \in \mathbb{R}\} = \{a + 0i : a \in \mathbb{R}\} \subset \mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$.

- The sum $u + v$ and scalar multiple cv for $u, v \in \mathbb{C}^n$ and $c \in \mathbb{C}$ are defined exactly as for vectors in $\mathbb{R}^n$, except we use the addition and multiplication operations from $\mathbb{C}$ instead of $\mathbb{R}$.

- If A is an $n \times n$ matrix and $v \in \mathbb{C}^n$ then we define Av in the same way as when $v \in \mathbb{R}^n$.

  Let A be an $n \times n$ matrix whose entries are all real numbers.

  Call $\lambda \in \mathbb{C}$ a *(complex) eigenvalue* of A if there exists a nonzero vector $v \in \mathbb{C}^n$ such that $Av = \lambda v$.

  Equivalently, $\lambda \in \mathbb{C}$ is an eigenvalue of A if λ is a root of the characteristic polynomial $\det(A - xI)$.

  This is no different from our first definition of an eigenvalue, except that now we permit $\lambda \in \mathbb{C}$.

1 Last time: methods to check diagonalizability

Let n be a positive integer and let A be an $n \times n$ matrix.

Remember that A is *diagonalizable* if $A = PDP^{-1}$ where P is an invertible $n \times n$ matrix and D is an $n \times n$ diagonal matrix. In other words, A is diagonalizable if A is similar to a diagonal matrix.

Suppose $v_1, v_2, \dots, v_n \in \mathbb{R}^n$ are linearly independent vectors and $\lambda_1, \lambda_2, \dots, \lambda_n$ are numbers. Define

$$P = \left[\begin{array}{cccc} v_1 & v_2 & \dots & v_n \end{array}\right] \qquad \text{and} \qquad D = \left[\begin{array}{cccc} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{array}\right].$$

If $A = PDP^{-1}$ then $Av_i = PDP^{-1}v_i = PDe_i = \lambda_i Pe_i = \lambda_i v_i$ for each $i = 1, 2, \dots, n$.

In other words, when $A = PDP^{-1}$, the columns of P are a basis for $\mathbb{R}^n$ made up of eigenvectors of A.

Matrices that are not diagonalizable.

Proposition. $\left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right]$ is not diagonalizable.

Proof. To check this directly, suppose $ad - bc \neq 0$ and compute

$$\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]\left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right]\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]^{-1} = \frac{1}{ad-bc}\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]\left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right]\left[\begin{array}{cc} d & -b \\ -c & a \end{array}\right] = \frac{1}{ad-bc}\left[\begin{array}{cc} -ac & a^2 \\ -c^2 & ac \end{array}\right].$$

The only way the last matrix can be diagonal is if $a = c = 0$, but then we would have $ad - bc = 0$ so $\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$ would not be invertible. Therefore $\left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right]$ is not similar to a diagonal matrix. $\square$

Here is a second family of examples.

Proposition. Let A be an $n \times n$ upper-triangular matrix with all entries on the diagonal equal to λ:

$$A = \left[\begin{array}{cccc} \lambda & * & \dots & * \\ & \lambda & \ddots & \vdots \\ & & \ddots & * \\ & & & \lambda \end{array}\right].$$

(All entries in A below the diagonal are zero, and the entries above the diagonal can be any numbers.)

If A is diagonalizable then A is equal to the diagonal matrix λI.

This means that if A is not diagonal then A is not diagonalizable.

Proof. Suppose $A = PDP^{-1}$ where D is diagonal.

Every diagonal entry of D is an eigenvalue for A.

But A has characteristic polynomial $(\lambda - x)^n$ so its only eigenvalue is λ.

Therefore $D = \lambda I$ so $A = P(\lambda I)P^{-1} = \lambda PIP^{-1} = \lambda PP^{-1} = \lambda I$. $\square$

The following result summarizes everything we need to know about diagonalizability: how to determine if a matrix A is diagonalizable, and then how to compute the decomposition $A = PDP^{-1}$ if it exists.

Theorem. Let A be an $n \times n$ matrix.

Suppose $\lambda_1, \lambda_2, \dots, \lambda_p$ are the distinct eigenvalues of A.

Let $d_i = \dim \mathrm{Nul}(A - \lambda_i I)$ for $i = 1, 2, \dots, p$.

By the definition of an eigenvalue, we have $1 \le d_i \le n$ for each i. Moreover, the following holds:

1. We always have $d_1 + d_2 + \cdots + d_p \le n$.
2. The matrix A is diagonalizable if and only if $d_1 + d_2 + \cdots + d_p = n$.
3. Suppose A is diagonalizable. Let $D_i = \lambda_i I_{d_i}$ and define D as the $n \times n$ diagonal matrix

$$D = \left[\begin{array}{cccc} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_p \end{array}\right].$$

Choose n vectors $v_1, v_2, \dots, v_n \in \mathbb{R}^n$ such that the first d_1 vectors are a basis for $\mathrm{Nul}(A - \lambda_1 I)$, the next d_2 vectors are a basis for $\mathrm{Nul}(A - \lambda_2 I)$, the next d_3 vectors are a basis for $\mathrm{Nul}(A - \lambda_3 I)$, and so on, so that the last d_p vectors are basis for $\mathrm{Nul}(A - \lambda_p I)$. Then $A = PDP^{-1}$ for

$$P = \left[\begin{array}{cccc} v_1 & v_2 & \dots & v_n \end{array}\right].$$

2 Complex numbers

For the rest of this lecture, let $i = \left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right]$. Recall that $I_2 = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]$.

Suppose $a, b \in \mathbb{R}$. Both i and I_2 are 2×2 matrices, so we can form the sum $aI_2 + bi$.

To simplify our notation, we will write 1 instead of I_2 and $a + bi$ instead of $aI_2 + bi$.

We consider $a = a + 0i$ and $bi = 0 + bi$ and $0 = 0 + 0i$. With this convention, we have

$$a + bi = a\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] + b\left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right] = \left[\begin{array}{cc} a & 0 \\ 0 & b \end{array}\right] + \left[\begin{array}{cc} 0 & -b \\ b & 0 \end{array}\right] = \left[\begin{array}{cc} a & -b \\ b & a \end{array}\right].$$

Define $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\} = \left\{\left[\begin{array}{cc} a & -b \\ b & a \end{array}\right] : a, b \in \mathbb{R}\right\}$. This is called the set of *complex numbers*.

According to our definition, each element of $\mathbb{C}$ is a 2×2 matrix, to be called a *complex number*.

Fact. We can add complex numbers together. If $a, b, c, d \in \mathbb{R}$ then

$$(a + bi) + (c + di) = \left[\begin{array}{cc} a & -b \\ b & a \end{array}\right] + \left[\begin{array}{cc} c & -d \\ d & c \end{array}\right] = \left[\begin{array}{cc} a + c & -b - d \\ b + d & a + c \end{array}\right] = (a + c) + (b + d)i \in \mathbb{C}.$$

Clearly $\boxed{(a + bi) + (c + di) = (c + di) + (a + bi) = (a + c) + (b + d)i}$.

Fact. We can subtract complex numbers. If $a, b, c, d \in \mathbb{R}$ then

$$(a + bi) - (c + di) = \left[\begin{array}{cc} a & -b \\ b & a \end{array}\right] - \left[\begin{array}{cc} c & -d \\ d & c \end{array}\right] = \left[\begin{array}{cc} a - c & -b + d \\ b - d & a - c \end{array}\right] = (a - c) + (b - d)i \in \mathbb{C}.$$

Fact. We can multiply complex numbers. If $a, b, c, d \in \mathbb{R}$ then

$$(a+bi)(c+di) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{bmatrix} = (ac-bd)+(ad+bc)i \in \mathbb{C}.$$

Note that $\boxed{(a+bi)(c+di) = (c+di)(a+bi) = (ac-bd)+(ad+bc)i}$.

Fact. We can multiply complex numbers by real numbers. If $a, b, x \in \mathbb{R}$ then define

$$(a+bi)x = x(a+bi) = x \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} ax & -bx \\ bx & ax \end{bmatrix} = (ax)+(bx)i \in \mathbb{C}.$$

Note that this is the same as the product $(a+bi)(x+0i)$.

Fact. We can divide complex numbers by nonzero real numbers. If $a, b, x \in \mathbb{R}$ and $x \neq 0$ then define

$$(a+bi)/x = (a+bi)(1/x) = (a/x)+(b/x)i.$$

We sometimes write $\frac{p}{q}$ instead of p/q. Both expressions means the same thing.

A complex number $a+bi$ is *nonzero* if $a \neq 0$ or $b \neq 0$. Since

$$\det(a+bi) = \det \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = a^2+b^2,$$

which is only zero if $a=b=0$, every nonzero complex number is invertible as a matrix.

Fact. This fact lets us divide complex numbers. If $a, b, c, d \in \mathbb{R}$ and $c+di \neq 0$ then define

$$(a+bi)/(c+di) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix}^{-1}.$$

We can write this more explicitly as

$$\begin{aligned} (a+bi)/(c+di) &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix}^{-1} \\ &= \frac{1}{c^2+d^2} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \\ &= \frac{1}{c^2+d^2} \begin{bmatrix} ac+bd & ad-bc \\ bc-ad & ac+bd \end{bmatrix} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2} i \in \mathbb{C}. \end{aligned}$$

The last formula is not so easy to remember.

It may be easier to divide complex numbers using the following method:

Example. We have $\frac{3-4i}{2+i} = \frac{(3-4i)(2-i)}{(2+i)(2-i)} = \frac{6-3i-8i+4i^2}{4-i^2} = \frac{6-11i-4}{5} = \frac{2-11i}{5} = \frac{2}{5} - \frac{11}{5}i.$

More generally, if $c+di \neq 0$ then we always have $\boxed{\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{c^2+d^2}}$ since

$$\frac{a+bi}{c+di} = (a+bi)(c+di)^{-1} = \frac{1}{c^2+d^2}(a+bi)(c-di) = \frac{(a+bi)(c-di)}{c^2+d^2}.$$

The *complex conjugate* of $c+di$ is defined to be the complex number

$$\overline{c+di} = (c+di)^\top = c-di \in \mathbb{C}.$$

When $c + di$ is nonzero, the complex conjugate is related to the inverse by the identity

$$(c+di)^{-1} = \begin{bmatrix} c & -d \\ d & c \end{bmatrix}^{-1} = \frac{1}{c^2+d^2}\begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \frac{1}{c^2+d^2}\cdot \overline{c+di}.$$

Since $x, y \in \mathbb{C}$ satisfy $xy = yx$ and $(xy)^\top = y^\top x^\top$ (since complex numbers are matrices), it follows that

$$\boxed{\overline{xy} = \overline{y}\cdot\overline{x} = \overline{x}\cdot\overline{y}.}$$

We can also add complex numbers $a + bi$ with real numbers c when $a, b, c \in \mathbb{R}$.

To do this, we set $c = c + 0i$ and define $(a + bi) + c = c + (a + bi) = (a + bi) + (c + 0i) = (a + c) + bi$.

Under this convention, we have

$$\begin{aligned} i^2 + 1 = (0+i)(0+i) + (1+0i) &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 + 0i = 0. \end{aligned}$$

Thus it makes sense to write $\boxed{i^2 = -1}$. In a similar way:

Theorem. Define the exponential function $\mathbb{C} \to \mathbb{C}$ by the convergent power series

$$e^x = 1 + \frac{1}{1}x + \frac{1}{1\cdot 2}x^2 + \frac{1}{1\cdot 2\cdot 3}x^3 + \frac{1}{1\cdot 2\cdot 3\cdot 4}x^4 + \dots$$

Then $e^1 = e = 2.71828\dots$ and $\boxed{e^{i\pi} + 1 = 0}$.

Proof. We need two facts from calculus:

$$\begin{aligned} -1 = \cos\pi &= 1 - \frac{1}{1\cdot 2}\pi^2 + \frac{1}{1\cdot 2\cdot 3\cdot 4}\pi^4 - \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\pi^6 + \dots \\ 0 = \sin\pi &= \frac{1}{1}\pi - \frac{1}{1\cdot 2\cdot 3}\pi^3 + \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5}\pi^5 - \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}\pi^7 + \dots \end{aligned}$$

We have

$$i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad i^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad i^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \text{and} \quad i^0 = i^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus $i^{n+4} = i^n$ for all n.

Also, we have $(i\pi)^n = \pi^n i^n$. It follows that

$$e^{i\pi} = \begin{bmatrix} 1 - \frac{1}{1\cdot 2}\pi^2 + \frac{1}{1\cdot 2\cdot 3\cdot 4}\pi^4 - \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\pi^6 + \dots & \frac{1}{1}\pi - \frac{1}{1\cdot 2\cdot 3}\pi^3 + \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5}\pi^5 - \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}\pi^7 + \dots \\ \frac{1}{1}\pi - \frac{1}{1\cdot 2\cdot 3}\pi^3 + \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5}\pi^5 - \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}\pi^7 + \dots & 1 - \frac{1}{1\cdot 2}\pi^2 + \frac{1}{1\cdot 2\cdot 3\cdot 4}\pi^4 - \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\pi^6 + \dots \end{bmatrix}.$$

By our two facts, this is just $e^{i\pi} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -1 + 0i$. Thus $e^{i\pi} + 1 = (-1 + 0i) + (1 + 0i) = 0$. $\square$

After a while, we tend to forget that complex numbers are 2×2 matrices and instead view the elements of $\mathbb{C}$ as formal expressions $a + bi$ where $a, b \in \mathbb{R}$ and i is a symbol that satisfies $i^2 = -1$.

We can add, subtract, and multiply such expressions just like polynomials, but substituting -1 for i^2. This convention gives the same operations as we saw above.

Moreover, this makes it clearer how to view $\mathbb{R}$ as a subset of $\mathbb{C}$, by setting $a = a + 0i$.

The *real part* of a complex number $a + bi \in \mathbb{C}$ is $\mathsf{Re}(a + bi) = a \in \mathbb{R}$.

The *imaginary part* of $a + bi \in \mathbb{C}$ is $\mathsf{Im}(a + bi) = b \in \mathbb{R}$.

Remark. It can be helpful to draw the complex number $a + bi \in \mathbb{C}$ as the vector $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$.

The number $i(a + bi) = -b + ai \in \mathbb{C}$ then corresponds to the vector $\begin{bmatrix} -b \\ a \end{bmatrix} \in \mathbb{R}^2$, which is given by *rotating* $\begin{bmatrix} a \\ b \end{bmatrix}$ *ninety degrees counterclockwise.* (Try drawing this yourself.)

The main reason it is useful to work with complex numbers is the following theorem about polynomials.

Suppose $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial with coefficients $a_0, a_1, \ldots, a_n \in \mathbb{C}$.

Assume $a_n \neq 0$ so that $p(x)$ has *degree* n.

Even though we think of complex numbers are 2×2 matrices, this expression for $p(x)$ still makes sense for $x \in \mathbb{C}$: if we plug in any complex number for x then $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{C}$.

Theorem (Fundamental theorem of algebra)**.** Define $p(x)$ as above. There are n (not necessarily distinct) complex numbers $r_1, r_2, \ldots, r_n \in \mathbb{C}$ such that $p(x) = a_n(x - r_1)(x - r_2)\cdots(x - r_n)$.

One calls the numbers $r_1, r_2, \ldots, r_n$ the *roots* of $p(x)$.

A root r has *multiplicity* m if exactly m of the numbers $r_1, r_2, \ldots, r_n$ are equal to r.

The use of complex numbers in this theorem is essential. The statement fails if we use $\mathbb{R}$ instead of $\mathbb{C}$.

Example: if $p(x) = x^2 + 1$ then there **do not exist** real numbers $r_1, r_2 \in \mathbb{R}$ with $p(x) = (x - r_1)(x - r_2)$.

However, we do have $x^2 + 1 = (x - i)(x + i)$.

3 Complex eigenvalues

The characteristic equation of an $n \times n$ matrix A is a degree n polynomial with real coefficients.

Counting multiplicities, $\det(A - xI)$ has exactly n roots but some roots may be complex numbers.

Define $\mathbb{C}^n$ to be the set of vectors $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ with n rows and entries $v_1, v_2, \ldots, v_n \in \mathbb{C}$.

Note that $\mathbb{R}^n \subset \mathbb{C}^n$ since $\mathbb{R} = \{a \in \mathbb{R}\} \subset \mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$.

The sum $u + v$ and scalar multiple cv for $u, v \in \mathbb{C}^n$ and $c \in \mathbb{C}$ are defined exactly as for vectors in $\mathbb{R}^n$, except we use the addition and multiplication operations from $\mathbb{C}$ instead of $\mathbb{R}$.

If A is an $n \times n$ matrix and $v \in \mathbb{C}^n$ then we define Av in the same way as when $v \in \mathbb{R}^n$.

Definition. Let A be an $n \times n$ matrix whose entries are all real numbers. Call $\lambda \in \mathbb{C}$ a *(complex) eigenvalue* of A if there exists a nonzero vector $v \in \mathbb{C}^n$ such that $Av = \lambda v$.

Equivalently, $\lambda \in \mathbb{C}$ is an eigenvalue of A if λ is a root of the characteristic polynomial $\det(A - xI)$.

This is no different from our first definition of an eigenvalue, except that now we permit λ to be in $\mathbb{C}$.

Example. Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then $\det(A - xI) = x^2 + 1 = (i - x)(-i - x)$.

The roots of this polynomial are the complex numbers i and $-i$. We have

$$A \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix} = i \begin{bmatrix} 1 \\ -i \end{bmatrix} \qquad \text{and} \qquad A \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = -i \begin{bmatrix} 1 \\ i \end{bmatrix}$$

so i and $-i$ are eigenvalues of A, with corresponding eigenvectors $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ i \end{bmatrix}$.

Example. Let $A = \begin{bmatrix} .5 & -.6 \\ .75 & 1.1 \end{bmatrix}$. Then $\det(A - xI) = \det \begin{bmatrix} .5 - x & -.6 \\ .75 & 1.1 - x \end{bmatrix} = x^2 - 1.6x + 1$.

Via the quadratic formula, we find that the roots of this characteristic polynomial are

$$x = \frac{1.6 \pm \sqrt{1.6^2 - 4}}{2} = .8 \pm .6i$$

since $i = \sqrt{-1}$. To find a basis for the $(.8 - .6i)$-eigenspace, we row reduce as usual

$$\begin{aligned} A - (.8 - .6i)I &= \begin{bmatrix} .5 & -.6 \\ .75 & 1.1 \end{bmatrix} - \begin{bmatrix} .8 - .6i & 0 \\ 0 & .8 - .6i \end{bmatrix} = \begin{bmatrix} -.3 + .6i & -.6 \\ .75 & .3 + .6i \end{bmatrix} \\ &\sim \begin{bmatrix} .5 - i & 1 \\ 1 & .8(.5 + i) \end{bmatrix} \sim \begin{bmatrix} 1 & .8(.5 + i) \\ .5 - i & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & .8(.5 + i) \\ 0 & 1 - .8(.5 + i)(.5 - i) \end{bmatrix} = \begin{bmatrix} 1 & .8(.5 + i) \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

The last equality holds since $.8(.5 + i)(.5 - i) = .8(.25 - i^2) = .8(1.25) = 1$.

This implies that $Ax = (.8 - .6i)x$ if and only if $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 + .8(.5 + i)x_2 = 0$, i.e., where $5x_1 = -4(.5 + i)x_2 = -(2 + 4i)x_2$. Satisfying these conditions is the vector

$$v = \begin{bmatrix} -2 - 4i \\ 5 \end{bmatrix}$$

which is therefore an eigenvector for A with eigenvalue $.8 - .6i$.

Similar calculations show that the vector $w = \begin{bmatrix} -2 + 4i \\ 5 \end{bmatrix}$ is an eigenvector for A with eigenvalue $.8 + .6i$.

Proposition. Suppose A is an $n \times n$ matrix with real entries. If A has a complex eigenvalue $\lambda \in \mathbb{C}$ with eigenvector $v \in \mathbb{C}^n$ then $\overline{v} \in \mathbb{C}^n$ is an eigenvector for A with eigenvalue $\overline{\lambda}$.

Proof. Since A has real entries, it holds that $\overline{A} = A$. Therefore $A\overline{v} = \overline{A}\overline{v} = \overline{Av} = \overline{\lambda v} = \overline{\lambda}\overline{v}$. □

4 Vocabulary

Keywords from today's lecture:

1. **Complex number**.

   We define a complex number to be either

   - A matrix $a + bi = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where $a, b \in \mathbb{R}$ and $i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
   - A formal expression "$a + bi$" where $a, b \in \mathbb{R}$ and i is a symbol that has $i^2 = -1$.

   The first definition makes it clear how to add, subtract, multiply, and divide complex numbers (use matrix operations). The second definition is secretly just a way of abbreviating the first definition.

   The set of complex numbers is denoted $\mathbb{C}$.

   Example:

   $1 + 2i$ corresponds to $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$.

   $(1 + 2i) + (2 + 3i) = 3 + 5i$ corresponds to $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ 5 & 3 \end{bmatrix}$.

   $(1 + 2i)(2 + 3i) = -4 + 7i$ corresponds to $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -4 & -6 \\ 7 & -4 \end{bmatrix}$.

   $(1 + 2i)^{-1} = \frac{1}{5} - \frac{2}{5}i$ corresponds to $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$.

2. **Complex conjugation**.

   If $a, b \in \mathbb{R}$ then *complex conjugate* of $a + bi \in \mathbb{C}$ is $\overline{a + bi} = a - bi \in \mathbb{C}$.

   If $y, z \in \mathbb{C}$ then $\overline{y + z} = \overline{y} + \overline{z}$ and $\overline{yz} = \overline{y} \cdot \overline{z}$ and $\overline{y^{-1}} = \overline{y}^{-1}$.

3. **Fundamental theorem of algebra**.

   Any polynomial
   $$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$
   with coefficients $a_0, a_1, \ldots, a_n \in \mathbb{C}$ and $a_n \neq 0$ can be factored as
   $$f(x) = a_n (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n)$$
   for some not necessarily distinct complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$.

4. **(Complex) eigenvalues and eigenvectors**.

   Let $\mathbb{C}^n$ be the set of vectors with n rows with entries in $\mathbb{C}$. Since $\mathbb{R} \subset \mathbb{C}$, we have $\mathbb{R}^n \subset \mathbb{C}^n$.

   If A is an $n \times n$ matrix and there exists a nonzero vector $v \in \mathbb{C}^n$ with $Av = \lambda v$ for some $\lambda \in \mathbb{C}$, then λ is an *eigenvalue* for A. The vector v is called an *eigenvector*.

   Example: The matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has eigenvalues i and $-i$.

   We have $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = -i \begin{bmatrix} 1 \\ i \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} i \\ -1 \end{bmatrix} = i \begin{bmatrix} 1 \\ -i \end{bmatrix}$.