

This document is intended as an **exact transcript** of the lecture, with extra summary and vocabulary sections for your convenience. By design, the material covered in lecture is exactly the same as what is in these notes. Due to time constraints, the notes may sometimes only contain limited illustrations, proofs, and examples; for a more thorough discussion of the course content, **consult the textbook**.

## Summary

Quick summary of today's notes. Lecture starts on next page.

- The characteristic equation of an  $n \times n$  matrix  $A$  is a degree  $n$  polynomial in one variable.

We can always factor this polynomial as

$$\det(A - xI) = (\lambda_1 - x)(\lambda_2 - x) \cdots (\lambda_n - x)$$

for some  $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$ . These complex numbers are the (*complex*) *eigenvalues* of  $A$ .

- Define  $\mathbb{C}^n$  to be the set of vectors  $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$  with  $n$  rows and entries  $v_1, v_2, \dots, v_n \in \mathbb{C}$ .

The sum  $u + v$  and scalar multiple  $cv$  for  $u, v \in \mathbb{C}^n$ ,  $c \in \mathbb{C}$  are defined just as for vectors in  $\mathbb{R}^n$ , except we use the addition and multiplication operations from  $\mathbb{C}$  instead of  $\mathbb{R}$ .

If  $A$  is an  $n \times n$  matrix and  $v \in \mathbb{C}^n$  then we define  $Av$  in the same way as when  $v \in \mathbb{R}^n$ .

A complex number  $\lambda \in \mathbb{C}$  an *eigenvalue* of  $A$  if and only if there exists  $0 \neq v \in \mathbb{C}^n$  with  $Av = \lambda v$ .

- The *trace* of a square matrix  $A$ , denoted  $\text{tr}A$ , is the sum of the diagonal entries of  $A$ .

If  $A$  and  $B$  are both  $n \times n$  then  $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$  and  $\text{tr}(AB) = \text{tr}(BA)$ .

But usually  $\text{tr}(AB) \neq \text{tr}(A)\text{tr}(B)$ .

- Let  $A$  be an  $n \times n$  matrix.

Suppose the roots of the characteristic polynomial  $\det(A - xI)$  are  $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$ .

These are the eigenvalues of  $A$ , repeated accordingly to their multiplicity.

Then  $\det A = \lambda_1 \lambda_2 \cdots \lambda_n$  and  $\text{tr}A = \lambda_1 + \lambda_2 + \dots + \lambda_n$ .

- Let  $A$  be an  $n \times n$  matrix.

The matrices  $A$  and  $A^T$  have the same characteristic polynomial and same eigenvalues.

If  $A$  is invertible, then  $A$  and  $A^{-1}$  have the same eigenvectors.

However,  $\lambda$  is an eigenvalue of  $A$  if and only if  $\lambda^{-1}$  is an eigenvalue for  $A^{-1}$ .

If  $A$  is diagonalizable then so is  $A^T$  and  $A^{-1}$  (when  $A$  is invertible).

# 1 Last time: complex numbers

Given  $a, b \in \mathbb{R}$ , we interpret  $a + bi$  as the matrix  $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ , so  $1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  and  $i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ .

Write  $\mathbb{C}$  for the set of *complex numbers*  $\{a + bi : a, b \in \mathbb{R}\}$ .

We view  $\mathbb{R} = \{a + 0i : a \in \mathbb{R}\}$  as a subset of  $\mathbb{C}$ .

According to our definition, every complex number is a  $2 \times 2$  matrix. It can also be helpful to think of a complex number  $a + bi$  as a polynomial with real coefficient in a variable  $i$  that satisfies  $i^2 = -1$ .

We can add, subtract, multiply, and invert complex numbers. These operations correspond to the usual ways of adding, subtracting, multiplying, and inverting matrices.

Let  $a, b, c, d \in \mathbb{R}$ . We add complex numbers in the following way:

$$(a + bi) + (c + di) = (a + c) + (b + d)i \in \mathbb{C}.$$

We multiply complex numbers like polynomials, but substituting  $-1$  for  $i^2$ :

$$(a + bi)(c + di) = ac + (ad + bc)i + bd(i^2) = (ac - bd) + (ad + bc)i \in \mathbb{C}.$$

The order of multiplication does not matter since  $(a + bi)(c + di) = (c + di)(a + bi)$ .

Given  $a, b \in \mathbb{R}$ , we define the *complex conjugate* of the complex number  $a + bi \in \mathbb{C}$  to be

$$\overline{a + bi} = a - bi \in \mathbb{C}.$$

If  $z = a + bi \in \mathbb{C}$ . Then  $\bar{z} = z$  if and only if  $b = 0$  and  $z \in \mathbb{R}$ .

If  $y, z \in \mathbb{C}$  then  $\overline{y + z} = \bar{y} + \bar{z}$  and  $\overline{yz} = \bar{y} \cdot \bar{z}$ .

If  $z = a + bi \in \mathbb{C}$  then  $z\bar{z} = (a + bi)(a - bi) = a^2 + b^2 \in \mathbb{R}$ .

This indicates how to invert complex numbers  $0 \neq a + bi$ :

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}^{-1} = (a + bi)^{-1} = \frac{a - bi}{(a + bi)(a - bi)} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i = \frac{1}{a^2 + b^2} \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

Finally, complex division is defined by

$$\frac{a + bi}{c + di} = (a + bi)(c + di)^{-1} = (c + di)^{-1}(a + bi).$$

**Example.** We have  $\frac{3 - 4i}{2 + i} = \frac{(3 - 4i)(2 - i)}{(2 + i)(2 - i)} = \frac{6 - 3i - 8i + 4i^2}{4 - i^2} = \frac{6 - 11i - 4}{5} = \frac{2 - 11i}{5} = \frac{2}{5} - \frac{11}{5}i$ .

**Theorem** (Fundamental theorem of algebra). Suppose

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

is a polynomial of degree  $n$  (meaning  $a_n \neq 0$ ) with coefficients  $a_0, a_1, \dots, a_n \in \mathbb{C}$ .

There are  $n$  (not necessarily distinct) numbers  $r_1, r_2, \dots, r_n \in \mathbb{C}$  such that

$$p(x) = a_n(x - r_1)(x - r_2) \cdots (x - r_n).$$

One calls the numbers  $r_1, r_2, \dots, r_n$  the *roots* of  $p(x)$ .

A root  $r$  has *multiplicity*  $m$  if exactly  $m$  of the numbers  $r_1, r_2, \dots, r_n$  are equal to  $r$ .

**Example.** We have  $9x^2 + 36 = 9(x - 2i)(x + 2i)$ .

## 2 Complex eigenvalues

Define  $\mathbb{C}^n$  to be the set of vectors  $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$  with  $n$  rows and entries  $v_1, v_2, \dots, v_n \in \mathbb{C}$ .

We have  $\mathbb{R}^n \subset \mathbb{C}^n$  since  $\mathbb{R} = \{a \in \mathbb{R}\} \subset \mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$ .

The sum  $u + v$  and scalar multiple  $cv$  for  $u, v \in \mathbb{C}^n$  and  $c \in \mathbb{C}$  are defined exactly as for vectors in  $\mathbb{R}^n$ , except we use the addition and multiplication operations from  $\mathbb{C}$  instead of  $\mathbb{R}$ .

If  $A$  is an  $n \times n$  matrix and  $v \in \mathbb{C}^n$  then we define  $Av$  in the same way as when  $v \in \mathbb{R}^n$ . For example:

$$\begin{bmatrix} i & 1 \\ 3 & 2i \end{bmatrix} \begin{bmatrix} 1 \\ 1 - i \end{bmatrix} = \begin{bmatrix} i + (1 - i) \\ 3 + 2i(1 - i) \end{bmatrix} = \begin{bmatrix} 1 \\ 3 + 2i - 2i^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 + 2i \end{bmatrix}.$$

**Definition.** Let  $A$  be an  $n \times n$  matrix with entries in  $\mathbb{R}$  or  $\mathbb{C}$ .

Let  $\lambda \in \mathbb{C}$ . The following statements are equivalent:

- $\lambda$  is an *eigenvalue* of  $A$ .
- $Av = \lambda v$  for some nonzero vector  $v \in \mathbb{C}^n$
- $\det(A - \lambda I) = 0$ .

This is no different from our first definition of an eigenvalue, except that now we permit  $\lambda$  to be in  $\mathbb{C}$ .

**Example.** The eigenvalues of  $A = \begin{bmatrix} i & 1 \\ 3 & 2i \end{bmatrix}$  are the solutions to

$$0 = \det(A - xI) = \det \begin{bmatrix} i - x & 1 \\ 3 & 2i - x \end{bmatrix} = (i - x)(2i - x) - 3 = 2i^2 - 3ix + x^2 - 3 = -5 - 3ix + x^2.$$

By the quadratic formula these solutions are

$$\lambda = \frac{3i \pm \sqrt{(-3i)^2 - 4(-5)}}{2} = \frac{3i \pm \sqrt{-9 + 20}}{2} = \pm \frac{\sqrt{11}}{2} + \frac{3}{2}i.$$

The fundamental theorem of algebra implies the following essential property:

**Fact.** If  $A$  is an  $n \times n$  matrix then  $A$  has  $n$  (not necessarily real or distinct) eigenvalues  $\lambda \in \mathbb{C}$ , counting repeated eigenvalues with their respective multiplicities.

If  $A$  is a matrix and  $v \in \mathbb{C}^n$  then we define  $\bar{A}$  and  $\bar{v}$  to be the matrix and vector given by replacing all entries of  $A$  and  $v$  by their complex conjugates.

**Proposition.** Suppose  $A$  is an  $n \times n$  matrix with real entries, so that  $A = \bar{A}$ . If  $A$  has a complex eigenvalue  $\lambda \in \mathbb{C}$  with eigenvector  $v \in \mathbb{C}^n$  then  $\bar{v} \in \mathbb{C}^n$  is an eigenvector for  $A$  with eigenvalue  $\bar{\lambda}$ .

This proposition does **not** apply to  $A = \begin{bmatrix} i & 1 \\ 3 & 2i \end{bmatrix}$  from above since  $A$  does not have all real entries.

### 3 Some final properties of eigenvalues of eigenvectors

We discuss a few more properties of eigenvalues and eigenvectors.

**Lemma.** Suppose we can write a polynomial in  $x$  in two ways as

$$(\lambda_1 - x)(\lambda_2 - x) \cdots (\lambda_n - x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

for some complex numbers  $\lambda_1, \lambda_2, \dots, \lambda_n, a_0, a_1, \dots, a_n \in \mathbb{C}$ . Then

$$a_n = (-1)^n \quad \text{and} \quad a_{n-1} = (-1)^{n-1}(\lambda_1 + \lambda_2 + \cdots + \lambda_n) \quad \text{and} \quad a_0 = \lambda_1 \lambda_2 \cdots \lambda_n.$$

*Proof.* The product  $(\lambda_1 - x)(\lambda_2 - x) \cdots (\lambda_n - x)$  is a sum of  $2^n$  monomials corresponding to a choice of either  $\lambda_i$  or  $-x$  for each of the  $n$  factors, multiplied together.

The only such monomial of degree  $n$  is  $(-x)^n = (-1)^n x^n = a_n x^n$  so  $a_n = (-1)^n$ .

The only such monomial of degree 0 is  $\lambda_1 \lambda_2 \cdots \lambda_n = a_0$ .

Finally, there are  $n$  monomials of degree  $n - 1$  that arise:

$$\lambda_1(-x)^{n-1} + (-x)\lambda_2(-x)^{n-2} + (-x)^2\lambda_3(-x)^{n-3} + \cdots + (-x)^{n-1}\lambda_n = (-1)^{n-1}(\lambda_1 + \cdots + \lambda_n)x^{n-1}.$$

This sum must be equal to  $a_{n-1}x^{n-1}$  so  $a_{n-1} = (-1)^{n-1}(\lambda_1 + \lambda_2 + \cdots + \lambda_n)$ .  $\square$

Let  $A$  be an  $n \times n$  matrix.

Define  $\text{tr}(A)$  to be the sum of the diagonal entries of  $A$ . Call  $\text{tr}(A)$  the *trace* of  $A$ .

**Example.**  $\text{tr} \left( \begin{bmatrix} 1 & 0 & 7 \\ -1 & 2 & 8 \\ 2 & 4 & 3 \end{bmatrix} \right) = 1 + 2 + 3 = 6 = \text{tr} \left( \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 4 \\ 7 & 8 & 3 \end{bmatrix} \right).$

We see in this example that  $\text{tr}(A^T) = \text{tr}(A)$  since  $A$  and  $A^T$  have the same diagonal entries. Additionally:

**Proposition.** If  $A, B$  are  $n \times n$  matrices then  $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$  and  $\text{tr}(AB) = \text{tr}(BA)$ .

However, usually  $\text{tr}(AB) \neq \text{tr}(A)\text{tr}(B)$ , unlike for the determinant.

*Proof.* The diagonal entries of  $A + B$  are given by adding together the diagonal entries of  $A$  with those of  $B$  in corresponding positions, so it follows that  $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$ .

Let  $E_{ij}$  be the  $n \times n$  matrix with 1 in position  $(i, j)$  and 0 in all other positions.

(In this proof, we use the symbol  $i$  to mean an integer index rather than a complex number.)

You can check that  $E_{ij}E_{kl}$  is the zero matrix if  $j \neq k$  and that  $E_{ij}E_{jk} = E_{ik}$ .

Moreover,  $\text{tr}(E_{ij}) = 0$  if  $i \neq j$  and  $\text{tr}(E_{ii}) = 1$ .

We conclude that  $\text{tr}(E_{ij}E_{kl})$  is 1 if  $i = l$  and  $j = k$  and is 0 otherwise.

This formula is symmetric so  $\text{tr}(E_{ij}E_{kl}) = \text{tr}(E_{kl}E_{ij})$ .

It follows that  $\text{tr}(AB) = \text{tr}(BA)$  since if  $A_{ij}$  and  $B_{ij}$  are the entries of  $A$  and  $B$  in positions  $(i, j)$ , then

$$A = \sum_{i=1}^n \sum_{j=1}^n A_{ij} E_{ij} \quad \text{and} \quad B = \sum_{k=1}^n \sum_{l=1}^n B_{kl} E_{kl}.$$

$\square$

**Theorem.** Let  $A$  be an  $n \times n$  matrix (with entries in  $\mathbb{R}$  or  $\mathbb{C}$ ).

Suppose the characteristic polynomial of  $A$  factors as

$$\det(A - xI) = (\lambda_1 - x)(\lambda_2 - x) \cdots (\lambda_n - x).$$

Then  $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$  and  $\text{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ . In other words:

- (a) The product of the (complex) eigenvalues of  $A$ , counted with multiplicity, is  $\det(A)$ .
- (b) The sum of the (complex) eigenvalues of  $A$ , counted with multiplicity is  $\text{tr}(A)$ .

**Remark.** The theorem is true for all matrices, but is much easier to prove for diagonalizable matrices.

If  $A = PDP^{-1}$  where  $D$  is a diagonal matrix, then  $\det(A) = \det(PDP^{-1}) = \det(D) = \lambda_1 \lambda_2 \cdots \lambda_n$  and

$$\text{tr}(A) = \text{tr}(PDP^{-1}) = \text{tr}(DP^{-1}P) = \text{tr}(D) = \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$

Before proving the theorem let's see an example.

**Example.** If  $A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & i \end{bmatrix}$  then  $\begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ , and  $\begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}$  are eigenvectors of  $A$ .

The corresponding eigenvalues are  $i$ ,  $i$ , and  $-i$ .

One can check that  $\det(A - xI) = -x^3 + ix^2 - x + i = (i - x)^2(-i - x)$ .

The theorem asserts that  $(i)(i)(-i) = -i^3 = i = \det(A)$  and  $i + i + (-i) = i = \text{tr}(A)$ .

*Proof of the theorem.* We can write  $\det(A - xI) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$  for some numbers  $a_0, a_1, \dots, a_n \in \mathbb{C}$ . By the lemma it suffices to show that  $a_0 = \det(A)$  and  $a_{n-1} = (-1)^{n-1} \text{tr}(A)$ .

The first claim is easy. The value of  $a_0$  is given by setting  $x = 0$  in  $\det(A - xI)$ , so  $a_0 = \det(A)$ .

Showing that  $a_{n-1} = (-1)^{n-1} \text{tr}(A)$  takes a little more work.

Consider the coefficient  $a_{n-1}$  of  $x^{n-1}$  in the characteristic polynomial  $\det(A - xI)$ . Remember our formula

$$\det(A - xI) = \sum_{Z \in S_n} (-1)^{\text{inv}(Z)} \text{prod}(Z, A - xI) \tag{*}$$

where  $\text{prod}(Z, A - xI)$  is the product of the entries of  $A - xI$  in the nonzero positions of the permutation matrix  $Z$ . The key observation to make is that if  $Z \in S_n$  is not the identity matrix then  $Z$  has at most  $n - 2$  nonzero entries on the diagonal, so  $\text{prod}(Z, A - xI)$  is a polynomial in  $x$  degree at most  $n - 2$ .

Therefore the formula (\*) implies that

$$\det(A - xI) = \text{prod}(I, A - xI) + (\text{polynomial terms of degree } \leq n - 2).$$

Let  $d_i$  be the diagonal entry of  $A$  in position  $(i, i)$ . Then  $\text{prod}(I, A - xI) = (d_1 - x)(d_2 - x) \cdots (d_n - x)$  and the coefficient of  $x^{n-1}$  in this polynomial must be equal to the coefficient of  $x^{n-1}$  in  $\det(A - xI)$ .

By the lemma, the coefficient of  $x^{n-1}$  in  $(d_1 - x)(d_2 - x) \cdots (d_n - x)$  is

$$(-1)^{n-1} (d_1 + d_2 + \cdots + d_n) = (-1)^{n-1} \text{tr}(A),$$

and so  $a_{n-1} = (-1)^{n-1} \text{tr}(A)$ . □

**Corollary.** Suppose  $A$  is a  $2 \times 2$  matrix. Let  $p = \det A$  and  $q = \operatorname{tr} A$ .

Then  $A$  has distinct eigenvalues if and only if  $q^2 \neq 4p$ .

*Proof.* Suppose  $a, b \in \mathbb{C}$  are the eigenvalues of  $A$  (repeated with multiplicity).

Then  $ab = p$  and  $a + b = q$  so  $a(q - a) = qa - a^2 = p$  and therefore  $a^2 - qa + p = 0$ .

The quadratic formula implies that  $a = \frac{q \pm \sqrt{q^2 - 4p}}{2}$  and  $b = \frac{q \mp \sqrt{q^2 - 4p}}{2}$  so  $a \neq b$  if and only if  $q^2 - 4p \neq 0$ .  $\square$

**Proposition.** If  $A$  is a square matrix then  $A$  and  $A^T$  have the same eigenvalues.

*Proof.* This follows since  $\det(A - xI) = \det((A - xI)^T) = \det(A^T - xI^T) = \det(A^T - xI)$ .  $\square$

**Proposition.** Let  $A$  be a square matrix. Then  $A$  is invertible if and only if  $0$  is not one of its eigenvalues. Assume  $A$  is invertible. Then  $A$  and  $A^{-1}$  have the same eigenvectors, but  $v$  is an eigenvector of  $A$  with eigenvalue  $\lambda$  if and only if  $v$  is an eigenvector of  $A^{-1}$  with eigenvalue  $1/\lambda$ .

*Proof.*  $0$  is an eigenvalue of  $A$  if and only if  $\det A = 0$  which occurs precisely when  $A$  is not invertible.

If  $A$  is invertible and  $Av = \lambda v$  then  $v = A^{-1}Av = A^{-1}\lambda v = \lambda A^{-1}v$  so  $A^{-1}v = \lambda^{-1}v$ .  $\square$

**Corollary.** If  $A$  is invertible and diagonalizable then  $A^{-1}$  is diagonalizable.

*Proof.* If  $A$  is invertible and diagonalizable, then  $\mathbb{R}^n$  has a basis consisting of eigenvectors of  $A$ , but this basis is then also made up of eigenvectors of  $A^{-1}$ , so  $A^{-1}$  is diagonalizable.  $\square$

**Corollary.** If  $A$  is diagonalizable then  $A^T$  is diagonalizable.

*Proof.* If  $A = PDP^{-1}$  then  $A^T = (PDP^{-1})^T = (P^{-1})^T D^T P^T = QEQ^{-1}$  for the invertible matrix  $Q = (P^{-1})^T = (P^T)^{-1}$  and the diagonal matrix  $E = D^T$ .  $\square$

## 4 Vocabulary

Keywords from today's lecture:

1. **(Complex) eigenvalues and eigenvectors.**

Let  $\mathbb{C}^n$  be the set of vectors with  $n$  rows with entries in  $\mathbb{C}$ . Since  $\mathbb{R} \subset \mathbb{C}$ , we have  $\mathbb{R}^n \subset \mathbb{C}^n$ .

If  $A$  is an  $n \times n$  matrix and there exists a nonzero vector  $v \in \mathbb{C}^n$  with  $Av = \lambda v$  for some  $\lambda \in \mathbb{C}$ , then  $\lambda$  is an *eigenvalue* for  $A$ . The vector  $v$  is called an *eigenvector*.

Example: The matrix  $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  has eigenvalues  $i$  and  $-i$ .

We have  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = -i \begin{bmatrix} 1 \\ i \end{bmatrix}$  and  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} i \\ -1 \end{bmatrix} = i \begin{bmatrix} 1 \\ -i \end{bmatrix}$ .

2. **Trace** of a square matrix.

The sum of the diagonal entries of a square matrix  $A$ , denote  $\text{tr}(A)$ .

The value of  $\text{tr}(A)$  is also the sum of the complex eigenvalues of  $A$ , counted with multiplicity.

Example:  $\text{tr} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1 + 4 = 5$ .