Instructions: Choose **2 problems** and write down detailed solutions, showing all necessary work. You can earn up to **8 extra credit points** by correctly solving additional problems.[1]

Feel free to discuss problems with other students but write up your own solutions. If your solutions appear to be copied from somewhere else, you will automatically receive zero credit.

To get full credit for the offline homework, you just need to make a good-faith attempt on two problems. The bar for receiving extra credit points is higher: your solutions need to be close to completely correct.

1. (a) For each $i = 1, 2, 3$, the map $T_i : \mathbb{R}^3 \to \mathbb{R}^2$ given by the formula

$$T_i(v) = (\text{the vector } v \text{ with row } i \text{ deleted})$$

   is a linear transformation. What are the standard matrices of T_1, T_2, and T_3?

   (b) Can you find two linearly independent vectors $v, w \in \mathbb{R}^3$ such that for each $i = 1, 2, 3$, if we delete row i from both vectors, we get linearly independent vectors in $\mathbb{R}^2$?

   In other words, such that the set $\{T_1(v), T_1(w)\}$ is linearly independent, $\{T_2(v), T_2(w)\}$ is linearly independent, and $\{T_3(v), T_3(w)\}$ is linearly independent. Justify your answer.

   (c) Can you find two linearly independent vectors $v, w \in \mathbb{R}^3$ such that for each $i = 1, 2, 3$, if we delete row i from both vectors, we get linearly dependent vectors in $\mathbb{R}^2$?

   In other words, such that the set $\{T_1(v), T_1(w)\}$ is linearly dependent, $\{T_2(v), T_2(w)\}$ is linearly dependent, and $\{T_3(v), T_3(w)\}$ is linearly dependent. Justify your answer.

2. Suppose A is an $m \times n$ matrix and B is an $m \times q$ matrix.

   (a) What relationship, if any, exists between $\mathsf{RREF}(A)$ and $\mathsf{RREF}(\begin{bmatrix} A & B \end{bmatrix})$ in general?

   (b) What relationship, if any, exists between $\mathsf{RREF}(B)$ and $\mathsf{RREF}(\begin{bmatrix} A & B \end{bmatrix})$ in general?

   (c) What relationship, if any, exists between $\mathsf{RREF}(\begin{bmatrix} A & B \end{bmatrix})$ and $\mathsf{RREF}(\begin{bmatrix} B & A \end{bmatrix})$ in general?

   For any relationship that you find, justify why it holds in general.

3. Suppose A is an $m \times n$ matrix and C is a $q \times n$ matrix.

   (a) What relationship, if any, exists between $\mathsf{RREF}(A)$ and $\mathsf{RREF}\left(\begin{bmatrix} A \\ C \end{bmatrix}\right)$ in general?

   (b) What relationship, if any, exists between $\mathsf{RREF}(C)$ and $\mathsf{RREF}\left(\begin{bmatrix} A \\ C \end{bmatrix}\right)$ in general?

   (c) What relationship, if any, exists between $\mathsf{RREF}\left(\begin{bmatrix} A \\ C \end{bmatrix}\right)$ and $\mathsf{RREF}\left(\begin{bmatrix} C \\ A \end{bmatrix}\right)$ in general?

   For any relationship that you find, justify why it holds in general.

4. Suppose $v_1, v_2, \dots, v_k \in \mathbb{R}^n$ are vectors and $V = \mathbb{R}\text{-span}\{v_1, v_2, \dots, v_k\}$.

   (a) If we add another vector v_{k+1} to this list, will it still have the same span?

   What can you say about when $V = \mathbb{R}\text{-span}\{v_1, v_2, \dots, v_k, v_{k+1}\}$?

   (b) If we delete one of the vectors from the list, say v_k, will it still have the same span?

   What can you say about when $V = \mathbb{R}\text{-span}\{v_1, v_2, \dots, v_{k-1}\}$?

   Explain and justify your answers to both parts.

[1] There will be ∼11 weeks of assignments, each with ∼10 practice problems, so you can earn up to ∼88 equally weighted extra credit points. The maximum amount of extra credit you can earn is 5% of your total grade for the semester.

5. Let $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ be two vectors in $\mathbb{R}^2$.

   The span $\mathbb{R}\text{-span}\{v, w\} = \{\alpha v + \beta w : \alpha, \beta \in \mathbb{R}\}$ is the smallest set of vectors in $\mathbb{R}^2$ containing v and w that is **closed under scalar multiplication and vector addition/subtraction**.

   (a) Explain why $\mathbb{R}\text{-span}\{v, w\} = \mathbb{R}^2$.

   (b) Draw a picture of the smallest set $\mathcal{S}$ of vectors in $\mathbb{R}^2$ that contains v and w and is closed under just scalar multiplication. This means that $v \in \mathcal{S}$, $w \in \mathcal{S}$, if $u \in \mathcal{S}$ then $cu \in \mathcal{S}$ for all $c \in \mathbb{R}$, and $\mathcal{S}$ is as small as possible with these properties.

   (c) Draw a picture of the smallest set $\mathcal{T}$ of vectors in $\mathbb{R}^2$ that contains v and w and is closed under just vector addition and subtraction. This means that $v \in \mathcal{T}$, $w \in \mathcal{T}$, if $x, y \in \mathcal{T}$ then $x + y \in \mathcal{T}$ and $x - y \in \mathcal{T}$, and $\mathcal{T}$ is as small as possible with these properties.

   (d) Draw pictures of $\frac{1}{2}\mathcal{S} = \{\frac{1}{2}u : u \in \mathcal{S}\}$ and $\frac{1}{2}\mathcal{T} = \{\frac{1}{2}u : u \in \mathcal{T}\}$.

   In general, how is $\mathcal{S}$ related to $\frac{1}{n}\mathcal{S}$ and how is $\mathcal{T}$ related to $\frac{1}{n}\mathcal{T}$ when n is a positive integer?

6. Suppose $v_1, v_2, \dots, v_k$ and $w_1, w_2, \dots, w_l$ are two lists of vectors in $\mathbb{R}^n$.

   Describe an algorithm to determine whether or not $\mathbb{R}\text{-span}\{v_1, v_2, \dots, v_k\} = \mathbb{R}\text{-span}\{w_1, w_2, \dots, w_l\}$.

7. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^4$ is a linear transformation.

   If $T\left(\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}\right)$ and $T\left(\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}\right) = T\left(\begin{bmatrix} 4 \\ 4 \\ 7 \end{bmatrix}\right)$ then what is $T\left(\begin{bmatrix} 5 \\ 9 \\ 16 \end{bmatrix}\right)$?

   Justify your answer.

8. Suppose $m \leq n$ are positive integers.

   Let X be a finite set with m elements and let Y be a finite set with n elements.

   Let $\mathbb{Z}$ be the infinite set of all integers.

   (a) Explain why any one-to-one map $f : Y \to X$ is also onto.

   Similarly, explain why any one-to-one linear map $f : \mathbb{R}^n \to \mathbb{R}^m$ is also onto.

   (b) Explain why any onto map $f : X \to Y$ is also one-to-one.

   Similarly, explain why any onto linear map $f : \mathbb{R}^m \to \mathbb{R}^n$ is also one-to-one.

   (c) These properties do not hold for arbitrary maps between infinite sets.

   To show this, give an example of a map $f : \mathbb{Z} \to \mathbb{Z}$ that is one-to-one but not onto.

   Then give an example of a map $g : \mathbb{Z} \to \mathbb{Z}$ that is onto but not one-to-one.

   (Suggestion: for parts (a) and (b) use the characterization of onto and one-to-one linear maps in terms of the reduced echelon form of the standard matrix.)

9. Consider the set of vectors $\mathcal{Q}$ in $\mathbb{R}^2$ whose endpoints are in the quadrilateral in the xy-plane with vertices $(0, 0)$, $(-20, 22)$, $(21, 21)$, and $(1, 43)$.

   If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with $\left\{T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) : 0 \leq a \leq 1 \text{ and } 0 \leq b \leq 1\right\} = \mathcal{Q}$, then what are the possibilities for the standard matrix A of T? Justify your answer.

10. The unit cube in $\mathbb{R}^n$ is the set $\mathcal{C} = \{v \in \mathbb{R}^n : 0 \leq v_i \leq 1 \text{ for all } i = 1, 2, \dots, n\}$.

    Find all $n \times n$ matrices A such that $A\mathcal{C} = \mathcal{C}$. Be sure to justify your answer.

    Here $A\mathcal{C}$ is defined to be the set $\{Av : v \in \mathcal{C}\}$.