**Instructions:** Choose **2 problems** and write down detailed solutions, showing all necessary work. You can earn up to **8 extra credit points** by correctly solving additional problems.[1]

Feel free to discuss problems with other students but write up your own solutions. If your solutions appear to be copied from somewhere else, you will automatically receive zero credit.

To get full credit for the offline homework, you just need to make a good-faith attempt on two problems. The bar for receiving extra credit points is higher: your solutions need to be close to completely correct.

Show all steps and provide justification for all answers.

1. Define the *two-sided reduced echelon form* of a matrix $A$ to be
$$\mathsf{TREF}(A) := \mathsf{RREF}(\mathsf{RREF}(A)^T)^T.$$
Here $M^T$ denotes the transpose of a matrix $M$.
As a warmup, calculate $\mathsf{TREF}(A)$ if
$$A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix}.$$
Describe all the possibilities for $\mathsf{TREF}(A)$ when $A$ is a general $m \times n$ matrix.

2. Does it always hold that $\mathsf{TREF}(A^T) = \mathsf{TREF}(A)^T$ for any matrix $A$?
Prove this property or find a counterexample.

3. Find a general formula for $A^n$ when $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $n$ is any positive integer.
*(Hint: write $A = I + B$ where $I$ the identity matrix. Then find a general formula for $B^n$ first.)*

4. Suppose $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \in \mathbb{R}$. When is
$$A = \begin{bmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{bmatrix}$$
invertible? Find a formula for $A^{-1}$ when $A$ is invertible.

5. Suppose $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \in \mathbb{R}$. When is
$$A = \begin{bmatrix} 0 & 0 & a_1 & b_1 \\ 0 & 0 & c_1 & d_1 \\ a_2 & b_2 & 0 & 0 \\ c_2 & d_2 & 0 & 0 \end{bmatrix}$$
invertible? Find a formula for $A^{-1}$ when $A$ is invertible.

6. Suppose $a, b \in \mathbb{R}$ and $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and $D = \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix}$.
Find an invertible matrix $P$ such that $AP = PD$.
Then find a general formula for $A^n$ when $n$ is any positive integer.
*(Hint: first express $A$, $A^2$, and $A^3$ in terms of $P$ and $D$.)*

[1] There will be ∼11 weeks of assignments, each with ∼10 practice problems, so you can earn up to ∼88 equally weighted extra credit points. The maximum amount of extra credit you can earn is 5% of your total grade for the semester.

7. Suppose $a, b, c, d \in \mathbb{R}$. Define $f : \mathbb{R} \to \mathbb{R}$ to be the function with formula

$$f(x) = ax^3 + bx^2 + cx + d.$$

Important note: we are defining $f$ to have domain $\mathbb{R}$ and also codomain $\mathbb{R}$.

What are the possibilities for $(a, b, c, d)$ if $f$ is an invertible function $\mathbb{R} \to \mathbb{R}$?

*(Hint: separately consider the cases when $a \neq 0$, $a = 0 \neq b$, $a = b = 0 \neq c$, etc., and use calculus.)*

8. Suppose $A$ is an $n \times n$ matrix and $I$ is the $n \times n$ identity matrix.

Choose positive real numbers $b$ and $c$.

(a) Assume $A^2 = bI + cA$. Find a formula for $A^{-1}$.

(b) Suppose $A$ is the $n \times n$ matrix with $b$ on the main diagonal and $c$ everywhere else.

For example, if $n = 4$ then we would have

$$A = \begin{bmatrix} b & c & c & c \\ c & b & c & c \\ c & c & b & c \\ c & c & c & b \end{bmatrix}.$$

When is $A$ invertible? Compute $A^{-1}$ when $A$ is invertible.

(Your answer should be for general $b, c > 0$ and general $n$, not just for $n = 4$.)

9. Suppose $A$ and $B$ are $n \times n$ matrices such that $v^T Aw = v^T Bw \in \mathbb{R}$ for all vectors $v, w \in \mathbb{R}^n$.

Does it always hold that $A$ and $B$ are equal? Prove that $A = B$ or find a counterexample.

10. The *commutator* of two invertible matrices $A$ and $B$ is $[A, B] := ABA^{-1}B^{-1}$.

Compute a formula for $[A, B]$ when $A = \begin{bmatrix} 1 & a & b & c \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & v & w & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Here $a, b, c, d, e, v, w, x, y, z$ are arbitrary real numbers.