Instructions: Choose **2 problems** and write down detailed solutions, showing all necessary work. You can earn up to **8 extra credit points** by correctly solving additional problems.[1]

Feel free to discuss problems with other students but write up your own solutions. If your solutions appear to be copied from somewhere else, you will automatically receive zero credit.

To get full credit for the offline homework, you just need to make a good-faith attempt on two problems. The bar for receiving extra credit points is higher: your solutions need to be close to completely correct.

Show all steps and provide justification for all answers.

1. Let $A = \begin{bmatrix} 1 & 1 & 1 & 11 \\ 6 & 8 & 16 & -26 \\ 3 & 1 & -7 & 25 \end{bmatrix}$.
   (a) Find a basis for the column space of A.
   (b) Find a basis for the null space of A.

   Do all steps by hand and show your work.
2. Let $A = \begin{bmatrix} 2 & 3 & 16 & -5 \\ -6 & 8 & 20 & -2 \\ 3 & -2 & -8 & 3 \end{bmatrix}$.
   (a) Find a basis for the column space of A.
   (b) Find a basis for the null space of A.

   Do all steps by hand and show your work.
3. Let $A = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$ be a one-row matrix. Here $a_1, a_2, \dots, a_n \in \mathbb{R}$ are arbitrary numbers.
   (a) Find a basis for the column space of A.
   (b) Find a basis for the null space of A.

   Your answer will depend on the entries in A. Remember to the consider the case when $A = 0$.
4. Suppose $v_1, v_2, \dots, v_k$ are linearly independent vectors in $\mathbb{R}^n$, where $k < n$.

   Describe an algorithm to find vectors $v_{k+1}, v_{k+2}, \dots, v_n$ such that $v_1, v_2, \dots, v_n$ is a basis for $\mathbb{R}^n$.
5. Show that the rank one $m \times n$ matrices (the $m \times n$ matrices A with $\mathrm{rank}(A) = \dim(\mathrm{Col}(A)) = 1$) are precisely the matrices that can be expressed as vw^T for vectors $0 \neq v \in \mathbb{R}^m$ and $0 \neq w \in \mathbb{R}^n$.

   One way to do this is by the following steps:
   (a) Explain why $\mathrm{rank}(vw^T) = 1$ if $0 \neq v \in \mathbb{R}^m$ and $0 \neq w \in \mathbb{R}^n$.
   (b) Explain why a rank one matrix A must have a nonzero column.
   (c) If v is a nonzero column of a rank one matrix A, explain how to find a nonzero vector w such that $A = vw^T$.
6. Let $A = \begin{bmatrix} 2 & 0 & 4 & 0 \\ 1 & 8 & 0 & 5 \\ 2 & -8 & 4 & -4 \\ 0 & 0 & x & -9 \end{bmatrix}$.

   Determine the values of x such that A is invertible and find a formula for A^{-1} in this case.
7. Suppose A is an $m \times n$ matrix and I_m is the $m \times m$ identity matrix.

[1] There will be ∼11 weeks of assignments, each with ∼10 practice problems, so you can earn up to ∼88 equally weighted extra credit points. The maximum amount of extra credit you can earn is 5% of your total grade for the semester.

(a) Explain why there exists a matrix B with $AB = I_m$ if A has a pivot position in every row.

(b) Prove that A has a pivot position in every row if there exists a matrix B such that $AB = I_m$

8. Suppose A is an $m \times n$ matrix and I_n is the $n \times n$ identity matrix.

   (a) Explain why there exists a matrix B with $BA = I_n$ if A has a pivot position in every column.

   (b) Prove that A has a pivot position in every column if there exists a matrix B such that $BA = I_n$.

9. Suppose A is an $m \times n$ matrix and B is an $n \times q$ matrix. If $\mathrm{rank}(A) = n$ and $\mathrm{rank}(B) = r$, then what is $\mathrm{rank}(AB)$ in terms of m, n, q, and r? Justify your answer.

10. Let $v = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ and $w \in \mathbb{R}^3$. Suppose there exists a 3×3 matrix A whose null space **and** column space contains both v and w. What are the possibilities for w? For each of these possibilities, give an example of a 3×3 matrix A whose null space and column space contains both v and w