Instructions: Choose **2 problems** and write down detailed solutions, showing all necessary work. You can earn up to **8 extra credit points** by correctly solving additional problems.[1]

Feel free to discuss problems with other students but write up your own solutions. If your solutions appear to be copied from somewhere else, you will automatically receive zero credit.

To get full credit for the offline homework, you just need to make a good-faith attempt on two problems. The bar for receiving extra credit points is higher: your solutions need to be close to completely correct.

Show all steps and provide justification for all answers.

1. Suppose $v_1, v_2, \dots, v_n$ and $w_1, w_2, \dots, w_n$ are vectors in $\mathbb{R}^n$. Define
$$A = \left[\begin{array}{cccc} v_1 & v_2 & \dots & v_n \end{array}\right] \quad\text{and}\quad B = \left[\begin{array}{cccc} w_1 & w_2 & \dots & w_n \end{array}\right].$$
Suppose A is invertible. Explain why there is a unique linear function $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f(v_i) = w_i$ for all $i = 1, 2, \dots, n$. What is the standard matrix of f?

   As an application, find the standard matrix of the unique linear function $f : \mathbb{R}^3 \to \mathbb{R}^3$ with
$$f\left(\left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right) = \left[\begin{array}{c} 2 \\ 1 \\ 2 \end{array}\right], \quad f\left(\left[\begin{array}{c} 0 \\ 1 \\ 1 \end{array}\right]\right) = \left[\begin{array}{c} 1 \\ 2 \\ 0 \end{array}\right], \quad\text{and}\quad f\left(\left[\begin{array}{c} 1 \\ 0 \\ 1 \end{array}\right]\right) = \left[\begin{array}{c} 2 \\ 2 \\ 1 \end{array}\right].$$

2. Define A and B as in the previous question. Suppose A is not invertible. Explain why there is never a **unique** linear function $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f(v_i) = w_i$ for all $i = 1, 2, \dots, n$. In other words, explain why there is either no such function, or more than one.

3. A matrix A is *skew-symmetric* if $A^\top = -A$. This holds if and only if A is square with $A_{ii} = 0$ and $A_{ji} = -A_{ij}$ for all rows i and columns j that have $i \neq j$.

   Suppose A is an $n \times n$ skew-symmetric matrix with all integer entries.

   (a) Prove that if n is odd then $\det(A) = 0$.

   (b) It can be shown that if n is even then $\det(A)$ is a perfect square.

   For example, $A = \left[\begin{array}{cc} 0 & -a \\ a & 0 \end{array}\right]$ then $\det(A) = a^2$. Check that this also holds if $n = 4$.

4. Suppose $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \in \mathbb{R}$ are all positive. Give a direct geometric argument to compute the volume of the parallelepiped with edges
$$\left[\begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array}\right], \left[\begin{array}{c} b_1 \\ b_2 \\ b_3 \end{array}\right], \left[\begin{array}{c} c_1 \\ c_2 \\ c_3 \end{array}\right]$$
and confirm that it is equal to the absolute value of $\det\left(\left[\begin{array}{ccc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array}\right]\right)$.

5. Let $\mathbb{R}^{n\times k}$ be the set of $n \times k$ matrices.

   The determinant is a function $\mathbb{R}^{n\times n} \to \mathbb{R}$ that is *alternating* (switching two columns of the input multiplies by the output value by -1) and *multilinear* (it is a linear as a function of the ith column of the input matrix, for any fixed column index i).

   The set of alternating multilinear maps $\mathbb{R}^{n\times n} \to \mathbb{R}$ is a vector space.

   Show that this vector space is 1-dimensional with basis $\{\det\}$ in two steps:

[1] There will be ∼11 weeks of assignments, each with ∼10 practice problems, so you can earn up to ∼88 equally weighted extra credit points. The maximum amount of extra credit you can earn is 5% of your total grade for the semester.

(a) Suppose $f : \mathbb{R}^{n\times n} \to \mathbb{R}$ is an alternating multilinear function with $f(I) \neq 0$.

Explain why f must be a scalar multiple of det.

(b) Suppose $f : \mathbb{R}^{n\times n} \to \mathbb{R}$ is an alternating multilinear function with $f(I) = 0$.

Explain why $f(A) = 0$ for all $A \in \mathbb{R}^{n\times n}$.

6. The set of alternating multilinear maps $\mathbb{R}^{n\times k} \to \mathbb{R}$ is a vector space for any $k \geq 1$.

   Explain why this vector space is 0-dimensional if $k > n$.

7. Assume $n \geq 2$. Check that the function $\mathbb{R}^{n\times 2} \to \mathbb{R}$ given by

$$f\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \vdots & \vdots \\ a_{n1} & a_{n2} \end{bmatrix}\right) = \det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right)$$

   is alternating and multilinear.

   Then find a basis for the vector space of alternating multilinear maps $\mathbb{R}^{n\times 2} \to \mathbb{R}$.

8. Let A be a matrix. A $k \times l$ submatrix of A is formed by choosing k (not necessarily adjacent) rows and l (not necessarily adjacent) columns and using the entries of A in these rows and columns as the corresponding entries of a $k \times l$ matrix.

   The 9 different 2×2 submatrices of $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$ include $\begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix}$ and $\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$.

   Explain why $\mathrm{rank}(A) \geq k$ if A has a $k \times k$ submatrix with nonzero determinant.

9. Explain why $\mathrm{rank}(A) \leq k$ if every $(k+1) \times (k+1)$ submatrix of A has zero determinant.

   (It may be helpful to use the nontrivial fact that $\boxed{\mathrm{rank}(A) = \mathrm{rank}(A^\top)}$ *for any matrix A.)*

10. Let $\mathcal{V}$ be the vector space of 3×3 matrices.

    Define $L : \mathcal{V} \to \mathcal{V}$ as the linear transformation $L(A) = A + A^T$.

    (a) Describe a basis for $\mathcal{V}$. What is $\dim(\mathcal{V})$?

    (b) Find a basis for the subspace $\mathcal{N} = \{A \in \mathcal{V} : L(A) = 0\}$. What is $\dim(\mathcal{N})$?

    (c) Find a basis for the subspace $\mathcal{R} = \{L(A) : A \in \mathcal{V}\}$. What is $\dim(\mathcal{R})$?

    (d) Find two numbers $\lambda, \mu \in \mathbb{R}$ and two nonzero matrices $A, B \in \mathcal{V}$ such that

$$L(A) = \lambda A \quad \text{and} \quad L(B) = \mu B.$$