**Instructions:** *Complete the following exercises. Solutions will be graded on clarity as well as correctness. Feel free to discuss the problems with other students, but be sure to acknowledge your collaborators in your solutions, and to write up your final solutions by yourself.*

Due on **Thursday, February 24**.

*Suppose $V$ is a finite-dimensional vector space over any algebraically closed field $\mathbb{F}$.*

*Let $X : V \to V$ be a linear map. The* characteristic polynomial *of $X$ is* $\det(tI - X) \in \mathbb{F}[t]$*. Suppose this polynomial factors as* $\det(tI - X) = \prod_{i=1}^{k}(t - a_i)^{m_i}$ *where $a_i \neq a_j \in \mathbb{F}$ for $i \neq j$ and $m_i > 0$.*

*The following three exercises prove the existence of the generalized eigenspace decomposition corresponding to $X$ and then the Jordan decomposition of $X$. Your solutions to these problems therefore should only use elementary linear algebra and should not involve the Jordan canonical form.*

1. Define $V_i = \{v \in V : (X - a_i I)^n v = 0 \text{ for some } n > 0\}$. Show that $V = \bigoplus_{i=1}^{k} V_i$.

   In other words, check that $V_i \cap V_j = 0$ if $i \neq j$ and $V = V_1 + V_2 + \cdots + V_k$.

2. By considering the matrix of $X$ in an appropriate basis, show that the characteristic polynomial of $X$ is the product of the characteristic polynomials of the restricted maps $X|_{V_i} : V_i \to V_i$ for $i = 1, 2, \ldots, k$. Use this to deduce that $\dim V_i = m_i$ and that $V_i = \ker(X - a_i I)^{m_i}$.

3. Look up the statement of the Chinese Remainder Theorem for the polynomial ring $\mathbb{F}[t]$ and explain why this implies that there is a polynomial $p(t) \in \mathbb{F}[t]$ with

   $$p(t) \equiv a_i \pmod{(t - a_i)^{m_i}} \text{ for } i = 1, 2, \ldots, k \quad \text{and} \quad p(t) \equiv 0 \pmod{T}.$$

   Show that if $X_s = p(X)$ and $X_n = X - X_s$ then $X_s$ is diagonalizable and $X_n$ is nilpotent.

4. Assume $\mathbb{F}$ has characteristic zero and let $L = \mathfrak{sl}(V)$.

   Use Lie's Theorem to prove that $\mathrm{Rad}(L) = Z(L)$. Then show that $Z(L) = 0$ so $L$ is semsimple.

5. Suppose $X, Y \in \mathfrak{gl}(V)$ commute. Show that $(X + Y)_s = X_s + Y_s$ and $(X + Y)_n = X_n + Y_n$.

   Show by example that this can fail if $XY \neq YX$.

6. Show that if $L$ is a nilpotent Lie algebra then the Killing form is identically zero.

7. Show that a Lie algebra $L$ is solvable if and only if $[L, L]$ is inside the radical of the Killing form.

8. Compute the basis of $\mathfrak{sl}_2(\mathbb{F})$ dual to the standard basis $\{E, F, H\}$, relative to the killing form.