Instructions: *Complete the following exercises. Solutions will be graded on clarity as well as correctness. Feel free to discuss the problems with other students, but be sure to acknowledge your collaborators in your solutions, and to write up your final solutions by yourself.*

Due on **Thursday, March 17**.

Except when mentioned otherwise, all Lie algebras and vector spaces below are defined over an algebraically closed field $\mathbb{F}$ *of characteristic zero.*

1. Let m be a nonnegative integer and let $V(m)$ be a vector space with basis $v_0, v_1, v_2, \ldots, v_m$. Define $Hv_i = (m-2i)v_i$ and $Yv_i = (i+1)v_{i+1}$ and $Xv_i = (m-i+1)v_{i-1}$ where $v_{-1} = v_{m+1} = 0$. Show that these formulas extend to a module structure for the Lie algebra $\mathfrak{sl}_2(\mathbb{F})$ where

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

   To check this, verify that the matrices describing the action of H, Y, and X on $V(m)$ satisfy the same Lie bracket equations as H, Y, and X do.

2. $M = \mathfrak{sl}_3(\mathbb{F})$ contains a copy of $\mathfrak{sl}_2(\mathbb{F})$ in its upper left 2×2 position. We can view M as an $\mathfrak{sl}_2(\mathbb{F})$-module via the adjoint representation. Decompose M into irreducible $\mathfrak{sl}_2(\mathbb{F})$-modules and show that $M \cong V(0) \oplus V(1) \oplus V(1) \oplus V(2)$ as $\mathfrak{sl}_2(\mathbb{F})$-modules.

3. Suppose (just for this exercise) that $\mathbb{F}$ has characteristic $p > 0$. What numbers can occur as p? Show that the $\mathfrak{sl}_2(\mathbb{F})$-module $V(m)$ in Exercise 1 is irreducible if $m < p$, but reducible when $m = p$.

4. Let $\lambda \in \mathbb{F}$ be an arbitrary scalar. Let $M(\lambda)$ be a vector space with a countably infinite basis $v_0, v_1, v_2, \ldots$. Define $Hv_i = (\lambda - 2i)v_i$ and $Yv_i = (i+1)v_{i+1}$ and $Xv_i = (\lambda - i + 1)v_{i-1}$ where $v_{-1} = 0$. Your solution to Exercise 1 should easily extend to an argument that that these formulas make $M(\lambda)$ into an $\mathfrak{sl}_2(\mathbb{F})$-module. For which values of λ is $M(\lambda)$ irreducible? Prove your answer.

   The last two exercises implicitly contain four parts, so are worth more than the previous questions.

5. If L is a classical linear Lie algebra of type A_n, B_n, C_n, or D_n, prove that the set H of all diagonal matrices in L is a maximal toral subalgebra.

6. For each Lie algebra in the previous exercise, determine the roots and root spaces corresponding to the root space decomposition of L relative to the maximal toral subalgebra of diagonal matrices H.