Instructions: *Complete the following exercises. Solutions will be graded on clarity as well as correctness. Feel free to discuss the problems with other students, but be sure to acknowledge your collaborators in your solutions, and to write up your final solutions by yourself.*

Due on **Wednesday, February 15**.

1. Let $\mathfrak{A}$ be an $\mathbb{F}$-algebra, not necessarily associative. Show that the vector space of derivations $\operatorname{Der}\mathfrak{A}$ is a Lie subalgebra of $\mathfrak{gl}(\mathfrak{A})$. (See §III.1.3 in the textbook.)
2. Prove that if there exists a non-degenerate alternating bilinear form B on a finite-dimensional $\mathbb{F}$-vector space V then $\dim V$ is even. (*Alternating* means that $B(v,v)=0$ for all $v \in V$. This implies that B is skew-symmetric. The field $\mathbb{F}$ can have arbitrary characteristic, so be sure to address the case when $\operatorname{char}(\mathbb{F})=2$.)
3. Show that there exists a unique 2-dimensional Lie algebra $L = \mathbb{F}\text{-span}\{X,Y\}$ with $[X,Y]=X$. Find a subalgebra of $\mathfrak{gl}_n(\mathbb{F})$ for some n that is isomorphic to L. Finally, show that L is solvable but not nilpotent.
4. Suppose $X \in \mathfrak{gl}_n(\mathbb{F})$ has n distinct eigenvalues $a_1, a_2, \ldots, a_n \in \mathbb{F}$. Prove that the eigenvalues of $\operatorname{ad} X$ are the n^2 scalars $a_i - a_j$ for $1 \le i,j \le n$ (which are not necessarily distinct).
5. Let L be a Lie algebra over an algebraically closed field and let $X \in L$. Prove that the subspace of L spanned by the eigenvectors of $\operatorname{ad} X$ is a Lie subalgebra.
6. Show that $[\mathfrak{sl}_n(\mathbb{F}), \mathfrak{sl}_n(\mathbb{F})] = \mathfrak{sl}_n(\mathbb{F})$ if $\mathbb{F}$ has characteristic zero. Check that $\mathfrak{sl}_2(\mathbb{F})$ is nilpotent if $\mathbb{F}$ has characteristic 2.
7. Prove that a Lie algebra L is solvable if and only if there exists a chain of Lie subalgebras $L = L_0 \supset L_1 \supset L_2 \supset \cdots \supset L_k = 0$ such that each L_{i+1} is an ideal of L_i with L_i/L_{i+1} abelian.
8. Show that $\mathfrak{sl}_4(\mathbb{F}) \cong \mathfrak{o}_6(\mathbb{F})$.
9. Let L be a nilpotent Lie algebra. Prove that L has an ideal of codimension 1.
10. Suppose $\mathbb{F}$ has characteristic zero and $L \subseteq \mathfrak{gl}(V)$ is a Lie algebra for some vector space V. Show that if $X \in L$ is such that $\operatorname{ad} X$ is nilpotent then the formula

$$\exp(\operatorname{ad} X) := \sum_{n=0}^{\infty} \frac{1}{n!} (\operatorname{ad} X)^n$$

defines an automorphism of L. Show that if X is nilpotent (as an element of $\mathfrak{gl}(V)$) then

$$\exp(X) := \sum_{n=0}^{\infty} \frac{1}{n!} X^n$$

is an invertible element of $\mathfrak{gl}(V)$ such that

$$(\exp X) Y (\exp X)^{-1} = \exp(\operatorname{ad} X)(Y)$$

for all $Y \in L$. (See §III.2.3 in the textbook.)