# MATH 5143 - Lecture 17

# Some constructions

is orthogonal complement of $\mathbb{R}(\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_{n+1})$

so inherits nondeg. form from $\mathbb{R}^{n+1}$

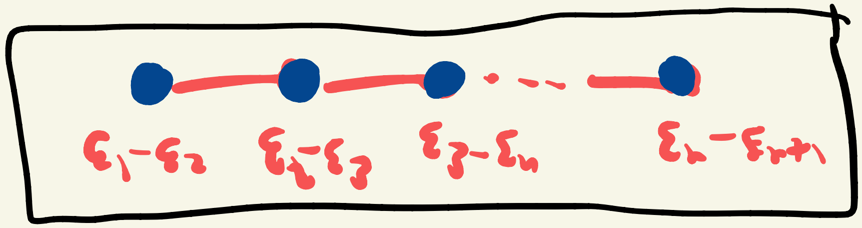
**Prop** Let $E = \{ v \in \mathbb{R}^{n+1} \mid v_1+v_2+v_3+\cdots+v_{n+1} = 0 \} \cong \mathbb{R}^n$

Write $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n+1}$ for standard basis of $\mathbb{R}^{n+1}$.

Define $\Phi_{A_n} = \{ \varepsilon_i - \varepsilon_j \mid 1 \leq i,j \leq n,\ i \neq j \}$

Then $\Phi_{A_n}$ is a root system with base $\Delta_{A_n} = \{ \varepsilon_i - \varepsilon_{i+1} \mid i = 1,2,\ldots,n \}$

and Dynkin diagram




Also, $r_{\varepsilon_i-\varepsilon_{i+1}}\left(\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}\right) = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} - (v_i - v_{i+1})(\varepsilon_i - \varepsilon_{i+1}) = \begin{bmatrix} v_1 \\ \vdots \\ v_{i+1} \\ v_i \\ \vdots \\ v_n \end{bmatrix}$

so it follows that Weyl group $W_{A_n} \cong S_{n+1}$ (symmetric group of $1,2,\ldots,n+1$)

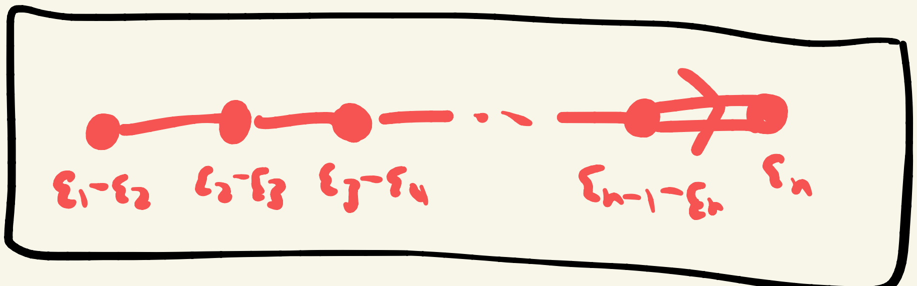
**Pf** All straightforward calculations. $\square$

**Prop** Let $\Phi_{B_n} \subseteq \mathbb{R}^n$ be set of $2n+4\binom{n}{2}$ vectors

$$\{\pm\varepsilon_i \mid i=1,2,\dots,n\} \sqcup \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le n\}.$$

Then $\Phi_{B_n}$ is a root system with base

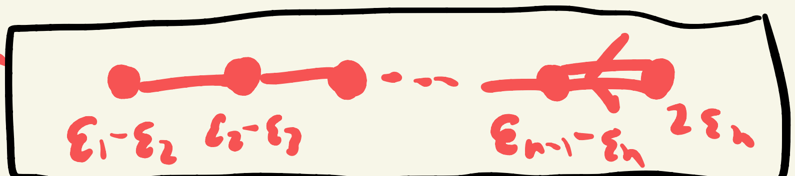
$$\Delta_{B_n} = \{\varepsilon_1-\varepsilon_2,\ \varepsilon_2-\varepsilon_3, \dots, \varepsilon_{n-1}-\varepsilon_n,\ \varepsilon_n\}$$

and Dynkin diagram 


The Weyl group $W_{B_n} \cong$ (signed $n \times n$ permutation matrices)

**Prop** Let $\Phi_{C_n} = \{\pm 2\varepsilon_i \mid i=1,2,\dots,n\} \sqcup \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le n\}$.

Then $\Phi_{C_n}$ is a root system with base $\Delta_{C_n} = \{\varepsilon_1-\varepsilon_2, \dots, \varepsilon_{n-1}-\varepsilon_n,\ 2\varepsilon_n\}$

and Dynkin diagram 

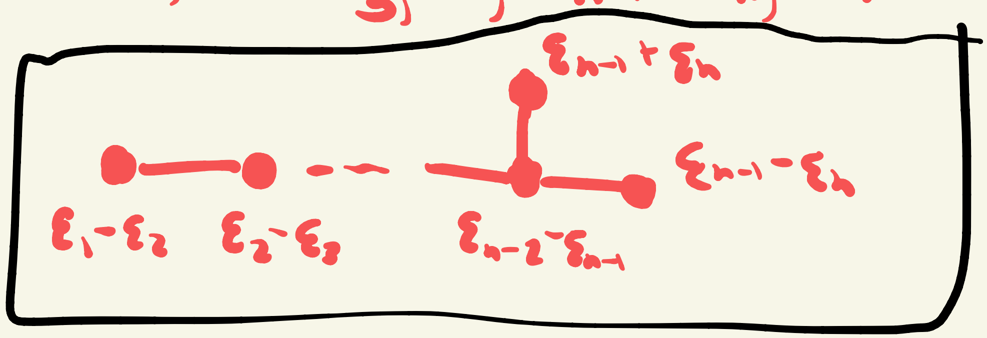

Also: $W_{C_n} = W_{B_n}$.

Prop Finally let $\Phi_{D_n} \subseteq \mathbb{R}^n$ be set of $4\binom{n}{2}$ vectors

$\{\pm \varepsilon_i \pm \varepsilon_j \mid 1 \leq i<j \leq n\}$. Then $\Phi_{D_n}$ is a root system

with base $\Delta_{D_n} = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \dots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n\}$

and Dynkin diagram




The Weyl group $W_{D_n}$ is an index two normal subgroup of $W_{B_n} = W_{C_n}$

(subgroup of even signed $n \times n$ permut. matrices)

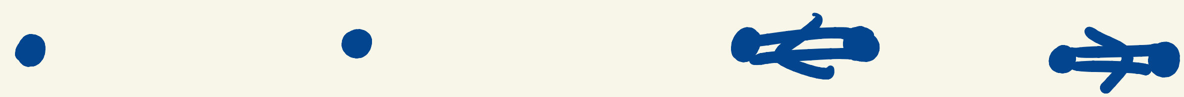
even # of $-1$ entries

$\Phi_{A_n}$ is irreducible $\forall n \geq 1$ (Dynkin diagram is connected)

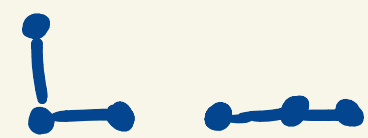
$\Phi_{B_1} \cong \Phi_{A_1}$ as Dynkin diagram is just an isolated vertex

So we only consider $\Phi_{B_n}$ for $n \geq 2$

$\Phi_{C_1} \cong \Phi_{B_1}$ and $\Phi_{C_2} \cong \Phi_{B_2}$ since Dynkin diagrams are isomorphic.



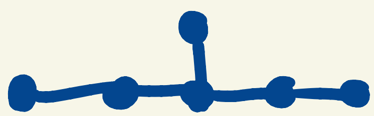
so we only consider $\Phi_{C_n}$ for $n \geq 3$.

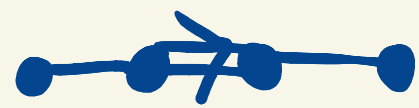


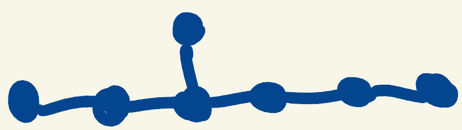
$\Phi_{D_1} \cong \Phi_{A_1}$, $\Phi_{D_2} \cong \Phi_{A_1 \times A_1}$ (not irreducible) $\Phi_{D_3} \cong \Phi_{A_3}$

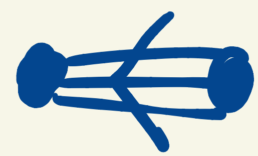
so we only consider $\Phi_{D_n}$ for $n \geq 4$.

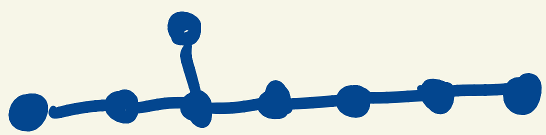
**Thm** Suppose $\Phi$ is an irreducible root system. Then the Dynkin diagram of $\Phi$ is either isomorphic to the Dynkin diagram of $\Phi_{A_n}$ (some $n \geq 1$), $\Phi_{B_n}$ (some $n \geq 2$), $\Phi_{C_n}$ (some $n \geq 3$), $\Phi_{D_n}$ (some $n \geq 4$), or to one of 5 "exceptional" diagrams:

$E_6$ 

$F_4$ 

$E_7$ 

$G_2$ 

$E_8$ 

Moreover, each of these exceptional diagrams does arise as the Dynkin diagram of an (irreducible) root system.

Proof that every irreducible root system has one of these Dynkin diagrams $\longrightarrow$ See the relevant sections of textbook if interested

## Constructions for $E_n$, $F_4$, $G_2$

$\Phi_{G_2}$: we've already seen

$$\Phi_{F_4} \overset{\text{def}}{=} \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq 4\} \sqcup \left\{\tfrac{1}{2}(a_1\varepsilon_1 + a_2\varepsilon_2 + a_3\varepsilon_3 + a_4\varepsilon_4) \;\middle|\; a_1, a_2, a_3, a_4 \in \{\pm 1\}\right\}$$

$$\sqcup \{\pm\varepsilon_i \mid 1 \leq i \leq 4\} \subseteq \mathbb{R}^4$$

has 48 elements, Weyl group has size 1152.

has base $\{\varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, \varepsilon_4, \tfrac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\}$

Suffices to construct $\Phi_{E_8}$ as $\Phi_{E_6}, \Phi_{E_7}$ can then be realized as subsystems. We can construct $\Phi_{E_8} \subseteq \mathbb{R}^8$ as the set of 240 vectors of the form

$$\{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq 8\} \sqcup \left\{ \begin{array}{l} \frac{1}{2}(a_1\varepsilon_1 + a_2\varepsilon_2 + \cdots + a_8\varepsilon_8) \\ a_1, a_2, \ldots, a_8 \in \{\pm 1\} \text{ with} \\ a_1 \cdot a_2 \cdot a_3 \cdots a_8 = +1 \end{array} \right\}$$

This is a root system with base

$$\Delta_{E_8} = \left\{ \begin{array}{l} \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8),\ \varepsilon_1 + \varepsilon_2 \\ \varepsilon_2 - \varepsilon_3,\ \varepsilon_3 - \varepsilon_4,\ \varepsilon_4 - \varepsilon_5,\ \varepsilon_5 - \varepsilon_6,\ \varepsilon_6 - \varepsilon_7,\ \varepsilon_7 - \varepsilon_8 \end{array} \right\}$$

# Isomorphism and conjugacy theorems

Recall that if $L$ is a semisimple Lie algebra (over an alg. closed, char. zero field $\mathbb{F}$), and $H \subseteq L$ is a maximal toral subalgebra then there is a finite set $\Phi \subseteq H^* \setminus \{0\}$ with $L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$ where

$$L_\alpha \overset{\text{def}}{=} \{x \in L \mid [h,x] = \alpha(h)x \ \forall h \in\} \neq 0 \text{ for } \alpha \in \Phi.$$

The set $\Phi$ is a root system in $E = \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{Q}\text{-span}\{\alpha \in \Phi\}$ with the bilinear form on $H^*$ dual to the Killing form of $L$ restricted to $H$.

**Prop** If $L$ is simple then $\Phi$ is irreducible

**Pf** If $\Phi = \Phi_1 \sqcup \Phi_2$ were reducible (with $\Phi_1, \Phi_2$ nonempty and orthogonal)

and $\alpha \in \Phi_1$, $\beta \in \Phi_2$, then $\alpha+\beta$ is neither in

$\Phi_1$ (since $(\beta, \alpha+\beta) = (\beta,\beta) \neq 0$) nor in $\Phi_2$

(since $(\alpha, \alpha+\beta) = (\alpha,\alpha) \neq 0$) so $\alpha+\beta \notin \Phi$

and it follows that the subalgebra of $L$ generated

by $L_\alpha$ for $\alpha \in \Phi_1$ is a proper nonzero ideal.

(since $[L_\alpha, L_\beta] = 0 \ \forall \alpha \in \Phi_1, \beta \in \Phi_2$) $\square$

<u>Prop</u> If $L = L_1 \oplus L_2 \oplus \cdots \oplus L_n$ is the decomposition of $L$ into simple ideals then $H_i \overset{\text{def}}{=} H \cap L_i$ is a maximal toral subalgebra of $L_i$ and the irreducible root system $\Phi_i$ determined by $H_i \subseteq L_i$ may be viewed as a subsystem of $\Phi$ relative to which

$$\Phi = \Phi_1 \cup \Phi_2 \cup \cdots \cup \Phi_n$$

is the decomposition of $\Phi$ into irreducible components.

<u>Pf</u> See discussion in §14.1 of textbook. □

Thm Suppose $L'$ is another semisimple Lie algebra with a maximal toral subalgebra $H'$ and root system $\Phi'$. Suppose there exists a root system isomorphism $f : \Phi \to \Phi'$.

Extend $f$ to a vector space isomorphism $f : H \xrightarrow{\sim} H'$ by setting $f(t_\alpha) = t'_{f(\alpha)}$ where for $\alpha \in \Phi$, $\alpha' \in \Phi'$, $t_\alpha \in H$ and $t'_{\alpha'} \in H'$ are the elements with $\begin{cases} \kappa(t_\alpha, h) = \alpha(h) \\ \kappa(t'_{\alpha'}, h') = \alpha'(h') \end{cases}$

Choose a base $\Delta \subseteq \Phi$ along with isomorphisms between the 1-dim root spaces $L_\alpha \xrightarrow{\sim} L'_{f(\alpha)}$ for $\alpha \in \Delta$. Then there is a unique Lie algebra isomorphism $L \xrightarrow{\sim} L'$ extending $f : H \to H'$ and these chosen isomorphisms.

This theorem does require some proof → see textbook.

Proof in §14 of textbook does not establish existence of a semisimple Lie algebra corresponding to any Dynkin diagram.

Existence of this is shown in §18, we may discuss in a few lectures. (For classical type ABCD, just use the classical algebras $\mathfrak{sl}_n$, $\mathfrak{o}_n$, $\mathfrak{sp}_n$ — main issue is about exceptional types)