The Mysterious Dilogarithm

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By analogy, we have:

**Definition (Leibnitz 1696; Euler 1768)**

The polylogarithm is defined by the power series

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\(\text{Li}_2(x)\) is called the dilogarithm function.
From the definition, it is clear that:

\[ \frac{d}{dx} \text{Li}_m(x) = \frac{1}{x} \text{Li}_{m-1}(x) \quad m \leq 2 \]
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Hence we can give an analytic continuation of the dilogarithm by:

\[ \text{Li}_2(z) = - \int_0^z \log(1 - u) \frac{du}{u} \quad \text{for } z \in \mathbb{C} \setminus [1, \infty) \]
Dilogarithm

Introduction

Definition
Reflection properties

Proposition

\[ \text{Li}_2\left(\frac{1}{z}\right) + \text{Li}_2(z) = -\frac{\pi^2}{6} - \frac{1}{2} \log^2(-z) \]

\[ \text{Li}_2(1 - z) + \text{Li}_2(z) = \frac{\pi^2}{6} - \log(z) \log(1 - z) \]
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Proof: Differentiating both sides.
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Proof: Differentiating both sides.
Applying these formula, we see that the 6 functions:

\[
\text{Li}_2(z), \text{Li}_2\left(\frac{1}{1 - z}\right), \text{Li}_2\left(\frac{z - 1}{z}\right), -\text{Li}_2\left(\frac{1}{z}\right), -\text{Li}_2(1 - z), -\text{Li}_2\left(\frac{z}{z - 1}\right)
\]

are equal modulo elementary functions.
Proposition (Duplication formula)

\[ \text{Li}_2(z^2) = 2(\text{Li}_2(z) + \text{Li}_2(-z)) \]
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and more generally the "distribution property":

\[ \text{Li}_2(x) = n \sum_{z^n = x} \text{Li}_2(z) \quad (n = 1, 2, 3...) \]
There are exactly 8 values of $z$ for which $z$ and $\text{Li}_2(z)$ can both be given in closed form:

\[
\begin{align*}
\text{Li}_2(0) &= 0 \\
\text{Li}_2(1) &= \frac{\pi^2}{6} \\
\text{Li}_2(-1) &= -\frac{\pi^2}{12}
\end{align*}
\]

\[
\begin{align*}
\text{Li}_2\left(\frac{1}{2}\right) &= \frac{\pi^2}{12} - \frac{1}{2} \log^2(2) \\
\text{Li}_2(\phi) &= \frac{\pi^2}{10} - \log^2(\phi^{-1}) \\
\text{Li}_2(-\phi) &= -\frac{\pi^2}{15} - \frac{1}{2} \log^2(\phi^{-1}) \\
\text{Li}_2(\phi^{-1}) &= \frac{\pi^2}{15} - \log^2(\phi^{-1}) \\
\text{Li}_2(-\phi^{-1}) &= -\frac{\pi^2}{10} - \frac{1}{2} \log^2(\phi^{-1})
\end{align*}
\]

where $\phi = \frac{\sqrt{5}-1}{2}$ is the golden ratio.
Let’s consider the recurrence relation

$$1 - z_n = z_{n-1}z_{n+1}$$
Five-Term Relation

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Five-Term Relation

Let’s consider the recurrence relation

\[ 1 - z_n = z_{n-1} z_{n+1} \]

If we let the initial values to be \( z_0 = x, z_1 = 1 - xy \) (so \( z_2 = y \)), then we have:

\[
\begin{align*}
  z_3 &= \frac{1 - z_2}{z_1} = \frac{1 - y}{1 - xy} \\
  z_4 &= \frac{1 - z_3}{z_2} = \frac{1 - x}{1 - xy} \\
  z_5 &= \frac{1 - z_4}{z_3} = x \\
  z_6 &= \frac{1 - z_5}{z_4} = 1 - xy \\
\end{align*}
\]

so this recurrence relation actually has period 5!
The most important functional equation is the following:

**Theorem (Spence(1809), Abel(1827), Hill(1828), Kummer(1840), Schaeffer(1846)...)**

\[
\text{Li}_2(x) + \text{Li}_2(1 - xy) + \text{Li}_2(y) + \text{Li}_2\left(\frac{1 - y}{1 - xy}\right) + \text{Li}_2\left(\frac{1 - x}{1 - xy}\right) = \frac{\pi^2}{6} - \log(x) \log(1 - x) - \log(y)(1 - y) + \log\left(\frac{1 - x}{1 - xy}\right) \log\left(\frac{1 - y}{1 - xy}\right)
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\]

\[
= \frac{\pi^2}{6} - \log(x) \log(1 - x) - \log(y)(1 - y) + \log\left(\frac{1 - x}{1 - xy}\right) \log\left(\frac{1 - y}{1 - xy}\right)
\]

The right hand side is a junk — they can be removed by giving an equivalent but modified definition of the dilogarithm function.
Bloch-Wigner function $D(z)$

- $\text{Li}_2(z)$ has a jump by $2\pi i \log |z|$ across the cut.
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**Definition**

The Bloch-Wigner function $D(z)$ is defined by

$$\Im(\text{Li}_2(z)) + \arg(1 - z) \log |z|$$
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- (Kummer)

$$D(z) = \frac{1}{2} \left[ D\left(\frac{z}{\bar{z}}\right) + D\left(\frac{1 - 1/\bar{z}}{1 - 1/z}\right) + D\left(\frac{1/(1-z)}{1/(1-\bar{z})}\right) \right]$$

i.e. $D(z)$ only depends on its value on the unit circle:

$$D(e^{i\theta}) = \Im[\text{Li}_2(e^{i\theta})] = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n^2}$$
Bloch-Wigner function $D(z)$

All functional equations for $\text{Li}_2(z)$ lose the elementary terms. In particular:

\[
D(z) + D(1 - \frac{1}{1 + z}) + D(y) + D(1 - \frac{1}{1 + y}) = 0
\]
Bloch-Wigner function $D(z)$

All functional equations for $\text{Li}_2(z)$ lose the elementary terms. In particular:

- (6-fold symmetry)

$$D(z) = D\left(\frac{1}{1-z}\right) = D\left(\frac{z-1}{z}\right)$$

$$= -D\left(\frac{1}{z}\right) = -D(1-z) = -D\left(\frac{z}{z-1}\right)$$
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- (5-term relation)

\[ D(x) + D(1-xy) + D(y) + D\left(\frac{1-y}{1-xy}\right) + D\left(\frac{1-x}{1-xy}\right) = 0 \]
Bloch-Wigner function \( D(z) \)

The relation become even nicer if we write \( D \) in terms of cross-ratio of 4 numbers:
Bloch-Wigner function $D(z)$

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\tilde{D}(z_0, z_1, z_2, z_3) = D \left( \frac{z_0 - z_2}{z_0 - z_3} \frac{z_1 - z_3}{z_1 - z_2} \right) \quad (z_0, z_1, z_2, z_3 \in \mathbb{C})
$$

Then:
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Then:

- ”6-fold symmetry” says that $\tilde{D}$ is (anti)invariant under (odd)even permutation of its 4 variables,

- ”5-term relation” becomes

$$\sum_{i=0}^{4} (-1)^i \tilde{D}(z_0, ..., \hat{z}_i, ..., z_4) = 0 \quad (z_0, ..., z_4 \in \mathbb{P}^1(\mathbb{C}))$$
5 term relation

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**Theorem**

\[ D(z) \text{ is the unique measurable function on } \mathbb{P}^1(\mathbb{C}) \text{ (up to constant) satisfying the 5 term relation.} \]
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**Theorem (Wojtkowiak)**

Every functional equation of the form \( \sum_i D(x_i(t)) = C \) is a formal consequence of the 5 term relation.
Here \( x_i(t) \) is a rational function in \( t \), and \( C \) is a constant.
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All applications onward will be related to THE 5-term relation.
Let’s realize the hyperbolic 3-space as $\mathbb{H}_3 = \mathbb{C} \times \mathbb{R}_+$ with standard hyperbolic metric.

(i.e. geodesics = vertical lines/semicircles in vertical planes with endpoints in $\mathbb{C} \times \{0\}$ etc.)
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**Definition**

An *ideal tetrahedron* is a tetrahedron whose vertices are all in $\partial \mathbb{H}_3 = \mathbb{C} \cup \{\infty\} = \mathbb{P}^1(\mathbb{C})$
How does Ideal Tetrahedra look like?
**Theorem (Lobachevsky)**

The hyperbolic volume of an ideal tetrahedron is finite, and is given by

$$Vol(\Delta) = \tilde{D}(z_0, z_1, z_2, z_3)$$
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Ideal Tetrahedra

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Then:

- ”6-fold symmetry” follows from the fact that renumbering the vertices leaves \( \Delta \) unchanged but may change the orientation.

- ”5-term relation” follows from the fact that the five \( \Delta \)'s spanned by 4 at a time of \( z_0, ..., z_4 \in \mathbb{P}^1(\mathbb{C}) \), with signs, add up algebraically to a zero 3-cycle.
It turns out that the group $SL_2(\mathbb{C})$ acts on $\mathbb{H}_3$ by isometries, and it can always bring \{\(z_0, z_1, z_2, z_3\}\} into the form \{\(\infty, 0, 1, z\)\}. 

Theorem (Jrgensen and Thurston)

The “volume spectrum” $\text{Vol}(M)$ is a countable and well-ordered subset of $\mathbb{R}_+$. 

**Volume of Hyperbolic 3-manifold**

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- Every complete oriented hyperbolic 3-manifold with finite volume can be triangulated into ideal tetrahedra.
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**Theorem (Jørgensen and Thurston)**

"volume spectrum"

$$Vol = \{ Vol(M) | M \text{ a hyperbolic 3-manifold} \} \subset \mathbb{R}_+$$

is a countable and well-ordered subset of $\mathbb{R}_+$. 
Bloch Group

- The parameters $z_v$ of the tetrahedra triangulation need to satisfy

$$\sum_{v=1}^{n} z_v \wedge (1 - z_v) = 0$$

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in the abelian group $\wedge^2 \mathbb{C}^\times$.

Here $\wedge^2 \mathbb{C}^\times$ is the set of all formal linear combinations $x \wedge y, x, y \in \mathbb{C}^\times$ subject to the relations

$$x \wedge x = 0$$

and

$$(x_1 x_2) \wedge y = x_1 \wedge y + x_2 \wedge y.$$
Consider the abelian group of formal sums $[z_1] + \ldots + [z_n]$ with $z_i \in \mathbb{C}^\times \setminus \{1\}$ satisfying $\sum_{v=1}^{n} z_v \wedge (1 - z_v) = 0$. 
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Then it contains the elements:

$$[x] + \left[\frac{1}{x}\right], \quad [x] + [1 - x],$$

$$[x] + [1 - xy] + [y] + \left[\frac{1 - y}{1 - xy}\right] + \left[\frac{1 - x}{1 - xy}\right] \quad (*)$$

corresponding to the symmetries and 5-term relation for $D(z)$. 

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corresponding to the symmetries and 5-term relation for $D(z)$.

**Definition**

The *Bloch Group* $\mathcal{B}_\mathbb{C}$ is defined as the quotient of this abelian group with the subgroup generated by the elements $(\ast)$.
It follows that $D$ extends to a linear map

$$D : \mathcal{B}_\mathbb{C} \longrightarrow \mathbb{R}$$

by

$$[z_1] + \ldots + [z_n] \mapsto D(z_1) + \ldots + D(z_n)$$
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**Theorem (Bloch)**

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The set $D(\mathcal{B}_C)$ coincides with $D(\mathcal{B}_\mathbb{Q})$.

In particular, **Vol** is countable.
Definition

The *Dedekind Zeta Function* of a number field $F$ is defined as

$$
\zeta_F(s) = \prod_{p \subset \mathcal{O}_F} \left(1 - \frac{1}{(Np)^s}\right)^{-1} = \sum_{a \subset \mathcal{O}_F} \frac{1}{(Na)^s}
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where $\mathcal{O}_F$ is the number ring of $F$, and $Na = |\mathcal{O}_F/a|$ is the *norm* of $a$. 
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When $F = \mathbb{Q}$, this is just the Riemann Zeta function:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{n \in \mathbb{Z}} \frac{1}{n^s}$$
Examples

- Let $F = \mathbb{Q}(\sqrt{-a})$ with $a \geq 1$ square free.
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- Then
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  where
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  is the L-series.
- Here $\left( \frac{d}{n} \right)$ is the Kronecker Symbol, taking values $\pm 1$ or 0 and periodic with period $|d|$ in $n$. 
Examples

For $a = -7$, we have:

$$\zeta_{\mathbb{Q}(\sqrt{-7})}(s) = \left( \sum_{n=1}^{\infty} n^{-s} \right) \left( \sum_{n=1}^{\infty} \left( \frac{-7}{n} \right) n^{-s} \right)$$

$$\left( \frac{-7}{n} \right) = \begin{cases} +1 & n \equiv 1, 2, 4 \mod 7 \\ -1 & n \equiv 3, 5, 6 \mod 7 \\ 0 & n \equiv 0 \mod 7 \end{cases}$$
One of the questions of interest is the evaluation of the Dedekind Zeta Function at integer arguments.
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It is well known that $\zeta_F(1)$ can be expressed using the usual logarithm (through a term called regulator).

We expect that $\zeta_F(2)$ can be expressed using dilogarithm also.
Examples

\[ \zeta_F(2) = \zeta(2) L(2) = \frac{\pi^2}{6} \sum_{n \geq 1} \left( \frac{d}{n} \right) n^{-2} \]
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Since \( \left( \frac{d}{n} \right) \) is periodic in \( n \), we can write it as finite linear combinations of \( e^{2\pi i kn/|d|} \) and obtain:

\[ \zeta_F(2) = \frac{\pi^2}{6 \sqrt{|d|}} \sum_{k=1}^{|d|-1} \left( \frac{d}{k'} \right) D(e^{2\pi ik/|d|}) \]
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For example:

\[ \zeta_{\mathbb{Q}(\sqrt{-7})}(2) = \frac{\pi^2}{3 \sqrt{7}} (D(e^{2\pi i/7}) + D(e^{4\pi i/7}) - D(e^{6\pi i/7})) \]

expressing \(\zeta_F(2)\) in closed form using \(D(z)\) at algebraic arguments \(z\).
Examples

By considering $\Gamma = SL_2(\mathcal{O}_F)$ as a discrete subgroup of $SL_2(\mathbb{C})$, hence acts on $\mathfrak{h}_3$: 

Theorem (Humbert, Zagier)

$$Vol(\mathfrak{h}_3) = \frac{1}{d} \frac{1}{3!} = 2$$

and $H_3$ can be triangulated into ideal tetrahedra with vertices on $P_1(\mathcal{O})$.

Hence

$$F(2) = 2\frac{1}{3!} \frac{1}{d}$$

where $n_v \in \mathbb{Z}$ and $z_v \in F$, a much smaller field than $\mathbb{Q}(e^{2\pi i/d})$.

For example:

$$\mathbb{Q}(\sqrt{7})(2) = 2\frac{1}{3!} \frac{1}{d} \sqrt{7} + D(1 + \sqrt{7}) + D(1 + \sqrt{7}^4)$$
Examples

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**Theorem (Humbert, Zagier)**

$$Vol(\mathfrak{H}_3/\Gamma) = |d|^{3/2} \zeta_F(2)/4\pi^2$$

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By considering $\Gamma = SL_2(O_F)$ as a discrete subgroup of $SL_2(\mathbb{C})$, hence acts on $\mathfrak{H}_3$:

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- For example:

$$\zeta_{\mathbb{Q}(\sqrt{-7})}(2) = \frac{4\pi^2}{21\sqrt{7}} \left(2D\left(\frac{1 + \sqrt{-7}}{2}\right) + D\left(\frac{-1 + \sqrt{-7}}{4}\right)\right)$$
Algebraic K Theory

For general number field $F$ with $r_1$ real and $r_2$ pairs of complex embeddings, the relation between $D(z)$ and $\zeta_F(2)$ is given nicely through the use of Algebraic K Theory by A. Borel.
Algebraic K Theory

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- The Bloch Group for $F$: $\mathcal{B}_F$ is isomorphic to a some quotient of $K_3(F)$. 
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- The Bloch Group for $F$: $B_F$ is isomorphic to a some quotient of $K_3(F)$.
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**Theorem (A. Borel)**

- The Bloch Group for $F$: $\mathcal{B}_F$ is isomorphic to a some quotient of $K_3(F)$.
- $\mathcal{B}_F/\{\text{torsion}\} \cong \mathbb{Z}^{r_2}$
- The image of the map $\mathcal{B}_F \rightarrow \mathcal{B}_C^{r_2} \xrightarrow{D} \mathbb{R}^{r_2}$ after torsion, has co-volume:

$$c|d|^{1/2}\zeta_F(2)/\pi^{2r_1+2r_2} \text{ for some } c \in \mathbb{Q}$$

Here the first map corresponds to the $r_2$ different complex embeddings of $F$ to $\mathbb{C}$. 
Conjecture (Lichtenbaum)

The rational multiple $c$ is related to

$$\frac{|K_3(F)_{\text{torsion}}|}{|K_2(\mathcal{O}_F)|}$$
Algebraic K Theory

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- So the mysterious dilogarithm gives, at least conjecturally, an effective way of calculating the orders of certain groups in algebraic K-theory!
Rogers dilogarithm \( L(z) \)

Another version of dilogarithm, taking real arguments, is more common in physical literature: 

\[
L(z) := \text{Li}_2(z) + \frac{1}{2} \log^2(1 - z)
\]

and has an analytic continuation to \( \mathbb{C} \) on \([-1, 1]). \)

Furthermore, \( L(z) \) belongs to the class \( C^1([0, 1]) \).
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and has an analytic continuation to \( \mathbb{C} \setminus ((-\infty, 0] \cup [1, \infty)) \). Furthermore, \( L(x) \) belongs to the class \( C^\infty((0, 1)) \).
The five-term relation is now simplified to:

\[ L(x) + L(1 - xy) + L(y) + L\left(\frac{1 - y}{1 - xy}\right) + L\left(\frac{1 - x}{1 - xy}\right) = \frac{\pi^2}{2} \]

where \(0 < x, y < 1\).
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$L(x)$ is the unique function in $C^3((0, 1))$ that satisfies the five-term relation.
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Definition

\(L(x)\) is extended to the rest of \(\mathbb{R}\) by setting \(L(0) = 0, L(1) = \frac{\pi^2}{6}\),

\[ L(x) = \begin{cases} 2L(1) - L(1/x) & \text{if } x > 1 \\ -L(x/(x-1)) & \text{if } x < 0 \end{cases} \]
Rogers dilogarithm

\[ L(z) \]

Definition

Rogers dilogarithm is defined by the integral

\[ L(z) = -\frac{\pi^2}{6} + \int_1^z \frac{\log(1-t/2)}{t} \, dt \]

modulo 2, this function is "continuous" at 1, and the five-term relation holds. Every "nice" functional equations is again a consequence of the five-term relation.
Rogers dilogarithm

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The Rogers dilogarithm $L(z)$ is well known in the physics literature, especially in rational conformal field theory.
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For example, consider the identity:

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\sum_{i=1}^{[k/2]} L \left( \frac{\sin^2 \frac{\pi}{k+2}}{\sin^2 \frac{(i+1)\pi}{k+2}} \right) = L(1) \frac{k - 1}{k + 2}
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on the left hand side, the expression (Jones indices):

$$\frac{\sin \frac{(i+1)\pi}{k+2}}{\sin \frac{\pi}{k+2}}$$

is the ”quantum dimensions” of the primary fields of this WZW theory.
a $q$ Hypergeometric series is (roughly) a series of the form
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\sum_{n=0}^{\infty} A_n(q) \text{ where } \frac{A_n(q)}{A_{n-1}(q)} \text{ is rational function of } q
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- Consider the $r$-fold $q$-hypergeometric series defined by:

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 f_{A,B,C}(z) = \sum_{n \in (\mathbb{Z}_{\geq 0})^r} \frac{q^{\frac{1}{2}n^t A n + B^t n + C}}{(q)_{n_1} \cdots (q)_{n_r}}
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Conjecture (Nahm)

*Given positive definite symmetric \( r \times r \) matrix \( A \), the following is equivalent:*
Modular Function

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**Conjecture (Nahm)**

*Given positive definite symmetric $r \times r$ matrix $A$, the following is equivalent:*

- The element $\xi_Q$ is torsion in $\mathcal{B}_F$ for every solution $Q = (Q_i)$
- There exists $B \in \mathbb{Q}^r, C \in \mathbb{Q}$ such that $f_{A,B,C}(z)$ is a modular function.*
Modular Function

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$L(x)$ also appears in the asymptotic analysis for $f_{A,B,C}$: $L(1) - L(Q)$ is the leading coefficient for the series when $q = e^{-\epsilon} \to 1$ as $\epsilon \searrow 0$. 
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- This conjecture is motivated from physics: all modular functions $f_{A,B,C}$ obtained in this way should be characters of rational conformal field theories.
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- $L(x)$ also appears in the asymptotic analysis for $f_{A,B,C}$: $L(1) - L(Q)$ is the leading coefficient for the series when $q = e^{-\epsilon} \longrightarrow 1$ as $\epsilon \searrow 0$.
- This conjecture is motivated from physics: all modular functions $f_{A,B,C}$ obtained in this way should be characters of rational conformal field theories.
- In some special cases, the proof uses Quantum Dilogarithm.
Quantum Dilogarithm

$q$ Generalization of the Dilogarithm function ($|q| < 1$)
Quantum Dilogarithm

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Definition (Faddeev)

\[ S_q(w) = \prod_{n=0}^{\infty} (1 + q^{2n+1}w) \]
# Quantum Dilogarithm

$q$ Generalization of the Dilogarithm function ($|q| < 1$)

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S_q(w) = \prod_{n=0}^{\infty} (1 + q^{2n+1}w) = 1 + \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k-1)/2} w^k}{(q - q^{-1}) \cdots (q^k - q^{-k})}
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They are the same because they satisfy: \(\frac{S_q(qw)}{S_q(q^{-1}w)} = \frac{1}{1+w}\) and \(S_q(0) = 1\).
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- The second expression says \( S_q(w) \) is like an Exponential function
- The last expression says \( S_q(w) \) is like a Dilogarithm function!
Quantum 5-term relation

**Theorem**

*If* $uv = q^2 vu$ *is a Weyl pair, then:*

\[
S_q(u) S_q(v) = S_q(u + v) \quad S_q(v) S_q(u) = S_q(u) S_q(v^1) S_q(v) S_q(u) = S_q(u) S_q(q^1 uv) S_q(v) S_q(u) \]

These are proven formally using the power series expansion. The last relation reduces to the 5-term relation in a suitable $q!$ limit.
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Knot Invariant

- The Quantum 5-term relation is used to prove braid relation:
- Define
  \[ \Theta(w) = S_q(qw) S_q(q^{-1}w^{-1}) \]
- Then for the Weyl pair it satisfies the Braid Relation:
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The Quantum 5-term relation is used to prove braid relation:

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Then for the Weyl pair it satisfies the Braid Relation:

$$\Theta(u)\Theta(v)\Theta(u) = \Theta(v)\Theta(u)\Theta(v)$$

Using this fact, Hikami constructed a Knot Invariant related to the complement of the hyperbolic volume of links.
More importantly, the Quantum Dilogarithm is used to construct the Universal $R$-matrix of $SL_q(2)$, the most important component of Quantum Group Theory:
More importantly, the Quantum Dilogarithm is used to construct the Universal $R$-matrix of $SL_q(2)$, the most important component of Quantum Group Theory:

**Theorem (Drinfeld)**

For the quantum group $SL_q(2) = \langle K = q^H, K', e, f \rangle$, the Universal $R$ Matrix is given by:

$$R = q^{-\frac{H \otimes H'}{2}} S_q(-(q - q^{-1})^2 e \otimes f) \in U_q \otimes U_q$$

satisfying the Yang-Baxter Relation:

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$
Many Other Applications...

- **Combinatorial formula for characteristic classes**
  (Gel’fand, MacPherson...)

- **Cohomology of $GL_n(\mathbb{C})$**
  (A. Borel, Dupont, Quillen...)

- **Rogers-Ramanujan’s type identities, asymptotic behavior of partitions**
  (Ramanujan, Hardy, Littlewood...)

- **Representation Theory of infinite dimensional Lie Algebra**
  (Lepowsky, Kac, Fuchs, E.Frenkel...)

- **Exactly Solvable Models**
  (Baxter, Kirillov, Reshetikhin...)

- **Feynman Integral of Ladder Diagrams**
  (Ussyukina, Davydchev...)

- and much more...
"The dilogarithm function is the only mathematical function with a sense of humor."

– Don Zagier
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